

# Integrated Unitarity for Scattering Amplitudes

[2403.18047](#)

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TUM-MPP Collider Phenomenology Seminar



**Universität  
Zürich**<sup>UZH</sup>



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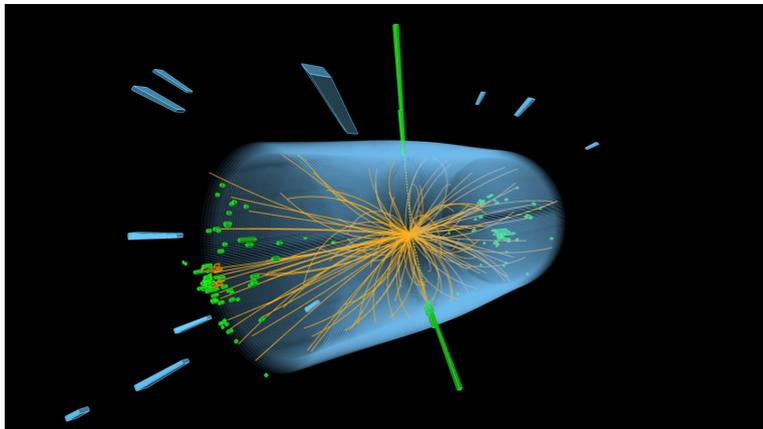
# Presentation plan

- Overview (motivation, state-of-the-art)
- Background (cuts, discontinuities, dispersion relation)
- Integrated Unitarity (derivation, example, application)
- Outlook (Multivariate Integrated Unitarity)
- Outro (what's next?)

# Overview

# Overview : motivation

precision HEP

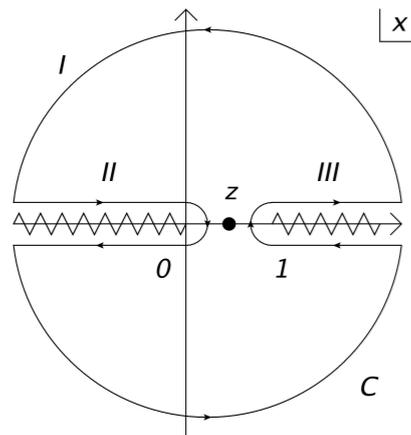


Scattering Amplitudes

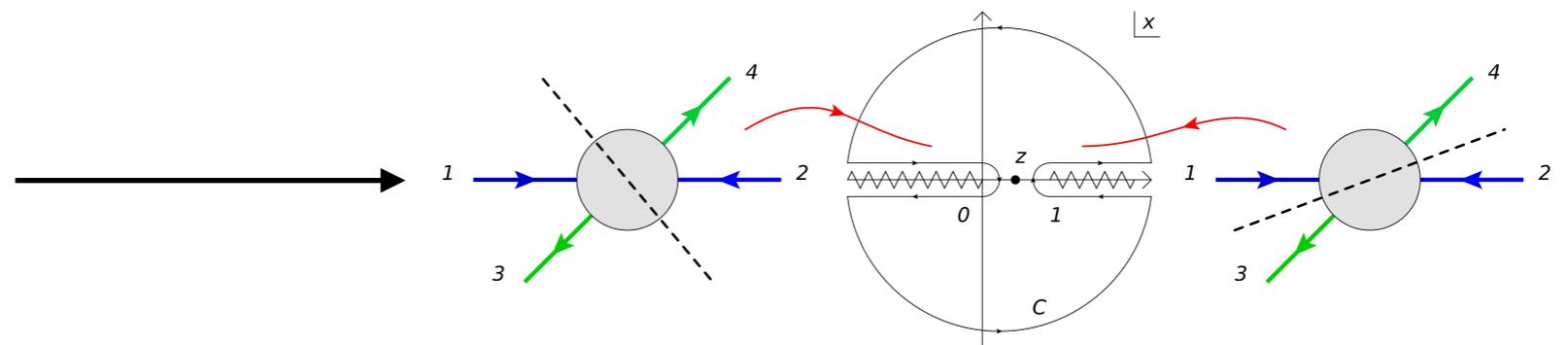
$$\leftarrow \begin{array}{c} 2 \\ \diagup \\ \text{---} \circ (z) \\ \diagdown \\ 1 \end{array} \begin{array}{c} 3 \\ \diagup \\ \text{---} \\ \diagdown \\ 4 \end{array} = \begin{array}{c} 2 \\ \diagup \\ \text{---} \circ \\ \diagdown \\ 1 \end{array} \begin{array}{c} 1 \\ \diagup \\ \text{---} \\ \diagdown \\ 2 \end{array} + \frac{1}{2\pi i} \int_1^\infty \sum_{\{c_j\}} \left( \frac{1}{x-z} - \frac{1}{x-z_0} \right) dx \begin{array}{c} 3 \\ \diagup \\ \text{---} \circ \text{---} \circ (x) \\ \diagdown \\ 2 \end{array} \begin{array}{c} 4 \\ \diagup \\ \text{---} \\ \diagdown \\ 1 \end{array}$$



mathematical structure



useful properties



# Overview : QCD amplitudes frontier

- 2-point : 5-loop ( $\beta$  function)

[Herzog, Ruijl, Ueda, Vermaseren, Vogt [1701.01404](#)]

- 3-point : 4-loop (form factors)

[Lee, Manteuffel, Schabinger, Smirnov, Smirnov, Steinhauser [2202.04660](#)]

- 4-point : 3-loop ( $\leq 1$ -off shell)

[**PB**, Caola, Chakraborty, Gambuti, Manteuffel, Tancredi]

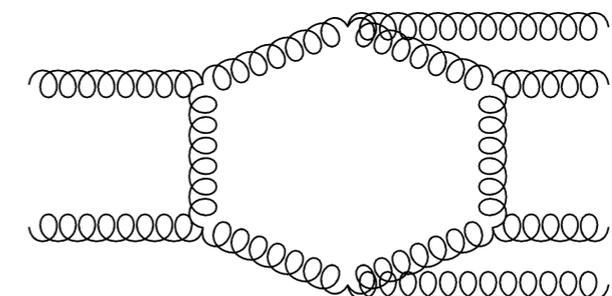
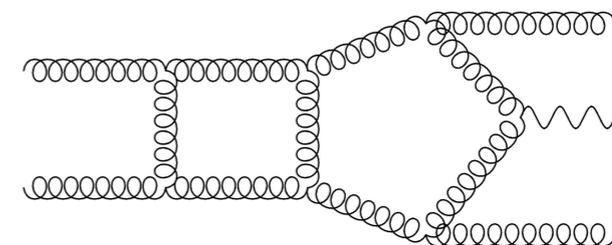
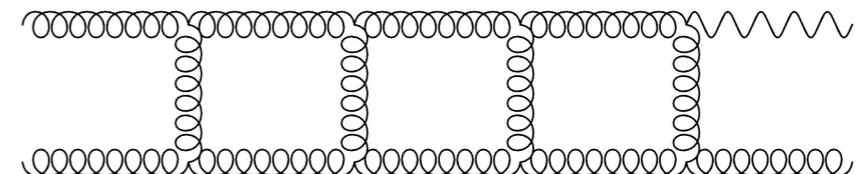
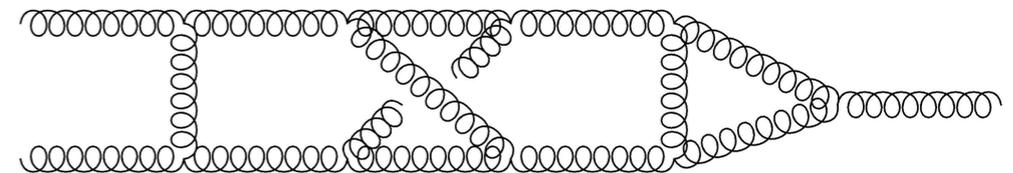
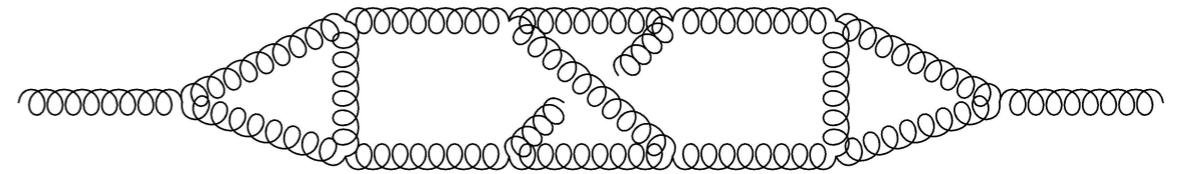
[Gehrmann, Jakubčík, Mella, Syrrakos, Tancredi]

- 5-point : 2-loop (multiscale)

[Abreu, Badger, Buccioni, Chawdhry, Chicherin, Cordero, Czakon, Devoto, Gambuti, Ita, Manteuffel, Mitov, Page, Peraro, Poncelet, Sotnikov, Tancredi, Zoia, ...]

