Feynman Parameter Integrals with Positive Integrands

Thomas Stone

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Stephen Jones, Anton Olsson, **TS**
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Gudrun Heinrich, Matthias Kerner, Vitaly Magerya, Johannes Schlenk [WIP]

Technische Universität München

Outline

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Feynman integrals can be difficult to compute analytically

Various methods to approximate/evaluate them numerically

Numerical differential equations	**ODE/PDE**
Series solutions of differential equations (AMFlow, DiffExp, Seasyde) Taylor expansion in Feynman parameters (TayInt)	**Series Solutions**
Numerical Mellin-Barnes (MB, Ambre) Tropical sampling (Feyntrop) Numerical Loop-Tree Duality (cLTD, Lotty) Sector decomposition (Sector_decomposition, FIESTA, pySecDec)	**~Monte Carlo Integration**

[Slide style inspired by Stephen Jones]

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$\hookrightarrow$ *This talk:* avoiding contour deformation in the context of **sector decomposition**

Minkowski Regime

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Minkowski regime $\Rightarrow$ poles on the contour of integration

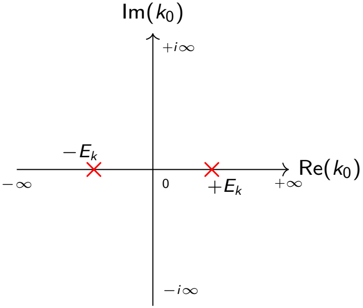

Figure: Tadpole k_0 integration

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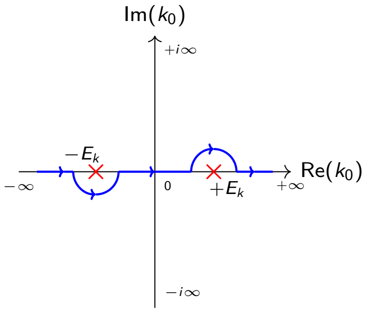

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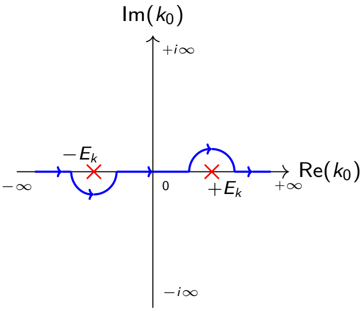

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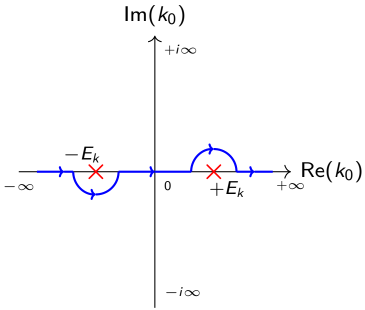
Methods being explored to remove the need for contour deformation in momentum space

[Buchta, Chachamis, Draggiotis, Rodrigo; Anastasiou, Haindl, Sterman, Yang, Zeng; Aguilera-Verdugo, Hernandez-Pinto, Sborlini, Torres Bobadilla; Capatti, Hirschi, Kermanschah, Pelloni, Ruijl...]



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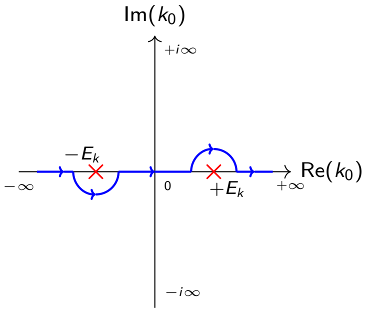

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Can we do the same in **Feynman parameter** space?

Feynman Parameter Integrals

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Loop integral in Feynman parameter space

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[1] *Generally*, interpret J as an integral over $\mathbb{RP}^{N-1}_{\geq 0} \Rightarrow$ *any* valid deprojectivisation

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$$\mathcal{U}(\mathbf{x}) = \sum_{T^1} \prod_{e\notin T^1} x_e \geq 0,$$

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We distinguish between 3 types of kinematic regime:

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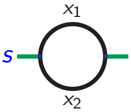

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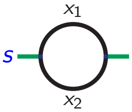

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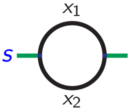

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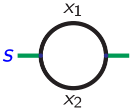

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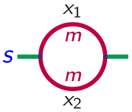

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Singularities in Feynman Parameter Space

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Landau equations in Feynman parameter space

$$\textbf{a)}\quad \mathcal{F}(\mathbf{x};\mathbf{s}) = 0\,, \qquad \textbf{b)}\quad \forall x_i : \; x_i \frac{\partial \mathcal{F}(\mathbf{x};\mathbf{s})}{\partial x_i} = 0$$

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Landau equations in Feynman parameter space

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Sector Decomposition

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$$\text{e.g.}\quad I = \int_0^1 \mathrm{d}x\, x^{-1+a\epsilon}\, \mathcal{I}(x,\epsilon) = \frac{\mathcal{I}(0,\epsilon)}{a\,\epsilon} + \underbrace{\int_0^1 \mathrm{d}x\, x^{-1+a\epsilon}\left[\mathcal{I}(x,\epsilon) - \mathcal{I}(0,\epsilon)\right]}_{\text{no problem as } x \to 0,\text{ finite!}}$$

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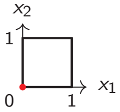

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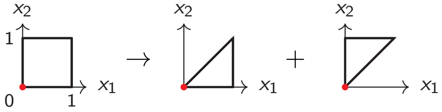

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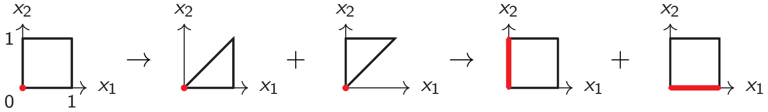

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TAKE HOME MESSAGE

Sector decomposition resolves issues on the *boundary* and can be considered 'solved'*

*for our purposes at least!

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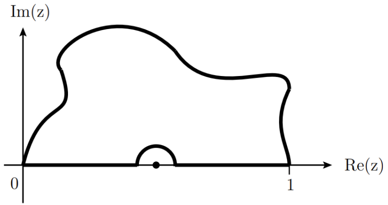

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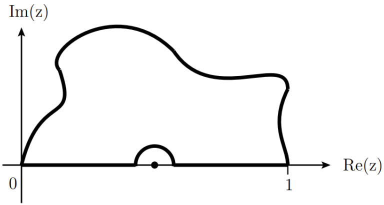

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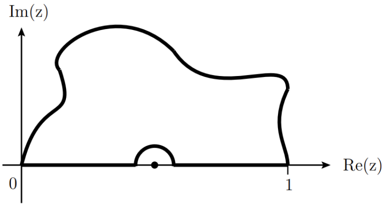

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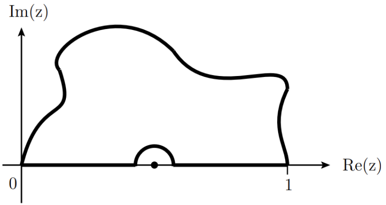

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If this works, why try to ***avoid*** it?