- $\geq 6$ -point : 1-loop (numerical, automated)

[Blackhat, GoSam, MadLoops, OpenLoops (Buccioni), Rocket (Zanderighi), ...]



# Overview : modern methods for QCD amplitudes

- form factor decomposition ~ project Feynman diagrams onto independent tensor structures  
[Garland, Gehrmann, Glover, Koukoutsakis, Remiddi [0206067](#)]
- Generalized Unitarity ~ constrain coefficients of Feynman Integrals with cuts  
[Badger, Frellesvig, Zhang [1310.1051](#)]
- Numerical Unitarity ~ constrain over finite fields the coefficients of Master Integrals with cuts  
[Abreu, Cordero, Ita, Jaquier, Page, Zeng [1703.05273](#)]
- ...

**Integrated Unitarity ~ Generalized Unitarity @ integrated level**

compatible with all above methods

# Overview : Integrated VS Generalized Unitarity

consider a toy 1-loop amplitude

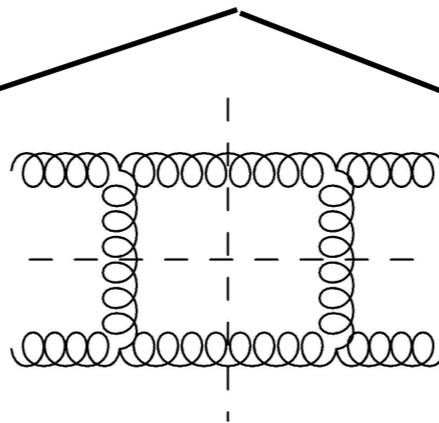
$$\mathcal{A}^{(1)} \sim \int \frac{d^d k}{(2\pi)^d} \frac{\mathcal{N}(k)}{\mathcal{D}_1(k) \mathcal{D}_2(k) \mathcal{D}_3(k) \mathcal{D}_4(k)} = r_{1234} \int \frac{d^d k}{(2\pi)^d} \frac{1}{\mathcal{D}_1(k) \mathcal{D}_2(k) \mathcal{D}_3(k) \mathcal{D}_4(k)} + \text{subsectors}$$

cut all 4 propagators

$$\text{Cut}_{1234} \mathcal{A}^{(1)} \sim \int \frac{d^d k}{(2\pi)^d} \mathcal{A}_1^{(0)}(k) \mathcal{A}_2^{(0)}(k) \mathcal{A}_3^{(0)}(k) \mathcal{A}_4^{(0)}(k) \delta^+(\mathcal{D}_1(k)) \delta^+(\mathcal{D}_2(k)) \delta^+(\mathcal{D}_3(k)) \delta^+(\mathcal{D}_4(k)) = r_{1234} \int \frac{d^d k}{(2\pi)^d} \delta^+(\mathcal{D}_1(k)) \delta^+(\mathcal{D}_2(k)) \delta^+(\mathcal{D}_3(k)) \delta^+(\mathcal{D}_4(k))$$

**Generalized Unitarity**

**Integrated Unitarity**



can compute integral coefficient

compute cut amplitude

$$r_{1234} = \frac{1}{2} \sum_{2 \text{ cut solns } k^*} \mathcal{A}_1^{(0)}(k^*) \mathcal{A}_2^{(0)}(k^*) \mathcal{A}_3^{(0)}(k^*) \mathcal{A}_4^{(0)}(k^*)$$

$$\text{Cut}_{1234} \mathcal{A}^{(1)} = r_{1234} \int \frac{d^d k}{(2\pi)^d} \delta^+(\mathcal{D}_1(k)) \delta^+(\mathcal{D}_2(k)) \delta^+(\mathcal{D}_3(k)) \delta^+(\mathcal{D}_4(k))$$

scalar integral remains to be computed

can reconstruct whole amplitude

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{\mathcal{D}_1(k) \mathcal{D}_2(k) \mathcal{D}_3(k) \mathcal{D}_4(k)}$$

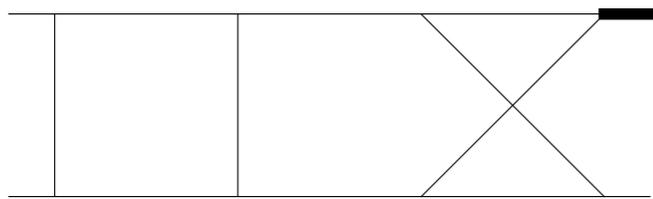
$$\mathcal{A}^{(1)}(s, u) \sim \int_0^\infty \frac{ds'}{s' - s} \int_0^\infty \frac{du'}{u' - u} \text{Cut}_{1234} \mathcal{A}^{(1)}(s', u')$$

# Overview : Feynman integrals frontier

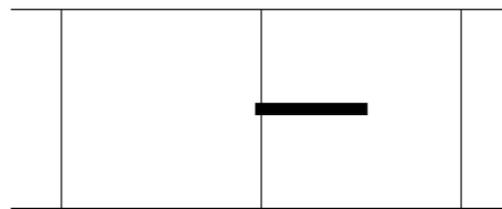
amplitude = f(color, helicity, **kinematics**)

← here, we focus on analytic properties

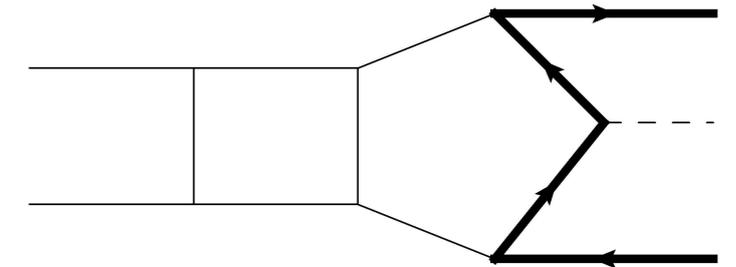
[Henn, Lim, Bobadilla [2302.12776](#)]



[Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia [2306.15431](#)]

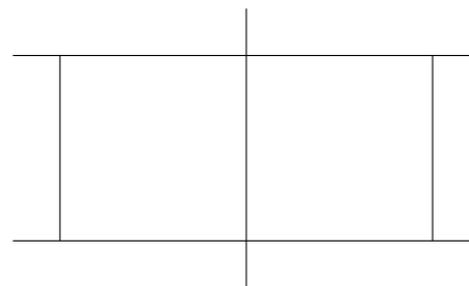
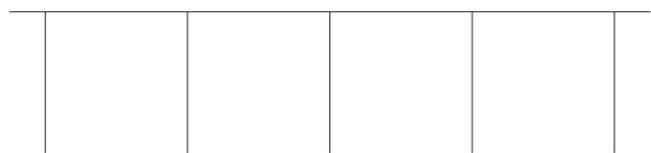


[Cordero, Figueiredo, Kraus, Page, Reina [2312.08131](#)]

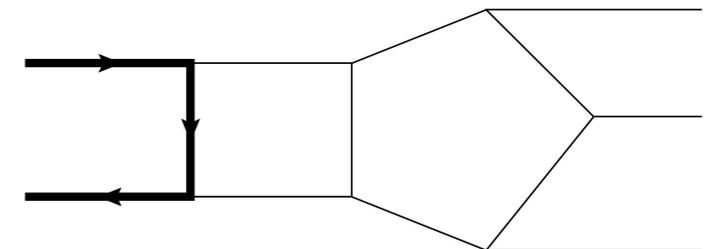


now available !

[Henn, Matijašić, Miczajka, Peraro, Xu, Zhang [2403.19742](#)]



[Badger, Becchetti, Giraud, Zoia [2404.12325](#)]



# Overview : analytic properties frontier

modern developments in analytic properties of amplitudes

- differential equations for cut integrals  
[Abreu, Britto, Duhr, Gardi, Grönqvist, Matthew]
- monodromy for PolyLog discontinuities  
[Bourjaily, Hannesdottir, McLeod, Schwartz, Vergu]
- dispersion relation applications  
[Tancredi, Remiddi, Primo]
- modern analytic S-matrix program  
[Mizera, Telen, Hannesdottir, Caron-Huot, Giroux, Fevola]

# Background

# Background : cuts

- integral definition :

$$I_{\{n_i\}} = \int \left( \prod_{l=1}^L D^d k_l \right) \prod_{i=1}^N \mathcal{D}_i^{-n_i} \quad D^d k_l = e^{\epsilon \gamma_E} \frac{d^d k_l}{i\pi^{d/2}}$$

- cut propagator :

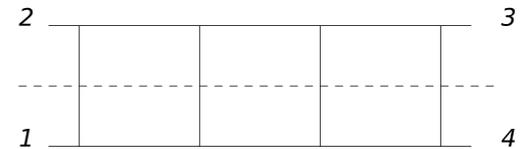
$$\frac{1}{\mathcal{D} + i\epsilon} \rightarrow 2\pi i \delta^+(\mathcal{D}) \quad \delta^+(q^2) = \delta(q^2) \theta(q_0)$$

- cut integral :

$$\text{Cut}_u I_{\{n_i\}} = \sum_{\{c_j\} \in \mathcal{C}_u} I_{\{n_i\}; \{c_j\}} = \sum_{\{c_j\} \in \mathcal{C}_u} \int \left( \prod_{l=1}^L D^d k_l \right) \left( \prod_{i \notin \{c_j\}} \mathcal{D}_i^{-n_i} \right) \prod_{m \in \{c_j\}} \delta_{1, n_m} 2\pi i \delta^+(\mathcal{D}_m)$$

- less subsectors :

$$/2^{\mathcal{C}}$$



- Integration-By-Parts identities (IBPs) :

[Laporta [0102033](#)]

$$\int \left( \prod_{l=1}^L D^d k_l \right) \frac{\partial}{\partial k_l^\mu} \left( q^\mu \prod_{i=1}^N \mathcal{D}_i^{-n_i} \right) = 0$$

[Chetyrkin, Tkachov 1981]

$$p_{j\mu} p_{l\nu} \left( p_n^\nu \frac{\partial}{\partial p_{n,\mu}} - p_n^\mu \frac{\partial}{\partial p_{n,\nu}} \right) I_{\{n_i\}} = 0$$

[Gehrmann, Remiddi [9912329](#)]

- Differential Equations (DEQ) :  $\partial_{x_n} M_i(\vec{x}, \epsilon) = A_{ij}(\vec{x}, \epsilon) M_j(\vec{x}, \epsilon)$   $\xrightarrow{\text{canonical}}$   $\partial_{x_n} M_i^c(\vec{x}, \epsilon) = \epsilon A_{ij}^c(\vec{x}) M_j^c(\vec{x}, \epsilon)$

Master Integrals (MIs)  $\nearrow$

[Kotikov 1991]

[Henn [1304.1806](#)]

$\rightarrow$  can solve cut integrals with DEQ

# Background : discontinuities

- discontinuity :

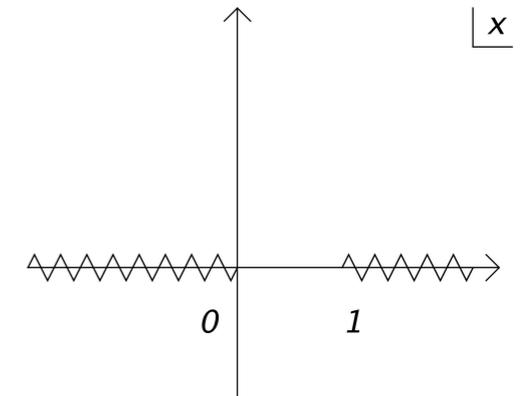
$$\text{Disc}_0 f(x) = f(x + i\epsilon) - f(x - i\epsilon)$$

- from now on, focus on specific kinematics :

$$x = -\frac{t}{s} \quad \text{4-point massless}$$

- branch cuts :

$$\begin{array}{lll} t > 0 & u > 0 & s > 0 \\ x < 0 & x > 1 & \text{pushed to } \infty \end{array}$$



- Harmonic Polylogarithms (HPLs) :  
[Remiddi, Vermaseren 9905237]

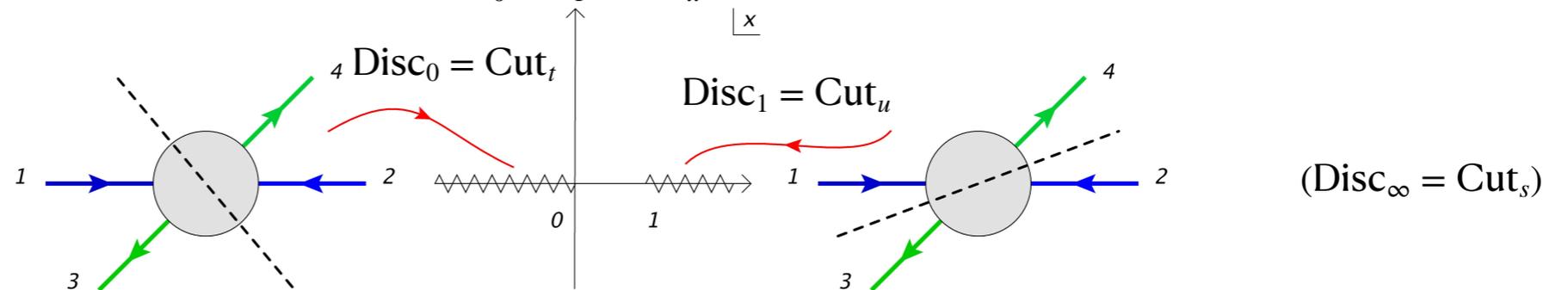
$$G(\alpha_n, \dots, \alpha_1; x) = \int_0^x \frac{dz}{z - \alpha_n} G(\alpha_{n-1}, \dots, \alpha_1; z) \quad \alpha_k \in \{0, 1\}$$

- discontinuities of HPLs algorithmic from monodromy matrices :

[Bourjaily, Hannesdottir, McLeod, Schwartz, Vergu 2007.13747]

$$\begin{aligned} \text{Disc}_0 &= (1 - \mathcal{M}_0) \cdot \mathcal{M}_{\rightarrow x}, \\ \text{Disc}_1 &= -(1 - \mathcal{M}_1) \cdot \mathcal{M}_{\rightarrow x}, \\ \text{Disc}_\infty &= (1 - \mathcal{M}_0 \cdot \mathcal{M}_1) \cdot \mathcal{M}_{\rightarrow x}. \end{aligned}$$

- unitarity :



# Background : dispersion relation

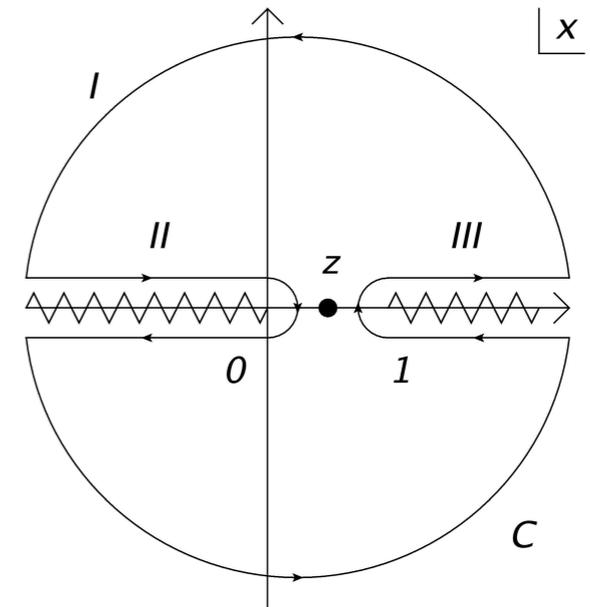
Cauchy's integral formula

$$\mathcal{A}(z) = \frac{1}{2\pi i} \oint_C \frac{\mathcal{A}(x)dx}{x-z}$$

[Cutkosky, Mandelstam, Eden,  
Landshoff, Olive, Polkinghorne,  
Remiddi, van Neerven, Kniehl, Sirlin]

piecewise contour

$$\mathcal{A}(z) = c_\infty + \frac{1}{2\pi i} \left( \int_0^\infty \text{Disc}_0 + \int_1^\infty \text{Disc}_1 \right) \frac{\mathcal{A}(x)dx}{x-z}$$

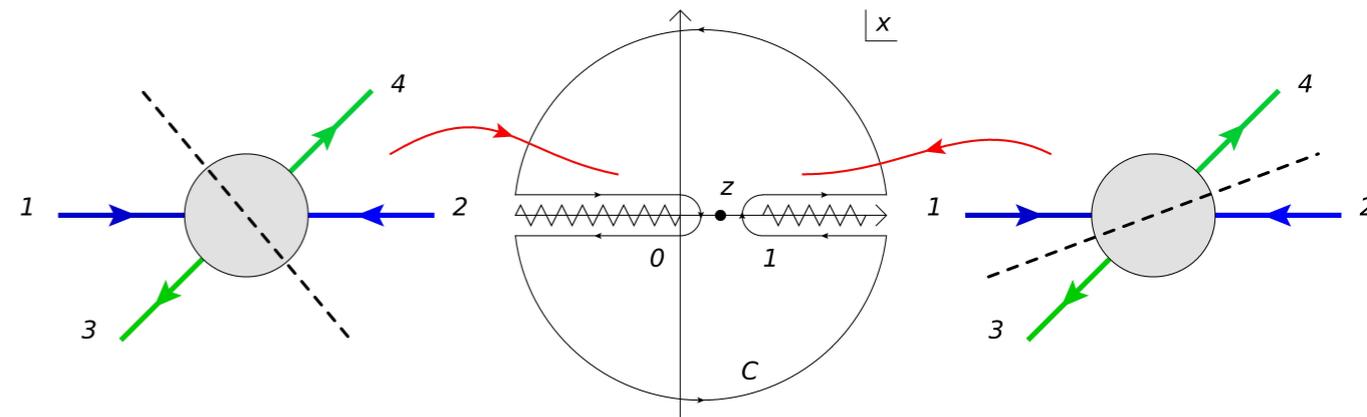


cancel constant from infinite arc

$$\mathcal{A}(z) = \mathcal{A}_0 + \frac{1}{2\pi i} \left( \int_0^\infty \text{Disc}_0 + \int_1^\infty \text{Disc}_1 \right) \left( \frac{1}{x-z} - \frac{1}{x-z_0} \right) \mathcal{A}(x)dx$$

unitarity

$$\mathcal{A}(z) = \mathcal{A}_0 + \frac{1}{2\pi i} \left( \int_0^\infty \text{Cut}_t + \int_1^\infty \text{Cut}_u \right) \left( \frac{1}{x-z} - \frac{1}{x-z_0} \right) \mathcal{A}(x)dx$$