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TAKE HOME MESSAGE

Contour deformation reduces precision, can be **very** slow and can break down entirely!

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$$J(\mathbf{s}) = \sum_{n_+=1}^{N_+} J^{+,n_+}(\mathbf{s}) + (-1-i\delta^+)^{LD/2-\nu} \sum_{n_-=1}^{N_-} J^{-,n_-}(\mathbf{s})$$

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Trivial analytic continuation in this representation: $\mathrm{Im}(J)$ controlled by $(-1-i\delta^+)^{LD/2-\nu} \times \mathbb{R}$

Constructing the Decomposition

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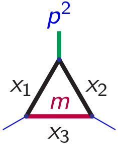

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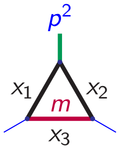

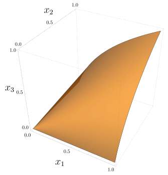

How to achieve this representation?

Analyse the (projective) *variety of $\mathcal{F}$*: codimension-1 hypersurface defined by $\mathcal{F}(\mathbf{x}; \mathbf{s}) = 0$

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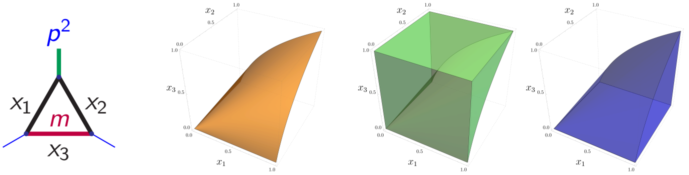

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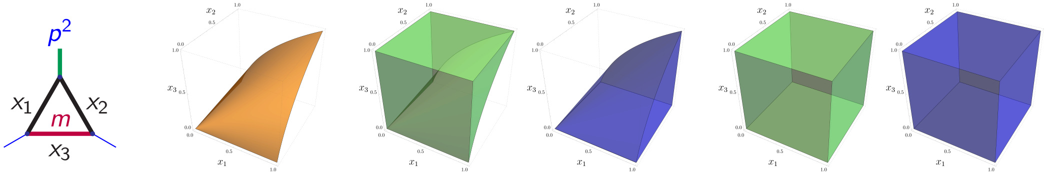

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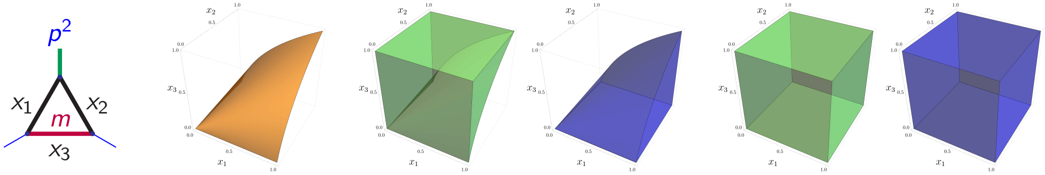

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Variety ($\mathcal{F} = 0$) mapped to *boundary* $\Rightarrow$ resolved by **sector decomposition**!

The Simplest Example

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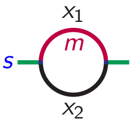

$$= \Gamma(\epsilon) \int_{\mathbb{R}^2_{\geq 0}} \mathrm{d}x_1 \mathrm{d}x_2 \, \frac{(x_1 + x_2)^{-2+2\epsilon}}{\left(-s\, x_1 x_2 + m^2\, x_1\, (x_1 + x_2) - i\delta^+\right)^{\epsilon}} \, \delta(1 - \alpha(\mathbf{x}))\,, \qquad s > m^2 > 0$$

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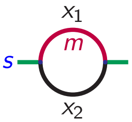

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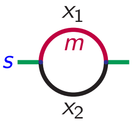
$$= \Gamma(\epsilon) \int_0^{\infty} \mathrm{d}x_2 \int_0^{\frac{s-m^2}{m^2} x_2} \mathrm{d}x_1 \, \frac{(x_1 + x_2)^{-2+2\epsilon}}{\left(-s\, x_1 x_2 + m^2\, x_1\, (x_1 + x_2) - i\delta^+\right)^{\epsilon}} \, \delta(\, 1 - \alpha(\mathbf{x})\,)$$

$$+ \Gamma(\epsilon) \int_0^{\infty} \mathrm{d}x_2 \int_{\frac{s-m^2}{m^2} x_2}^{\infty} \mathrm{d}x_1 \, \frac{(x_1 + x_2)^{-2+2\epsilon}}{\left(-s\, x_1 x_2 + m^2\, x_1\, (x_1 + x_2) - i\delta^+\right)^{\epsilon}} \, \delta(\, 1 - \alpha(\mathbf{x})\,)$$

The Simplest Example



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$$= (-1-i\delta^+)^{-\epsilon}\, \Gamma(\epsilon) \int_0^\infty \mathrm{d}x_2 \int_0^{\frac{s-m^2}{m^2} x_2} \mathrm{d}x_1 \, \frac{(x_1+x_2)^{-2+2\epsilon}}{\left(+s\, x_1 x_2 - m^2\, x_1\,(x_1+x_2)\right)^{\epsilon}} \, \delta(1-\alpha(\mathbf{x}))$$

$$+ \; \Gamma(\epsilon) \int_0^\infty \mathrm{d}x_2 \int_{\frac{s-m^2}{m^2} x_2}^{\infty} \mathrm{d}x_1 \, \frac{(x_1+x_2)^{-2+2\epsilon}}{\left(-s\, x_1 x_2 + m^2\, x_1\,(x_1+x_2)\right)^{\epsilon}} \, \delta(1-\alpha(\mathbf{x}))$$

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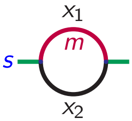

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$$+\ \Gamma(\epsilon)\, (m^2)^{2-2\epsilon} \int_{\mathbb{R}^2_{\geq 0}} \mathrm{d}x_1 \mathrm{d}x_2 \frac{x_1^{-\epsilon}\, \left(m^2 x_1 + (s - m^2)\, x_2\right)^{-\epsilon}}{(s x_2 + m^2 x_1)^{2-2\epsilon}}\, \delta(\,1 - \alpha^+(\mathbf{x})\,)$$

Outline

Univariate Bisectable Integrals

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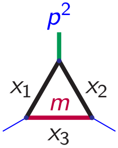
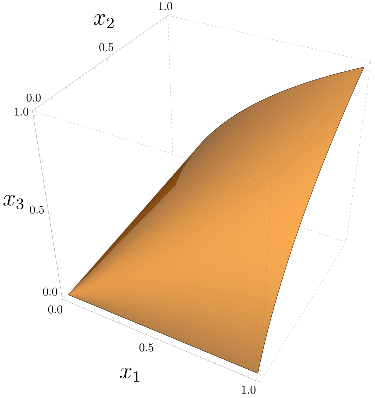


is UB w.r.t. x_3 for $p^2, m^2 > 0$



Univariate Bisectable Integrals

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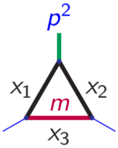
e.g. 