# Integrated Unitarity

# Integrated Unitarity

$$\mathcal{A}(z) = \mathcal{A}_0 + \frac{1}{2\pi i} \left( \int_0^\infty \text{Cut}_t + \int_1^\infty \text{Cut}_u \right) \left( \frac{1}{x-z} - \frac{1}{x-z_0} \right) \mathcal{A}(x) dx$$

expressing cuts as phase space integrals

$$\begin{aligned} \mathcal{A}(z) = \mathcal{A}_0 + \frac{1}{2\pi i} & \left( \int_0^\infty \sum_{\{c_j\} \in \mathcal{C}_t} \int d\text{PS}_{t,\{c_j\}} \mathcal{A}_{t,\{c_j\},L} \mathcal{A}_{t,\{c_j\},R}^* \right. \\ & \left. + \int_1^\infty \sum_{\{c_j\} \in \mathcal{C}_u} \int d\text{PS}_{u,\{c_j\}} \mathcal{A}_{u,\{c_j\},L} \mathcal{A}_{u,\{c_j\},R}^* \right) \left( \frac{1}{x-z} - \frac{1}{x-z_0} \right) dx \end{aligned}$$

diagrammatically, in the planar case

$$\begin{array}{c} 2 \\ \diagdown \\ \text{---} \circ \text{---} \\ \diagup \\ 3 \end{array} \begin{array}{c} 1 \\ \diagdown \\ \text{---} \circ \text{---} \\ \diagup \\ 4 \end{array} (z) = \begin{array}{c} 2 \\ \diagdown \\ \text{---} \circ \text{---} \\ \diagup \\ 1 \end{array} \begin{array}{c} 1 \\ \diagdown \\ \text{---} \circ \text{---} \\ \diagup \\ 2 \end{array} + \frac{1}{2\pi i} \int_1^\infty \sum_{\{c_j\}} \left( \frac{1}{x-z} - \frac{1}{x-z_0} \right) dx \begin{array}{c} 3 \\ \diagdown \\ \text{---} \circ \text{---} \\ \diagup \\ 2 \end{array} \begin{array}{c} 4 \\ \diagdown \\ \text{---} \circ \text{---} \\ \diagup \\ 1 \end{array} \begin{array}{c} | \\ \{c_j\} \\ | \\ \vdots \end{array}$$

# Integrated Unitarity : properties

$$\begin{array}{c} 2 \\ \diagdown \\ \text{---} \circ \text{---} \\ \diagup \\ 1 \end{array} \begin{array}{c} 3 \\ \diagup \\ \text{---} \circ \text{---} \\ \diagdown \\ 4 \end{array} (z) = \begin{array}{c} 2 \\ \diagdown \\ \text{---} \circ \text{---} \\ \diagup \\ 1 \end{array} \begin{array}{c} 1 \\ \diagup \\ \text{---} \circ \text{---} \\ \diagdown \\ 2 \end{array} + \frac{1}{2\pi i} \int_1^\infty \sum \{c_j\} \left( \frac{1}{x-z} - \frac{1}{x-z_0} \right) dx \begin{array}{c} 3 \\ \diagdown \\ \text{---} \circ \text{---} \\ \diagup \\ 2 \end{array} \begin{array}{c} 4 \\ \diagup \\ \text{---} \circ \text{---} \\ \diagdown \\ 1 \end{array} (x)$$

- Generalized Unitarity @ integrated level : constrains both MIs and their coefficients with cuts
- algorithmic in dimReg : cut canonical differential equations
- compatible with other amplitude methods
- less subsectors : by a factor of  $2^{\{\# \text{ cut propagators}\}}$
- less MIs : simpler DEQ
- simpler IBPs : propagator powers  $\{0, -1, -2, \dots\}$  unsupported on a cut

# Integrated Unitarity : 3 methods

$$\mathcal{A}(z) = \mathcal{A}_0 + \frac{1}{2\pi i} \left( \int_0^\infty \text{Cut}_t + \int_1^\infty \text{Cut}_u \right) \left( \frac{1}{x-z} - \frac{1}{x-z_0} \right) \mathcal{A}(x) dx$$

- A. explicit integration : convergent e.g. for canonical MIs  
subtraction terms needed for full amplitude

$$\mathcal{A}(x) \rightarrow S(x) \mathcal{A}(x) \quad S(x) = \frac{(1-x)^p x^q}{(x-z_1)^r} \quad -\text{Res}_{x \rightarrow z_1} \mathcal{A}(x) S(x) \left( \frac{1}{x-z} - \frac{1}{x-z_0} \right)$$

- B. ansatz reconstruction with 2 discontinuities + # evaluations :

$$\begin{cases} \text{Disc}_0 \mathcal{A} &= \text{Cut}_t \mathcal{A} \\ \text{Disc}_1 \mathcal{A} &= \text{Cut}_u \mathcal{A} \\ \mathcal{A}(z_i) &= \mathcal{A}_i \end{cases}$$

$$\mathcal{A}(z) = \epsilon^\# \sum_{n \geq 0} \epsilon^n \sum_{\vec{\alpha}: |\vec{\alpha}| \leq n} r_{n, \vec{\alpha}}(z) G(\vec{\alpha}, z)$$

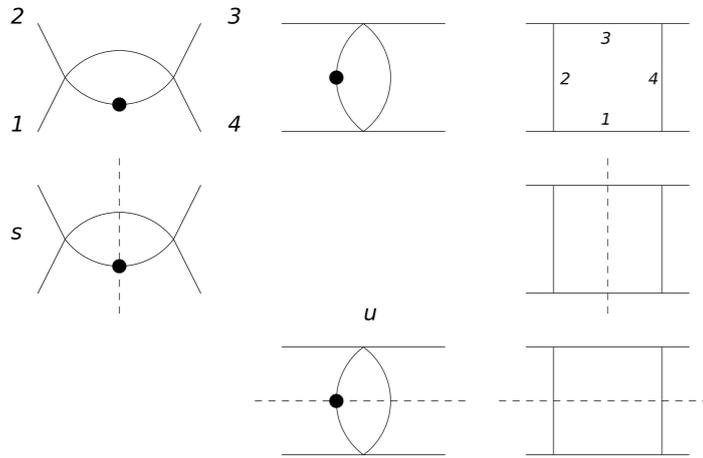
↑  
unknowns

- C. ansatz reconstruction with 3 discontinuities :

$$\begin{cases} \text{Disc}_0 \mathcal{A} &= \text{Cut}_t \mathcal{A} \\ \text{Disc}_1 \mathcal{A} &= \text{Cut}_u \mathcal{A} \\ \text{Disc}_\infty \mathcal{A} &= \text{Cut}_s \mathcal{A} \end{cases}$$

$$\mathcal{A}(z) = \epsilon^\# \sum_{n \geq 0} \epsilon^n \sum_{\vec{\alpha}: |\vec{\alpha}| \leq n} r_{n, \vec{\alpha}}(z) G(\vec{\alpha}, z)$$

# Integrated Unitarity : example



cut MIs

$$M_i^c \in \{\epsilon(2\epsilon - 1)I_{1,0,1,0}, \epsilon(2\epsilon - 1)I_{0,1,0,1}, \epsilon^2(x - 1)I_{1,1,1,1}\}$$

$$\text{Cut}_s M_i^c \in \{\epsilon(2\epsilon - 1)I_{1,0,1,0;1,3}, 0, \epsilon^2(x - 1)I_{1,1,1,1;1,3}\}$$

$$\text{Cut}_u M_i^c \in \{0, \epsilon(2\epsilon - 1)I_{0,1,0,1;2,4}, \epsilon^2(x - 1)I_{1,1,1,1;2,4}\}$$

$$\text{Cut}_t M_i^c \in \{0, 0, 0\}$$

cut DEQ

$$\partial_x M_i^c(x, \epsilon) = \epsilon A_{ij}^c(x) M_j^c(x, \epsilon)$$

$$\partial_x \text{Cut}_s M_i^c(x, \epsilon) = \epsilon A_{ij}^{c,s}(x) \text{Cut}_s M_j^c(x, \epsilon)$$

$$\partial_x \text{Cut}_u M_i^c(x, \epsilon) = \epsilon A_{ij}^{c,u}(x) \text{Cut}_u M_j^c(x, \epsilon)$$

$$\partial_x \text{Cut}_t M_i^c(x, \epsilon) = 0$$

$$A^c(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{1-x} & 0 \\ \frac{2}{x} & \frac{2}{x} + \frac{2}{1-x} & \frac{1}{x} + \frac{1}{1-x} \end{pmatrix}$$

3 methods

$$M_i^c(z) = \epsilon^{-2} \sum_{n \geq 0} \epsilon^n \sum_{\vec{\alpha}: |\vec{\alpha}| \leq n} c_{n,\vec{\alpha}} G(\vec{\alpha}, z) \quad \alpha_k \in \{0,1\}$$

unknowns

A. explicit integration

$$M_i^c(z) = M_{i,0}^c + \frac{1}{2\pi i} \int_1^\infty \left( \frac{dx}{x-z} - \frac{dx}{x} \right) \text{Cut}_u M_i^c(x)$$



B. ansatz + 2cuts + 1pt

ansatz	computed
$\text{Disc}_0 M_i^c$	$= \text{Cut}_t M_i^c$
$\text{Disc}_1 M_i^c$	$= \text{Cut}_u M_i^c$
$M_i^c(0)$	$= M_{i,0}^c$



C. ansatz + 3cuts

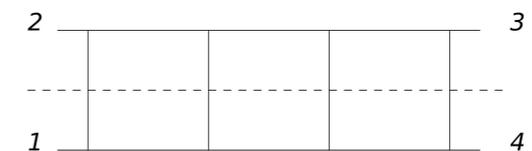
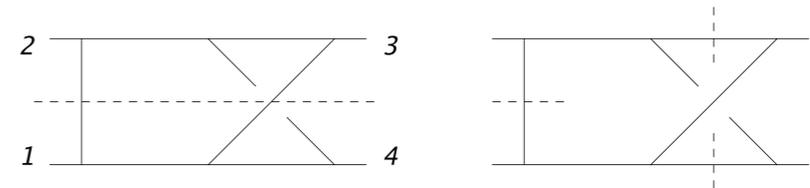
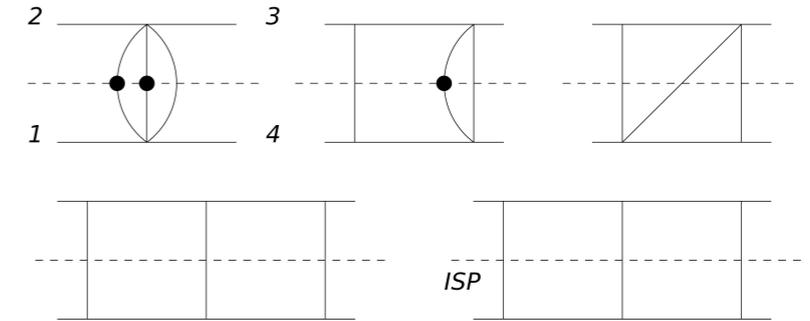
$\text{Disc}_0 M_i^c$	$= \text{Cut}_t M_i^c$
$\text{Disc}_1 M_i^c$	$= \text{Cut}_u M_i^c$
$\text{Disc}_\infty M_i^c$	$= \text{Cut}_s M_i^c$



# Integrated Unitarity : further checks

## Master Integrals (from DEQ)

- 2-loop planar (#MIs : 8  $\rightarrow$  5) ✓
- 2-loop nonplanar (#MIs : 12  $\rightarrow$  2+6) ✓
- 3-loop planar ladder (#MIs : 26  $\rightarrow$  17) ✓



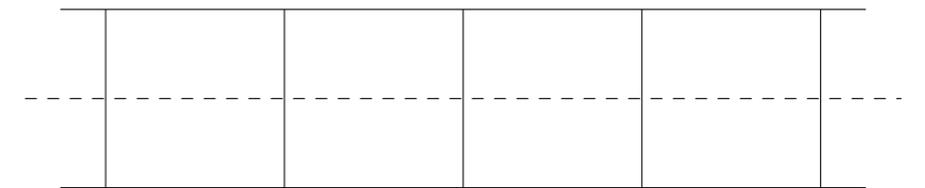
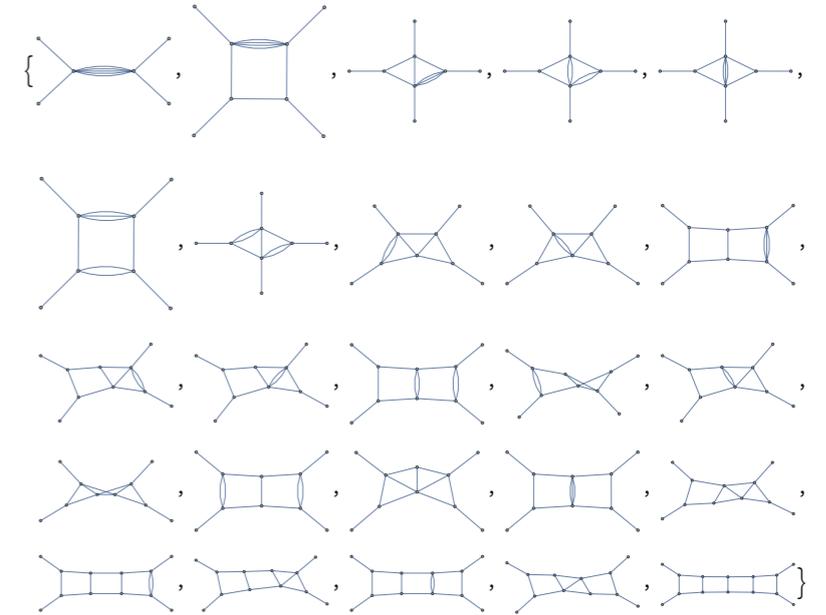
## Amplitudes (from form factor method)

- 1-loop gg  $\rightarrow$  gg (#INTs :  $\sim 2$  per cut) ✓
- 2-loop gg  $\rightarrow$  gg planar (#INTs :  $\sim 8$  per cut) ✓
- 2-loop gg  $\rightarrow$  gg nonplanar (#INTs :  $\sim 4$  per cut) ✓

# Integrated Unitarity : four-loop ladder

## procedure

- 22 generalized propagators = 13 denominators + 9 ISPs
- cut u : 5 propagators
- IBP : 59 MIs  $\text{Cut}_u M_i^c$  (LiteRed + Kira)
- canonical DEQ :  $A_{ij}^c(x) = \frac{a_{ij}}{x} + \frac{b_{ij}}{1-x}$  (CANONICA + MultivariateApart + FiniteFlow)
- canonical general solution :  $M_u^c(x, \epsilon) = \mathbb{P} e^{\epsilon \int A^c(x) dx} M_{u,0}^c(\epsilon)$  (PolyLogTools + in-house)
- regularity constraints on BCs  $M_{u,0}^c(\epsilon) : 59 \rightarrow 5$  (in-house)
- 5 remaining BCs : weight 7 (AMFlow)
- method B :  $\text{Disc}_0 M_i^c = \text{Cut}_t M_i^c = 0$  &  $\text{Disc}_1 M_i^c = \text{Cut}_u M_i^c$  (in-house)



fixed all HPL coefficients to weight 8

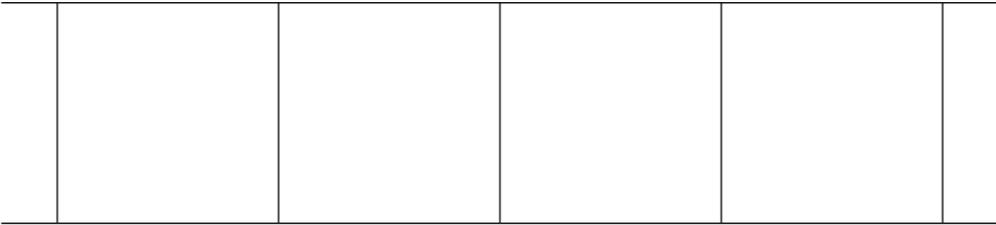
# Integrated Unitarity : four-loop ladder