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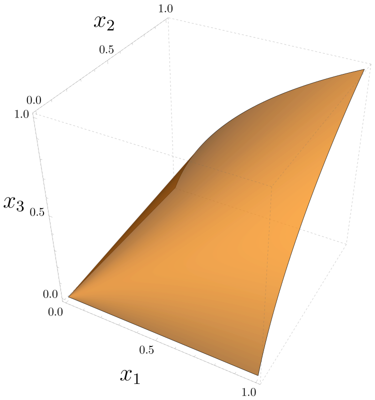

Same-sign decomposition for UB integrals

$$J(\mathbf{s}) \;=\; J^{+}(\mathbf{s}) \;+\; (-1-i\delta^{+})^{LD/2-\nu}\; J^{-}(\mathbf{s})$$

UB Algorithm

UB Algorithm

UB integrals $\rightarrow$ *algorithmic* procedure

UB Algorithm

Algorithm: Univariate Bisection (UB)

Input: $\mathcal{I}(\mathbf{x};\mathbf{s};\delta)$, $\mathbf{s}_R$
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foreach $x_i \in \mathbf{x}$ **do**
  Let $r = \mathtt{Reduce}[\{\mathcal{F}(\mathbf{x};\mathbf{s}) < 0\} \cup \{0 < \mathbf{x}\} \cup \mathbf{s}_R,\ x_i]$;
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Essentially just *parameter transformations*:

$$x_i \rightarrow y_i = \frac{x_i}{x_i+x_j} f(\mathbf{x}_{\neq i}) \tag{1}$$
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(1) $\mathcal{F} = 0$ hypersurface $\rightarrow \infty$
(2) $\mathcal{F} = 0$ hypersurface $\rightarrow 0$

1-Loop Box with One Off-Shell Leg

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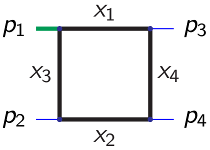

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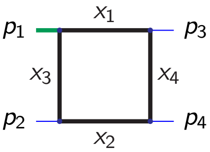
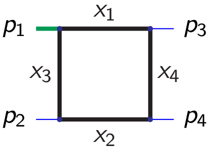
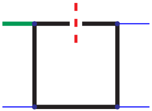


$\mathcal{U}(\mathbf{x})$

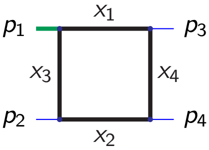
1-Loop Box with One Off-Shell Leg

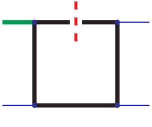
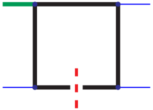


$\mathcal{U}(\mathbf{x}) =$ 

$= x_1$

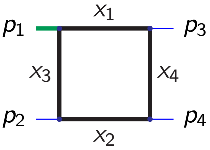
1-Loop Box with One Off-Shell Leg



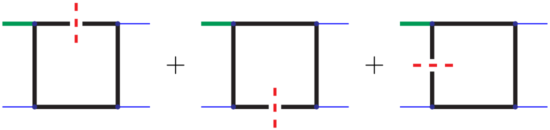
$\mathcal{U}(\mathbf{x}) =$  $+$ 

$= x_1 + x_2$

1-Loop Box with One Off-Shell Leg

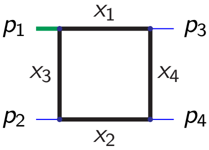


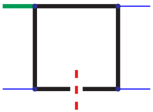
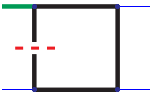
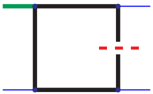
$\mathcal{U}(\mathbf{x}) =$



$$= x_1 + x_2 + x_3$$

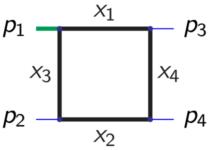
1-Loop Box with One Off-Shell Leg

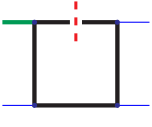
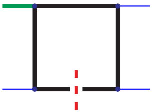
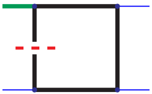
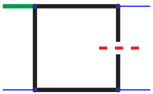


$\mathcal{U}(\mathbf{x}) =$  $+$  $+$  $+$ 

$$= x_1 + x_2 + x_3 + x_4$$

1-Loop Box with One Off-Shell Leg

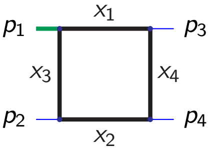


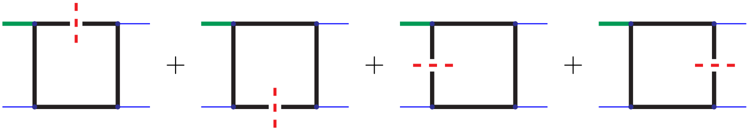
$\mathcal{U}(\mathbf{x}) =$  $+$  $+$  $+$ 

$$= x_1 + x_2 + x_3 + x_4$$

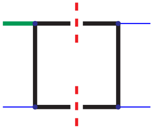
$\mathcal{F}(\mathbf{x}; \mathbf{s})$

1-Loop Box with One Off-Shell Leg



$\mathcal{U}(\mathbf{x}) =$ 

$$= x_1 + x_2 + x_3 + x_4$$

$\mathcal{F}(\mathbf{x}; s) = -s$ 

$$= -s\, x_1 x_2$$

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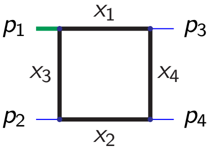
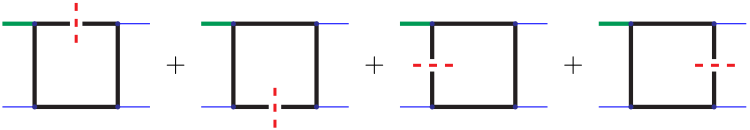