2

3

1

4



$$\begin{aligned} & ((1.29535880081736032 \times 10^7 - 2.0215509232332160170 \times 10^8)G(1,x) - (4916.0033092753716635 - 925.89113880652441994)G(0,1,x) + (10416.08095707170424 - 15906.0928817291276941)G(1,1,x) + (224.32883087376640869 - 58.937124477165316921)G(0,0,1,x) - (4872.8054889885424783 - 648.30836924881848621)G(0,1,1,x) - (6848.2665281193484352 + 512.29962045536006251)G(1,0,1,x) + (18326.117849612906352 - 517.3999485351147531)G(1,1,1,x) - (150.44292948584820862 + 306.58552387758111421)G(0,0,0,1,x) \\ & + (907.2273861240918352 + 1689.5773142537034608)G(0,0,0,1,1,x) + (1714.4000021984428954 + 4022.536718372227758)G(0,0,0,1,1,x) - (7177.7271696981104554 + 7412.88060061641419044)G(0,1,0,1,1,x) + (267.3338387266511333 + 5190.69480849200685731)G(1,0,0,1,x) - (10918.717297952120107 + 11669.2727220873333721)G(1,0,1,1,x) - (10996.404054505164026 + 14149.9302755313469971)G(1,0,1,1,x) + (31185.34010730261758 + 23864.4403698033029211)G(1,1,1,1,x) - (56.897360082887462842 - 34.4514185336664668621)G(0,0,0,0,1,x) \\ & + (256.61688851155190596 - 289.391915682798321644)G(0,0,0,1,1,x) + (540.12423515304436557 - 1005.981421183060832361)G(0,0,0,1,1,x) - (1197.0705930427841165 - 1998.18227495265507891)G(0,0,0,1,1,x) + (2155.6887129995390852 - 1750.132061510265616661)G(1,0,0,1,x) - (3242.5262359599841109 - 7028.08938086795923984)G(0,1,0,1,x) - (3612.5816796289673591 - 6669.79462811782798444)G(0,1,0,1,x) + (5714.3113938644535452 - 12402.51067211992807021)G(0,1,1,1,x) \\ & + (821.40555049238942836 - 654.576952139662870371)G(1,0,0,0,1,x) - (4298.733682028809563 - 5622.47150469436739181)G(1,0,0,0,1,x) - (5357.9683030166982615 - 11754.82400368699849321)G(1,0,0,0,1,x) + (11201.389512405700832 - 21635.4908391425411891)G(1,0,0,1,1,x) - (6079.2024449124548327 - 9439.68867822461192011)G(1,0,0,1,x) + (14337.78575879297053 - 28443.0911413950350411)G(1,0,0,1,x) + (13524.742903082981836 - 32580.7065072883777111)G(1,1,0,0,1,x) \\ & - (24674.399448026777157 - 58402.0446982713946241)G(1,1,1,1,x) + 16.44934068482264365)G(0,0,0,0,1,x) - 86.267653283595875335)G(0,0,0,0,1,x) - 348.72602217182400453)G(0,0,0,0,1,x) + 426.95177557305077284)G(0,0,0,0,1,x) - 875.10492356325646420)G(0,0,0,0,1,x) + 1612.7664673187500084)G(0,0,0,0,1,x) + 1921.282990078284778)G(0,0,0,0,1,x) - 2035.3317520468721774)G(0,0,0,0,1,x) - 875.10492356325646420)G(0,0,0,0,1,x) + 4218.3420292063406838)G(0,0,0,0,1,x) + 6092.8357836058307207)G(0,0,0,0,1,x) \\ & - 7322.5153838008159945)G(0,1,0,1,1,x) + 5448.0216294013259576)G(0,1,0,0,1,x) - 9025.9359952489094821)G(0,1,0,0,1,x) - 11277.667962311440448)G(0,1,0,0,1,x) + 9451.4256072061650537)G(0,1,0,0,1,x) - 312.53747270116302293)G(1,0,0,0,1,x) + 1520.6501595752493279)G(1,0,0,0,1,x) + 4967.7008818816438381)G(1,0,0,0,1,x) + 7164.601713383862566)G(1,0,0,1,1,x) + 9481.3999613131771798)G(1,0,0,0,1,x) - 14943.312145056777050)G(1,0,0,0,1,x) - 19252.308318391642212)G(1,0,0,0,1,x) \\ & + 18563.629255737851411)G(1,0,0,0,1,1,x) + 6296.8076078950107988)G(1,1,0,0,0,1,x) - 14737.147075345132670)G(1,1,0,0,1,x) - 25160.911486510471572)G(1,1,0,0,1,x) + 28307.487585939256720)G(1,1,0,1,1,x) - 25884.682475923691204)G(1,1,0,0,1,x) + 39194.02678124463337)G(1,1,0,1,1,x) + 51332.49446221237802)G(1,1,1,0,1,x) - 49783.746762709701075)G(1,1,1,1,x) - 3.49065850398865915381)G(0,0,0,0,0,1,x) + 13.962634015954636615)G(0,0,0,0,0,1,x) + 60.039326268604937446)G(0,0,0,0,0,1,x) \\ & - 55.850536063818546462)G(0,0,0,0,1,1,x) + 185.7030324121966698)G(0,0,0,0,1,x) - 240.15730507441974978)G(0,0,0,0,1,x) - 332.3106895972035145)G(0,0,0,0,1,x) + 223.40214425527418585)G(0,0,0,0,1,1,x) + 185.7030324121966698)G(0,0,0,0,1,x) - 742.8121296487866794)G(0,0,0,0,1,x) - 1027.6498635742612549)G(0,0,0,0,1,x) + 960.62922029767899914)G(0,0,0,0,1,x) - 1010.8947027551156910)G(0,0,0,0,1,x) + 1329.2427583188814058)G(0,0,0,0,1,1,x) \\ & + 1840.2751633028211059)G(0,0,0,0,1,1,x) - 893.60857702109674338)G(0,0,0,0,1,1,1,x) + 185.7030324121966698)G(0,0,0,0,1,1,x) - 742.8121296487866794)G(0,0,0,0,1,1,x) - 2435.0833723824886257)G(0,0,0,0,1,1,x) + 2971.2485185951466718)G(0,0,0,0,1,1,x) - 3415.2602803205041161)G(0,0,0,0,1,1,x) + 4110.5994542970450196)G(0,0,0,0,1,1,x) + 5936.9119835839114889)G(0,0,0,0,1,1,x) - 3842.5168811907159966)G(0,0,0,0,1,1,1,x) - 2317.797246484696782)G(0,0,0,0,0,1,1,x) \\ & + 4043.5788110204627638)G(0,1,0,0,0,1,x) + 6615.4959967093068284)G(0,1,0,0,0,1,x) - 5316.971032755256231)G(0,1,0,0,0,1,x) + 6816.357926589053596)G(0,1,0,0,0,1,x) - 7361.1006532112844236)G(0,1,0,0,0,1,x) - 10041.92638427457654)G(0,1,0,0,0,1,x) + 3574.4343080843869735)G(0,1,0,0,0,1,x) + 66.322511575784523923)G(1,0,0,0,0,1,x) - 265.29004630313809569)G(1,0,0,0,0,1,x) - 914.55252804502869831)G(1,0,0,0,0,1,x) + 1061.16018502125523828)G(1,0,0,0,0,1,1,x) \\ & - 2171.189584908459937)G(1,0,0,0,0,1,x) + 3658.2101121801147932)G(1,0,0,0,0,1,x) + 5484.52264146698126251)G(1,0,0,0,0,1,x) - 4244.640740850295311)G(1,0,0,0,0,1,x) - 4156.6761465496953204)G(1,0,0,0,0,1,x) + 6975.7319543709364530)G(1,0,0,0,0,1,x) + 10703.75236630824429)G(1,0,0,0,0,1,x) - 9003.1064134875496896)G(1,0,0,0,0,1,x) + 10988.592970556299016)G(1,0,0,0,0,1,x) - 12890.303732529320523)G(1,0,0,0,0,1,x) - 17598.503913709223990)G(1,0,0,0,0,1,1,x) \\ & + 8936.0857702109674338)G(1,0,0,0,0,1,1,x) - 2631.956512007449002)G(1,0,0,0,0,1,1,x) + 4697.0300829671397574)G(1,0,0,0,0,1,1,x) + 9304.6993082321698405)G(1,0,0,0,0,1,1,x) - 9941.3954193597012702)G(1,0,0,0,0,1,1,x) + 14230.71658906965638)G(1,0,0,0,0,1,1,x) - 16911.542320124255869)G(1,0,0,0,0,1,1,x) - 23714.137612697354828)G(1,0,0,0,0,1,1,x) + 15638.150097869193009)G(1,0,0,0,0,1,1,x) + 12295.495514449653004)G(1,0,0,0,0,1,1,x) - 18570.30324121966698)G(1,0,0,0,0,1,1,x) \\ & - 28261.069379992982241)G(1,1,0,0,0,1,x) + 23915.199542527101595)G(1,1,0,0,0,1,x) - 30550.243226980744914)G(1,1,0,0,0,1,x) + 33934.785712376148830)G(1,1,0,0,0,1,x) + 45780.68441512062535)G(1,1,1,0,0,1,x) - 