$\mathcal{U}(\mathbf{x}) =$



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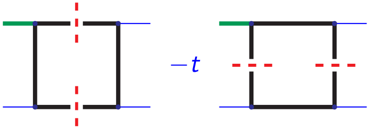
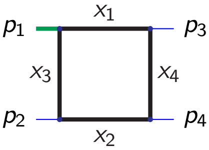
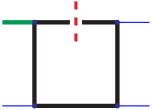
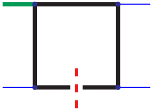
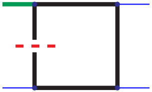
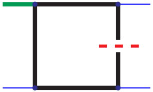
$\mathcal{F}(\mathbf{x}; \mathbf{s}) = -s$ ... $-t$



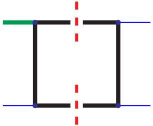
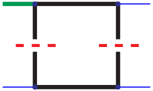
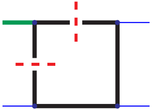
$$= -s\, x_1 x_2 - t\, x_3 x_4$$

1-Loop Box with One Off-Shell Leg



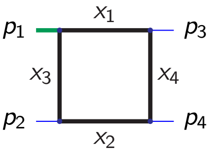
$\mathcal{U}(\mathbf{x}) =$  $+$  $+$  $+$ 

$$= x_1 + x_2 + x_3 + x_4$$

$\mathcal{F}(\mathbf{x}; \mathbf{s}) = -s$  $-t$  $-p_1^2$ 

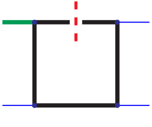
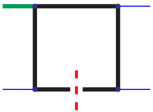
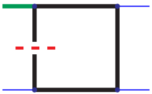
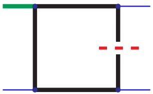
$$= -s\, x_1 x_2 - t\, x_3 x_4 - p_1^2\, x_1 x_3$$

1-Loop Box with One Off-Shell Leg

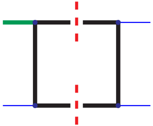
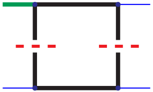
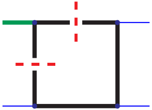


Minkowski $\mathbf{s}_R$:

$$0 < p_1^2 < s$$
$$-s < t < 0$$

$\mathcal{U}(\mathbf{x}) =$  $+$  $+$  $+$ 

$$= x_1 + x_2 + x_3 + x_4$$

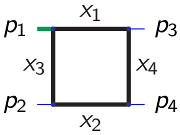
$\mathcal{F}(\mathbf{x}; \mathbf{s}) = -s$  $-t$  $-p_1^2$ 

$$= -s\, x_1 x_2 - t\, x_3 x_4 - p_1^2\, x_1 x_3$$

1-Loop Box with One Off-Shell Leg

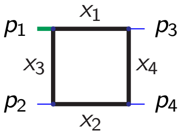
1-Loop Box with One Off-Shell Leg

$$J(\mathbf{s}) = \Gamma(2+\epsilon)\int_{\mathbb{R}^4_{\geq 0}} \prod_{i=1}^{4} \mathrm{d}x_i \, \frac{(x_1+x_2+x_3+x_4)^{2\epsilon}}{\left(-s\, x_1 x_2 - t\, x_3 x_4 - p_1^2\, x_1 x_3 - i\delta^+\right)^{2+\epsilon}} \, \delta(1-\alpha(\mathbf{x}))$$



1-Loop Box with One Off-Shell Leg

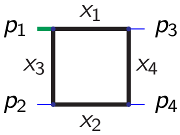
$$J(\mathbf{s}) = \Gamma(2+\epsilon) \int_{\mathbb{R}^4_{\geq 0}} \prod_{i=1}^{4} \mathrm{d}x_i \, \frac{(x_1+x_2+x_3+x_4)^{2\epsilon}}{(-s\, x_1x_2 - t\, x_3x_4 - p_1^2\, x_1x_3 - i\delta^+)^{2+\epsilon}} \, \delta(\,1-\alpha(\mathbf{x})\,)$$



Algorithm $\rightarrow$ UB in $\mathbf{s}_R = \{0 < p_1^2 < s,\ -s < t < 0\}$ w.r.t x_1

1-Loop Box with One Off-Shell Leg

$$J(\mathbf{s}) = \Gamma(2+\epsilon) \int_{\mathbb{R}^4_{\geq 0}} \prod_{i=1}^{4} \mathrm{d}x_i \, \frac{(x_1+x_2+x_3+x_4)^{2\epsilon}}{\left(-s\,x_1x_2 - t\,x_3x_4 - p_1^2\,x_1x_3 - i\delta^+\right)^{2+\epsilon}} \, \delta(\,1-\alpha(\mathbf{x})\,)$$

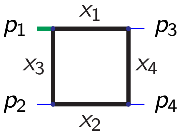


Algorithm $\rightarrow$ UB in $\mathbf{s}_R = \{0 < p_1^2 < s, \; -s < t < 0\}$ w.r.t x_1

$$x_1 \rightarrow y_1' = x_1 + \left(\frac{-t\,x_3x_4}{s\,x_2 + p_1^2\,x_3}\right)$$

1-Loop Box with One Off-Shell Leg

$$J(\mathbf{s}) = \Gamma(2+\epsilon) \int_{\mathbb{R}^4_{\geq 0}} \prod_{i=1}^{4} \mathrm{d}x_i \, \frac{(x_1 + x_2 + x_3 + x_4)^{2\epsilon}}{\left(-s\, x_1 x_2 - t\, x_3 x_4 - p_1^2\, x_1 x_3 - i\delta^+\right)^{2+\epsilon}} \, \delta(\, 1 - \alpha(\mathbf{x})\,)$$



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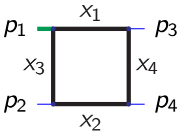
$$x_1 \rightarrow y_1' = x_1 + \left(\frac{-t\, x_3 x_4}{s\, x_2 + p_1^2\, x_3} \right)$$

$$\mathcal{I}^- = \frac{\left[\left(s\, x_2 + p_1^2\, x_3 \right) \left(x_1 + x_2 + x_3 + x_4 \right) - t\, x_3 x_4 \right]^{2\epsilon}}{x_1^{2+\epsilon} \left(s\, x_2 + p_1^2\, x_3 \right)^{2+3\epsilon}}$$

1-Loop Box with One Off-Shell Leg

$$J(\mathbf{s}) = \Gamma(2+\epsilon)\int_{\mathbb{R}^4_{\geq 0}} \prod_{i=1}^{4} \mathrm{d}x_i \, \frac{(x_1+x_2+x_3+x_4)^{2\epsilon}}{\left(-s\,x_1x_2 - t\,x_3x_4 - p_1^2\,x_1x_3 - i\delta^+\right)^{2+\epsilon}} \,\delta(\,1-\alpha(\mathbf{x})\,)$$



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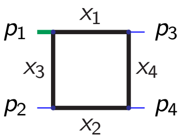
$$x_1 \rightarrow y_1 = \frac{x_1}{x_1+x_4}\left(\frac{-t\,x_3x_4}{s\,x_2 + p_1^2\,x_3}\right)$$

1-Loop Box with One Off-Shell Leg

$$J(\mathbf{s}) = \Gamma(2+\epsilon)\int_{\mathbb{R}^4_{\geq 0}} \prod_{i=1}^{4} \mathrm{d}x_i \, \frac{(x_1+x_2+x_3+x_4)^{2\epsilon}}{(-s\,x_1x_2 - t\,x_3x_4 - p_1^2\,x_1x_3 - i\delta^+)^{2+\epsilon}}\,\delta(\,1-\alpha(\mathbf{x})\,)$$



Algorithm $\rightarrow$ UB in $\mathbf{s}_R = \{0 < p_1^2 < s,\ -s < t < 0\}$ w.r.t x_1

$$x_1 \rightarrow y_1' = x_1 + \left(\frac{-t\,x_3x_4}{s\,x_2 + p_1^2\,x_3}\right) \qquad \mathcal{I}^- = \frac{\left[\left(s\,x_2 + p_1^2\,x_3\right)(x_1+x_2+x_3+x_4) - t\,x_3x_4\right]^{2\epsilon}}{x_1^{2+\epsilon}\left(s\,x_2 + p_1^2\,x_3\right)^{2+3\epsilon}}$$

$$x_1 \rightarrow y_1 = \frac{x_1}{x_1+x_4}\left(\frac{-t\,x_3x_4}{s\,x_2 + p_1^2\,x_3}\right) \qquad \mathcal{I}^+ = \frac{\left[\left(s\,x_2 + p_1^2\,x_3\right)(x_1+x_4)(x_2+x_3+x_4) - t\,x_1x_3x_4\right]^{2\epsilon}}{x_4^{2+2\epsilon}(x_1+x_4)^{\epsilon}(-t\,x_3)^{1+\epsilon}\left(s\,x_2 + p_1^2\,x_3\right)^{1+2\epsilon}}$$

1-Loop Box with One Off-Shell Leg

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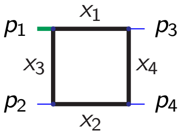

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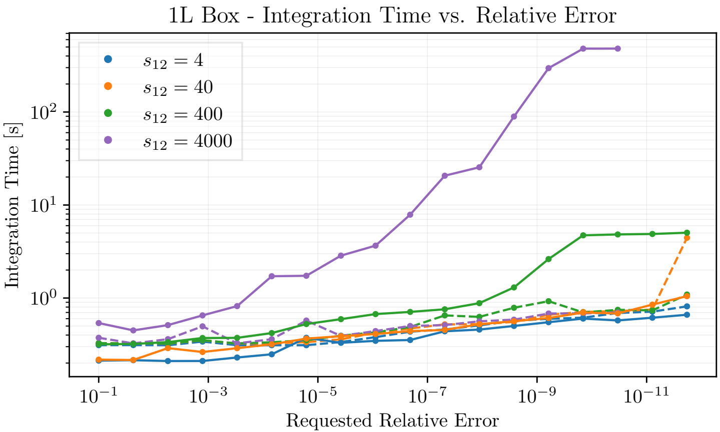
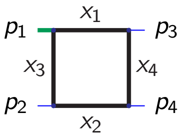
$$x_1 \rightarrow y_1 = \frac{x_1}{x_1+x_4}\left(\frac{-t\,x_3x_4}{s\,x_2 + p_1^2\,x_3}\right) \qquad \mathcal{I}^+ = \frac{\left[\left(s\,x_2 + p_1^2\,x_3\right)(x_1+x_4)(x_2+x_3+x_4) - t\,x_1x_3x_4\right]^{2\epsilon}}{x_4^{2+2\epsilon}(x_1+x_4)^{\epsilon}\left(-t\,x_3\right)^{1+\epsilon}\left(s\,x_2 + p_1^2\,x_3\right)^{1+2\epsilon}}$$

$$J(\mathbf{s}) \ = \ J^+(\mathbf{s}) \ + \ \left(-1 - i\delta^+\right)^{-2-\epsilon} J^-(\mathbf{s})$$

1-Loop Box with One Off-Shell Leg

1-Loop Box with One Off-Shell Leg





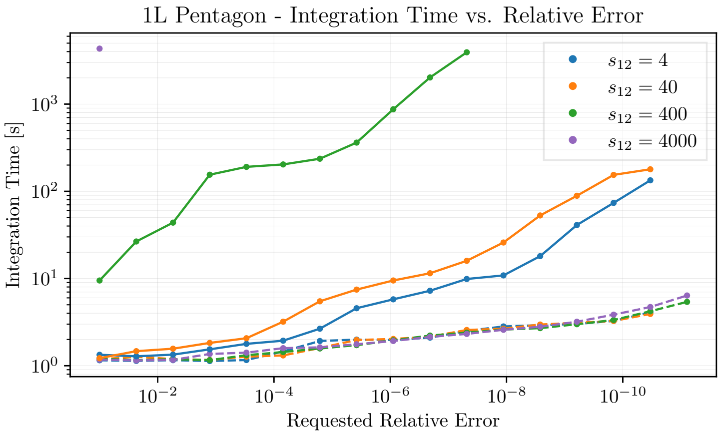
pySecDec
[Heinrich, Jones, Kerner, Magerya, Olsson, Schlenk]

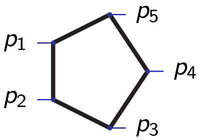
——— CD
- - - - - NOCD

$$s = s_{12}$$
$$t = -1$$
$$p_1^2 = 2$$

1-Loop Pentagon

1-Loop Pentagon





pySecDec

CD

NOCD

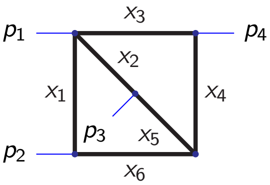
$s_{23} = -3.0, \quad s_{34} = 2.5$

$s_{45} = -3.0, \quad s_{51} = 5.0$

2-Loop Non-Planar 6 Propagator Box (BNP6)

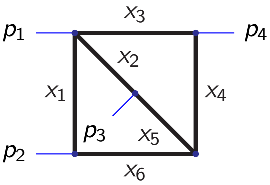
2-Loop Non-Planar 6 Propagator Box (BNP6)

$$J(\mathbf{s}) = \Gamma(2+2\epsilon) \int_{\mathbb{R}^6_{\geq 0}} \prod_{i=1}^{6} \mathrm{d}x_i \, \frac{\mathcal{U}(\mathbf{x})^{3\epsilon}}{(\mathcal{F}(\mathbf{x};\mathbf{s}) - i\delta +)^{2+2\epsilon}} \, \delta(\,1 - \alpha(\mathbf{x})\,)$$



2-Loop Non-Planar 6 Propagator Box (BNP6)