25021.040156590708815)G(1,1,1,0,0,1,x) - 1.1111111111111111)G(0,0,0,0,0,1,x) + 8.888888888888889)G(0,0,0,0,0,1,x) + 19.111111111111111)G(0,0,0,0,0,1,x) - 53.333333333333333)G(0,0,0,0,0,1,1,x) + 59.111111111111111)G(0,0,0,0,0,1,1,x) \\ & - 152.888888888888889)G(0,0,0,0,1,1,x) - 105.777777777777778)G(0,0,0,0,1,1,x) + 284.444444444444444)G(0,0,0,0,1,1,1,x) + 59.111111111111111)G(0,0,0,0,0,1,x) - 472.888888888888889)G(0,0,0,0,0,1,1,x) - 327.111111111111111)G(0,0,0,0,0,1,x) + 917.333333333333333)G(0,0,0,0,0,1,1,x) - 321.777777777777778)G(0,0,0,0,0,1,1,x) + 846.222222222222222)G(0,0,0,0,0,1,1,x) + 585.777777777777778)G(0,0,0,0,0,1,1,x) - 1422.222222222222222)G(0,0,0,0,0,1,1,1,x) \\ & + 59.111111111111111)G(0,0,0,0,0,1,1,x) - 472.888888888888889)G(0,0,0,0,0,1,1,x) - 775.111111111111111)G(0,0,0,0,0,1,1,x) + 2837.333333333333333)G(0,0,0,0,0,1,1,1,x) - 1087.111111111111111)G(0,0,0,0,0,1,1,x) + 2616.888888888888889)G(0,0,0,0,0,1,1,x) + 1889.777777777777778)G(0,0,0,0,0,1,1,x) - 4892.444444444444444)G(0,0,0,0,0,1,1,x) + 2574.222222222222222)G(0,0,0,0,0,1,1,x) + 2105.777777777777778)G(0,0,0,0,0,1,1,x) \\ & - 5077.333333333333333)G(0,0,0,0,0,1,1,1,x) + 2169.777777777777778)G(0,0,0,0,0,1,1,1,x) - 4686.222222222222222)G(0,0,0,0,0,1,1,1,x) - 3196.444444444444444)G(0,0,0,0,0,1,1,1,x) + 6826.666666666666667)G(0,0,0,0,0,1,1,1,x) + 21.111111111111111)G(1,0,0,0,0,0,1,x) - 168.888888888888889)G(1,0,0,0,0,0,1,x) - 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11822.222222222222222)G(1,0,0,0,0,0,1,1,x) - 8995.777777777777778)G(1,0,0,0,0,0,1,1,x) + 22837.333333333333333)G(1,0,0,0,0,0,1,1,1,x) \\ & - 9724.444444444444444)G(1,1,0,0,0,0,1,x) + 21693.555555555555556)G(1,1,0,0,0,0,1,x) + 14572.444444444444444)G(1,1,0,0,0,0,1,x) - 34588.444444444444444)G(1,1,0,0,0,0,1,x) - 34588.444444444444444)G(1,1,0,0,0,0,1,x) + (1721.2146729154373062 + 2370.5621574616726010)G(1,x) + (224.32883087376640869 - 58.937124477165316921)G(0,1,x) - (3774.667143614267159 + 1560.7003924049354116)G(1,1,x) - (150.44292948584820862 + 306.58552387758111421)G(0,0,1,x) + (907.2273861240918352 + 1689.57731425370346081)G(0,0,1,1,x) \\ & + (1081.2409104774270533 + 2461.6358220914884859)G(0,1,x) - (2871.1630227614915914 + 5457.5585101141977681)G(1,1,x) - (56.897360082887462842 - 34.4514185336664668621)G(0,0,1,x) + (256.61688851155190596 - 289.391915682798321644)G(0,0,1,x) + (540.12423515304436557 - 1005.981421183060832361)G(0,0,1,x) - (1197.0705930427841165 - 1998.18227495265507891)G(0,0,1,x) + (821.40555049238942836 - 654.576952139662870371)G(1,0,0,1,x) - (2195.9353589423610197 - 3472.7029881935798597)G(0,0,0,1,x) \\ & - (2501.881101095022393 - 4313.31760041504165111)G(1,0,0,1,x) + (4344.5007717750314351 - 9729.0805939074102417)G(1,1,0,0,1,x) + 16.44934068482264365)G(0,0,0,0,0,1,x) - 86.267653283595875335)G(0,0,0,0,0,0,1,x) - 348.72602217182400453)G(0,0,0,0,0,0,1,x) + 426.95177557305077284)G(0,0,0,0,0,0,1,x) - 875.10492356325646420)G(0,0,0,0,0,0,1,x) + 1612.7664673187500084)G(0,0,0,0,0,0,1,x) + 1921.282990078284778)G(0,0,0,0,0,0,1,x) - 2035.3317520468721774)G(0,0,0,0,0,0,1,x) - 875.10492356325646420)G(0,0,0,0,0,0,1,x) + 4218.3420292063406838)G(0,0,0,0,0,0,1,x) \\ & + 6092.8357836058307207)G(0,0,0,0,0,0,1,x) - 7322.5153838008159945)G(0,0,0,0,0,0,1,x) + 5448.0216294013259576)G(0,0,0,0,0,0,1,x) - 9025.9359952489094821)G(0,0,0,0,0,0,1,x) - 11277.667962311440448)G(0,0,0,0,0,0,1,x) + 9451.4256072061650537)G(0,0,0,0,0,0,1,x) - 312.53747270116302293)G(1,0,0,0,0,0,0,1,x) + 1520.6501595752493279)G(1,0,0,0,0,0,0,1,x) + 4967.7008818816438381)G(1,0,0,0,0,0,0,1,x) + 7164.601713383862566)G(1,0,0,0,0,0,0,1,x) + 9481.3999613131771798)G(1,0,0,0,0,0,0,1,x) - 14943.312145056777050)G(1,0,0,0,0,0,0,1,x) - 19252.308318391642212)G(1,0,0,0,0,0,0,1,x) \\ & - 18570.30324121966698)G(1,0,0,0,0,0,0,1,x) + 4043.5788110204627638)G(0,1,0,0,0,0,0,1,x) + 6615.4959967093068284)G(0,1,0,0,0,0,0,1,x) - 5316.971032755256231)G(0,1,0,0,0,0,0,1,x) + 6816.357926589053596)G(0,1,0,0,0,0,0,1,x) - 7361.1006532112844236)G(0,1,0,0,0,0,0,1,x) - 10041.92638427457654)G(0,1,0,0,0,0,0,1,x) + 3574.4343080843869735)G(0,1,0,0,0,0,0,1,x) + 66.322511575784523923)G(1,0,0,0,0,0,0,0,1,x) - 265.29004630313809569)G(1,0,0,0,0,0,0,0,1,x) - 914.55252804502869831)G(1,0,0,0,0,0,0,0,1,x) + 1061.16018502125523828)G(1,0,0,0,0,0,0,0,1,1,x) \\ & - 2171.189584908459937)G(1,0,0,0,0,0,0,0,1,x) + 3658.2101121801147932)G(1,0,0,0,0,0,0,0,1,x) + 5484.52264146698126251)G(1,0,0,0,0,0,0,0,1,x) - 4244.640740850295311)G(1,0,0,0,0,0,0,0,1,x) - 4156.6761465496953204)G(1,0,0,0,0,0,0,0,1,x) + 6975.7319543709364530)G(1,0,0,0,0,0,0,0,1,x) + 10703.75236630824429)G(1,0,0,0,0,0,0,0,1,x) - 9003.1064134875496896)G(1,0,0,0,0,0,0,0,1,x) + 10988.592970556299016)G(1,0,0,0,0,0,0,0,1,x) - 12890.303732529320523)G(1,0,0,0,0,0,0,0,1,x) - 17598.503913709223990)G(1,0,0,0,0,0,0,0,1,1,x) \\ & + 8936.0857702109674338)G(1,0,0,0,0,0,0,0,1,1,x) - 2631.956512007449002)G(1,0,0,0,0,0,0,0,1,1,x) + 4697.0300829671397574)G(1,0,0,0,0,0,0,0,1,1,x) + 9304.6993082321698405)G(1,0,0,0,0,0,0,0,1,1,x) - 9941.3954193597012702)G(1,0,0,0,0,0,0,0,1,1,x) + 14230.71658906965638)G(1,0,0,0,0,0,0,0,1,1,x) - 16911.542320124255869)G(1,0,0,0,0,0,0,0,1,1,x) - 23714.137612697354828)G(1,0,0,0,0,0,0,0,1,1,x) + 15638.150097869193009)G(1,0,0,0,0,0,0,0,1,1,x) + 12295.495514449653004)G(1,0,0,0,0,0,0,0,1,1,x) - 18570.30324121966698)G(1,0,0,0,0,0,0,0,1,1,x) \\ & - 28261.069379992982241)G(1,1,0,0,0,0,0,0,1,x) + 23915.199542527101595)G(1,1,0,0,0,0,0,0,1,x) - 30550.243226980744914)G(1,1,0,0,0,0,0,0,1,x) + 33934.785712376148830)G(1,1,0,0,0,0,0,0,1,x) + 45780.68441512062535)G(1,1,1,0,0,0,0,0,1,x) - 25021.040156590708815)G(1,1,1,0,0,0,0,0,1,x) - 1.1111111111111111)G(0,0,0,0,0,0,0,0,1,x) + 8.888888888888889)G(0,0,0,0,0,0,0,0,1,x) + 19.111111111111111)G(0,0,0,0,0,0,0,0,1,x) - 53.333333333333333)G(0,0,0,0,0,0,0,0,1,1,x) + 59.111111111111111)G(0,0,0,0,0,0,0,0,1,1,x) \\ & - 152.888888888888889)G(0,0,0,0,0,0,0,0,1,1,x) - 10$$