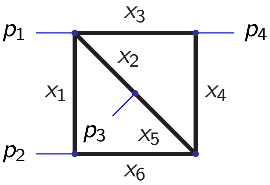
$$J(\mathbf{s}) = \Gamma(2+2\epsilon)\int_{\mathbb{R}^6_{\geq 0}}\prod_{i=1}^{6}\mathrm{d}x_i\,\frac{\mathcal{U}(\mathbf{x})^{3\epsilon}}{(\mathcal{F}(\mathbf{x};\mathbf{s})-i\delta^+)^{2+2\epsilon}}\,\delta(\,1-\alpha(\mathbf{x})\,)$$



$$\begin{aligned}\mathcal{U}(\mathbf{x}) &= x_1x_2 + x_1x_3 + x_1x_4 + x_1x_5 + x_2x_3 + x_2x_4 + x_2x_6 + \\ &\quad x_3x_5 + x_3x_6 + x_4x_5 + x_4x_6 + x_5x_6 \\ \mathcal{F}(\mathbf{x};\mathbf{s}) &= -s_{12}\,x_2x_3x_6 - s_{23}\,x_1x_2x_4 + \left(s_{12} + s_{23}\right)x_1x_3x_5\end{aligned}$$

2-Loop Non-Planar 6 Propagator Box (BNP6)

$$J(\mathbf{s}) = \Gamma(2+2\epsilon) \int_{\mathbb{R}^6_{\geq 0}} \prod_{i=1}^{6} \mathrm{d}x_i \, \frac{\mathcal{U}(\mathbf{x})^{3\epsilon}}{\left(\mathcal{F}(\mathbf{x};\mathbf{s}) - i\delta^+\right)^{2+2\epsilon}} \, \delta(\, 1 - \alpha(\mathbf{x})\,)$$

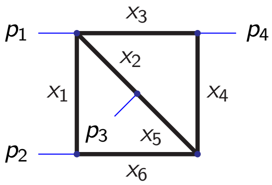


$$\begin{aligned}\mathcal{U}(\mathbf{x}) &= x_1x_2 + x_1x_3 + x_1x_4 + x_1x_5 + x_2x_3 + x_2x_4 + x_2x_6 + \\ &\quad x_3x_5 + x_3x_6 + x_4x_5 + x_4x_6 + x_5x_6 \\ \mathcal{F}(\mathbf{x};\mathbf{s}) &= -s_{12}\,x_2x_3x_6 - s_{23}\,x_1x_2x_4 + (s_{12} + s_{23})\,x_1x_3x_5\end{aligned}$$

$\nexists$ *same-sign regime!*

2-Loop Non-Planar 6 Propagator Box (BNP6)

$$J(\mathbf{s}) = \Gamma(2+2\epsilon)\int_{\mathbb{R}^6_{\geq 0}} \prod_{i=1}^{6} \mathrm{d}x_i \, \frac{\mathcal{U}(\mathbf{x})^{3\epsilon}}{(\mathcal{F}(\mathbf{x};\mathbf{s}) - i\delta^+)^{2+2\epsilon}} \, \delta(\, 1-\alpha(\mathbf{x})\,)$$



$$\mathcal{U}(\mathbf{x}) = x_1x_2 + x_1x_3 + x_1x_4 + x_1x_5 + x_2x_3 + x_2x_4 + x_2x_6 + x_3x_5 + x_3x_6 + x_4x_5 + x_4x_6 + x_5x_6$$

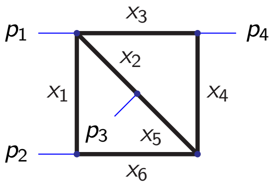
$$\mathcal{F}(\mathbf{x};\mathbf{s}) = -s_{12}\, x_2x_3x_6 - s_{23}\, x_1x_2x_4 + (s_{12} + s_{23})\, x_1x_3x_5$$

$\nexists$ *same-sign regime!*

Algorithm $\rightarrow$ UB in $\mathbf{s}_R = \{0 < s_{12},\ -s_{12} < s_{23} < 0\}$ w.r.t x_1

2-Loop Non-Planar 6 Propagator Box (BNP6)

$$J(\mathbf{s}) = \Gamma(2+2\epsilon)\int_{\mathbb{R}^6_{\geq 0}} \prod_{i=1}^{6} \mathrm{d}x_i \, \frac{\mathcal{U}(\mathbf{x})^{3\epsilon}}{(\mathcal{F}(\mathbf{x};\mathbf{s}) - i\delta^+)^{2+2\epsilon}} \, \delta(1-\alpha(\mathbf{x}))$$



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$$\mathcal{F}(\mathbf{x};\mathbf{s}) = -s_{12}\, x_2x_3x_6 - s_{23}\, x_1x_2x_4 + (s_{12}+s_{23})\, x_1x_3x_5$$

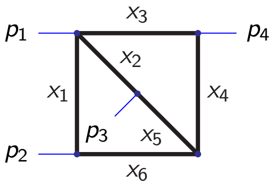
$\nexists$ *same-sign regime!*

Algorithm $\rightarrow$ UB in $\mathbf{s}_R = \{0 < s_{12},\ -s_{12} < s_{23} < 0\}$ w.r.t x_1

$$f(\mathbf{x}_{\neq 1}) = \frac{s_{12}x_2x_3x_6}{(s_{12}+s_{23})\,x_3x_5 - s_{23}x_2x_4}$$

2-Loop Non-Planar 6 Propagator Box (BNP6)

$$J(\mathbf{s}) = \Gamma(2+2\epsilon)\int_{\mathbb{R}^6_{\geq 0}} \prod_{i=1}^{6} \mathrm{d}x_i \, \frac{\mathcal{U}(\mathbf{x})^{3\epsilon}}{(\mathcal{F}(\mathbf{x};\mathbf{s}) - i\delta^+)^{2+2\epsilon}} \, \delta(1-\alpha(\mathbf{x}))$$



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$$\mathcal{F}(\mathbf{x};\mathbf{s}) = -s_{12}\, x_2x_3x_6 - s_{23}\, x_1x_2x_4 + (s_{12} + s_{23})\, x_1x_3x_5$$

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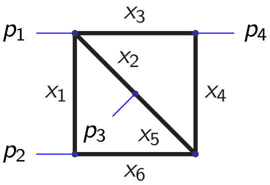
$$f(\mathbf{x}_{\neq 1}) = \frac{s_{12}x_2x_3x_6}{(s_{12}+s_{23})\,x_3x_5 - s_{23}x_2x_4} \qquad x_1 \rightarrow y_1 = \frac{x_1}{x_1+x_6} f(\mathbf{x}_{\neq 1})$$