# Conclusions

# Conclusions

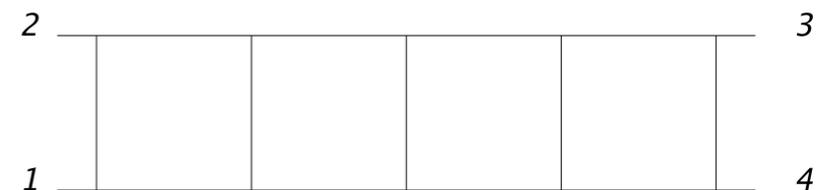
## Integrated Unitarity :

- ~ Generalized Unitarity @ integrated level
- allows algorithmic usage of dispersion relation

$$\begin{array}{c} 2 \\ \diagup \\ \text{---} \circ (z) \\ \diagdown \\ 1 \end{array} \begin{array}{c} 3 \\ \diagdown \\ \text{---} \\ \diagup \\ 4 \end{array} = \begin{array}{c} 2 \\ \diagup \\ \text{---} \circ \\ \diagdown \\ 1 \end{array} \begin{array}{c} 1 \\ \diagdown \\ \text{---} \\ \diagup \\ 2 \end{array} + \frac{1}{2\pi i} \int_1^\infty \sum_{\{c_j\}} \left( \frac{1}{x-z} - \frac{1}{x-z_0} \right) dx \begin{array}{c} 3 \\ \diagdown \\ \text{---} \circ \\ \diagup \\ 2 \end{array} \begin{array}{c} 4 \\ \diagdown \\ \text{---} \circ \\ \diagup \\ 1 \end{array} \begin{array}{c} \{c_j\} \\ \vdots \\ \vdots \\ \vdots \end{array} (x)$$

- **beneficial for IBP computational complexity**
  - **requires understanding of analytic structure**
- trade-off

- formulated for massless 4-point kinematics
- checked for 2-loop nonplanar amplitude
- provided new 4-loop ladder integral result



# Outlook

## kinematic limits

e.g.  $\text{Disc}_1 \mathcal{A} = \text{Cut}_u \mathcal{A}$  alone gives  $u \rightarrow 0$  limit at fixed Log accuracy to any subleading power

$$\lim_{x \rightarrow 1} \mathcal{A}(x) \sim c_{1,1}(x) G(1,1,x) + c_{0,1}(x) G(0,1,x) + c_{\dots,0}(x) G(\dots,0,x)$$

Leading Log
Next-to-Leading Log
suppressed

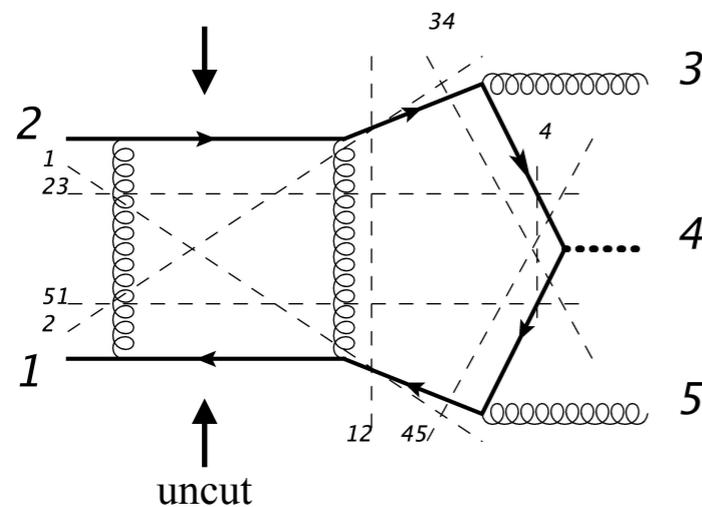
## recursive approach

## Multivariate Integrated Unitarity

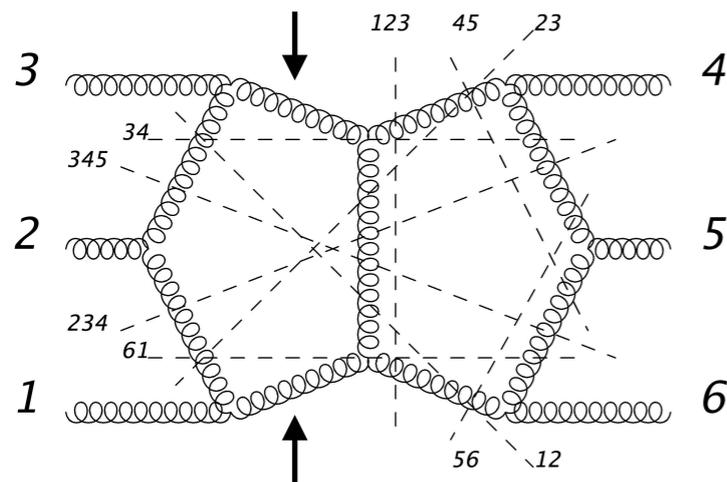
# Outlook into Multivariate Integrated Unitarity

examples

$pp \rightarrow t\bar{t}H$  @ 2 loops



$gg \rightarrow gggg$  @ 2 loops



properties

- many cut propagators → simpler IBPs
- iterative Cauchy formula → multivariate complex analysis
- Landau singularities → nontrivial analytic structure

[Helmer, Papathanasiou, Tellander [2402.14787](#)]

[Correia [2212.06157](#)]

- Steinmann relations → possible simplifications

[Caron-Huot, Dixon, McLeod, Hippel [1609.00669](#)]

THANK YOU

# Appendix

# Appendix : monodromies

following :

[Bourjaily, Hannesdottir, McLeod, Schwartz, Vergu 2007.13747]

- Harmonic Polylog :

$$G(\alpha_n, \dots, \alpha_1; x) \quad \alpha_k \in \{0,1\}$$

- vector of derivatives :

$$\mathcal{V}_i = \begin{cases} 1 & \text{if } i = 0, \\ (-1)^{\# \text{ of } 1} G(\alpha_{n+1-i}, \dots, \alpha_n, x) & \text{if } n \geq i > 0 \end{cases}$$

- connection matrix :

$$\omega_{ij} = \frac{dx}{x - \alpha_{n-i}} \delta_{i+1,j} \quad \text{s.t.} \quad d\mathcal{V} = \mathcal{V} \cdot \omega$$

- variation matrix :

$$\mathcal{M}_\gamma = \mathcal{P} e^{\int_\gamma \omega} \quad \text{collects all } n + 1 \text{ solutions for } \mathcal{V}$$

- general solution :

$$(\mathcal{M}_{\rightarrow x})_{ij} = \sum_{k=0}^n (-1)^{\# \text{ of } 1} G(\alpha_{n-i}, \dots, \alpha_{n-i-k+1}, x) \delta_{i+k,j}$$

$$G(x) = 1$$

- monodromy matrices :

$$\mathcal{M}_0 = \mathcal{M}_{\cup_0},$$

$$\mathcal{M}_1 = \mathcal{M}_{\rightarrow 1} \mathcal{M}_{\cup_1} \mathcal{M}_{\rightarrow 1}^{-1}.$$

- discontinuities :

$$\text{Disc}_0 = (1 - \mathcal{M}_0) \cdot \mathcal{M}_{\rightarrow x},$$

$$\text{Disc}_1 = -(1 - \mathcal{M}_1) \cdot \mathcal{M}_{\rightarrow x},$$

$$\text{Disc}_\infty = (1 - \mathcal{M}_0 \cdot \mathcal{M}_1) \cdot \mathcal{M}_{\rightarrow x}.$$

# Appendix : ansatz matching

- example ansatz :  $c_{1,1} G(1,1;x) + c_{1,0} G(1,0;x) + c_{0,1} G(0,1;x) + c_{0,0} G(0,0;x) + c_1 G(1;x) + c_0 G(0;x) + c$
- impose e.g.  $\text{Disc}_0 = 0$  :  $c_{1,1} G(1,1;x) + 0 + c_{0,1} G(0,1;x) + 0 + c_1 G(1;x) + 0 + c$
- $\text{Disc}_1 = 2\pi i (2 G(1;x) + 3 G(0;x) + 5 \pi i)$   $= c_{1,1} (2\pi i (-G(1;x) + i\pi)) + c_{0,1} (-2\pi i G(0,x)) + c_1 (-2\pi i) + 0$
- now only constant unconstrained :  $-2 G(1,1;x) - 3 G(0,1;x) - 7 i\pi G(1;x) + c$
- impose fixed value e.g.  $\zeta_2$  at  $x=0$  :  $-2 G(1,1;x) - 3 G(0,1;x) - 7 i\pi G(1;x) + \zeta_2$