2-Loop Non-Planar 6 Propagator Box (BNP6)

$$J(\mathbf{s}) = \Gamma(2+2\epsilon)\int_{\mathbb{R}^6_{\geq 0}} \prod_{i=1}^{6} \mathrm{d}x_i \, \frac{\mathcal{U}(\mathbf{x})^{3\epsilon}}{(\mathcal{F}(\mathbf{x};\mathbf{s}) - i\delta^+)^{2+2\epsilon}} \, \delta(\, 1-\alpha(\mathbf{x})\,)$$



$$\mathcal{U}(\mathbf{x}) = x_1x_2 + x_1x_3 + x_1x_4 + x_1x_5 + x_2x_3 + x_2x_4 + x_2x_6 + x_3x_5 + x_3x_6 + x_4x_5 + x_4x_6 + x_5x_6$$

$$\mathcal{F}(\mathbf{x};\mathbf{s}) = -s_{12}\, x_2x_3x_6 - s_{23}\, x_1x_2x_4 + (s_{12} + s_{23})\, x_1x_3x_5$$

$\nexists$ *same-sign regime!*

Algorithm $\rightarrow$ UB in $\mathbf{s}_R = \{0 < s_{12},\ -s_{12} < s_{23} < 0\}$ w.r.t x_1

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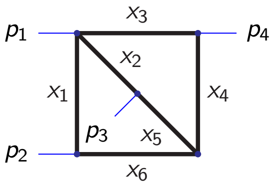

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2-Loop Non-Planar 6 Propagator Box (BNP6)

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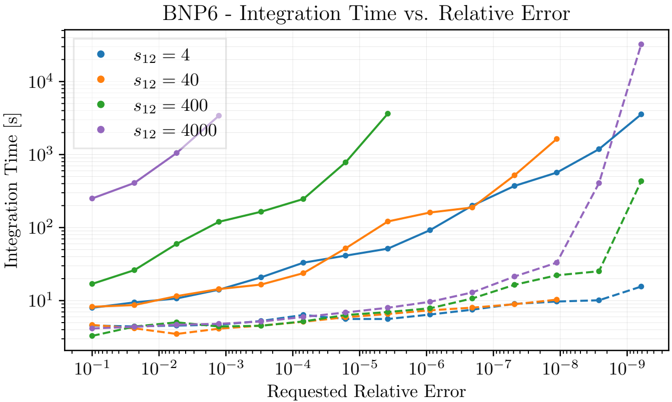

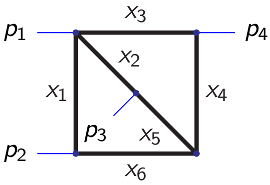

pySecDec

CD

NOCD

$s_{23} = -1$

2-Loop Non-Planar 7 Propagator Box (BNP7)

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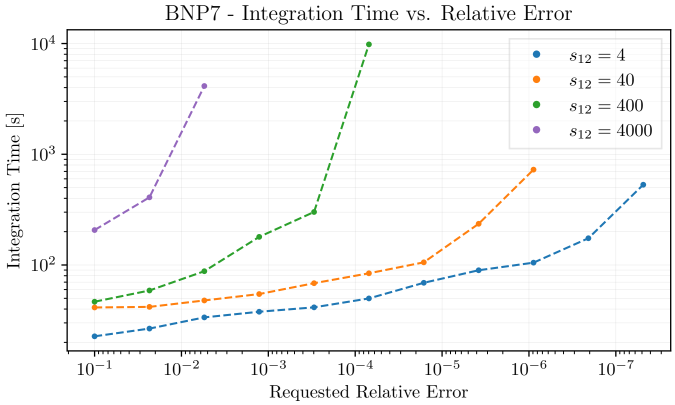

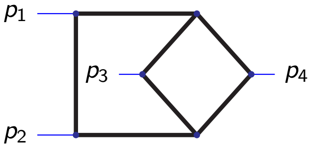

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3-Loop Non-Planar Box (Crown)

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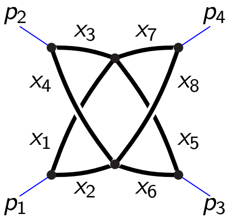
$$J(\mathbf{s}) = \Gamma(2+3\epsilon) \int_{\mathbb{R}^8_{\geq 0}} \prod_{i=1}^{8} \mathrm{d}x_i \, \frac{\mathcal{U}(\mathbf{x})^{4\epsilon}}{(\mathcal{F}(\mathbf{x};\mathbf{s}) - i\delta^+)^{2+3\epsilon}} \, \delta(\,1-\alpha(\mathbf{x})\,)$$



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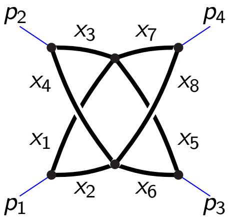

$$\mathcal{F}(\mathbf{x};\mathbf{s}) \;=\; -s_{12}\,(x_2x_5 - x_1x_6)\,(x_4x_7 - x_3x_8)\, - s_{13}\,(x_2x_3 - x_1x_4)\,(x_6x_7 - x_5x_8)$$

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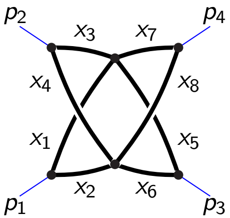

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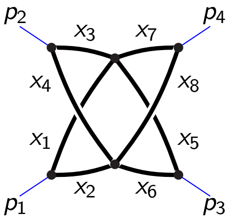

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Dissect integration domain on *Landau locus*
[Gardi, Herzog, Jones, Ma]

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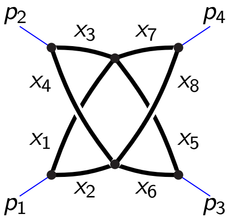

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Leading Landau singularity for **arbitrary kinematics** $\rightarrow$ CD breaks down!
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$$J(\mathbf{s}) = 4\,\Gamma(2+3\epsilon) \sum_K \int_{\mathbb{R}^8_{\geq 0}} \prod_{i=1}^{8} \mathrm{d}x_i \, \frac{\mathcal{U}_K(\mathbf{x})^{4\epsilon}}{(x_2x_4x_6)^{1+3\epsilon}\, x_8^{1+9\epsilon}\, (\mathcal{F}_K(\mathbf{x};\mathbf{s}) - i\delta^+)^{2+3\epsilon}} \, \delta(1-\alpha(\mathbf{x}))$$

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Verified numerical result against known analytic result ✓

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J^{\text{analytic}} &= \epsilon^{-4}\,[8.3400403922376 8 - 52.35987755984493i] && + \mathcal{O}\left(\epsilon^{-3}\right)
\end{aligned}$$

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CD: *struggled* to get $\mathcal{O}\left(\epsilon^{-3}\right)$ $\longrightarrow$ **NOCD**: **deeper** in ϵ

Outline

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Case-by-case (⚠ sub-optimal!!) treatment given here $\rightarrow$ **proof-of-concept**

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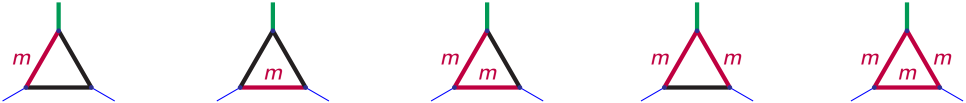

Figure: *Independent* equal-mass triangles with an off-shell leg

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- *Quadratic* dependence on x_i
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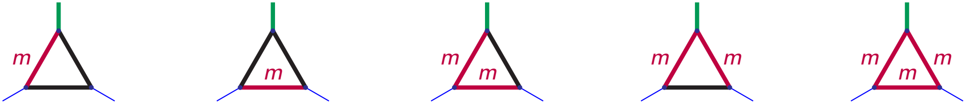


Figure: *Independent* equal-mass triangles with an off-shell leg

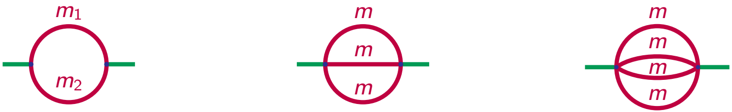
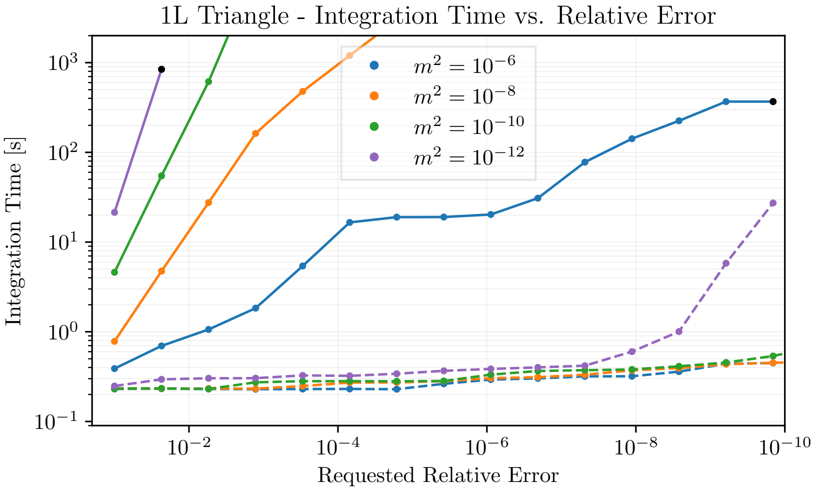
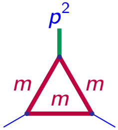


Figure: Banana-type integrals at 1, 2, and 3 loops

1-Loop Massive Triangle with One Off-Shell Leg

1-Loop Massive Triangle with One Off-Shell Leg





pySecDec

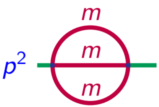
CD

NOCD

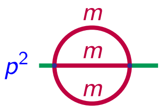
$p^2 = 1$

Massive Elliptic Sunrise

Massive Elliptic Sunrise

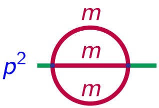


Massive Elliptic Sunrise



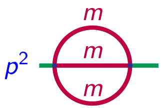
- Simplest *elliptic* loop integral

Massive Elliptic Sunrise



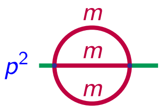
- Simplest *elliptic* loop integral
- Rational transformations
  $\hookrightarrow$ **Algebraic** transformations

Massive Elliptic Sunrise



- Simplest *elliptic* loop integral
- Rational transformations
  $\hookrightarrow$ **Algebraic** transformations
- $\mathcal{F} = -p^2 x_1 x_2 x_3$

Massive Elliptic Sunrise



- Simplest *elliptic* loop integral
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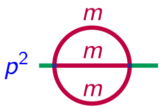
  $\hookrightarrow$ **Algebraic** transformations
- $\mathcal{F} = -p^2 x_1 x_2 x_3 + m^2 (x_1 + x_2 + x_3)(x_1 x_2 + x_1 x_3 + x_2 x_3)$

Massive Elliptic Sunrise



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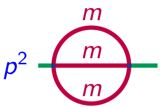
Massive Elliptic Sunrise



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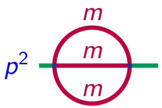
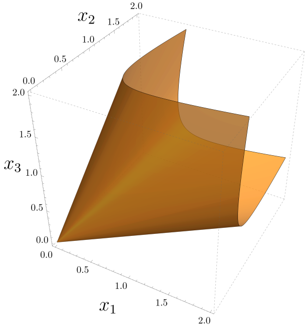
- $\widetilde{\mathcal{F}} = \frac{\mathcal{F}}{m^2} = (x_1 + x_2 + x_3)(x_1 x_2 + x_1 x_3 + x_2 x_3) - (9 + \beta^2) x_1 x_2 x_3$

Massive Elliptic Sunrise





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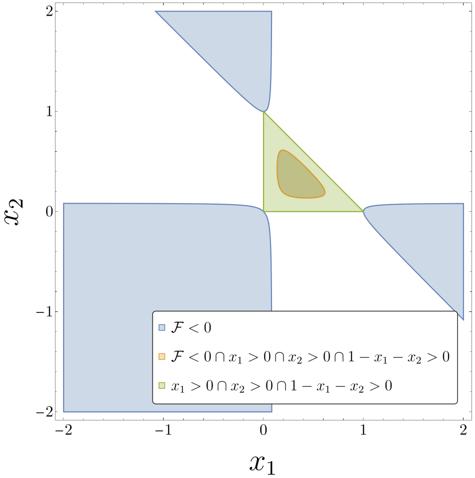
Reduce the dimension of the problem from 3 variables to 2 by integrating out x_3 using $\delta(1 - x_1 - x_2 - x_3)$ in the integrand

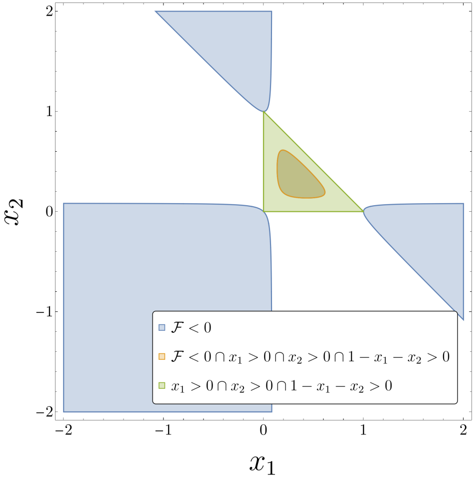
Massive Elliptic Sunrise

Massive Elliptic Sunrise



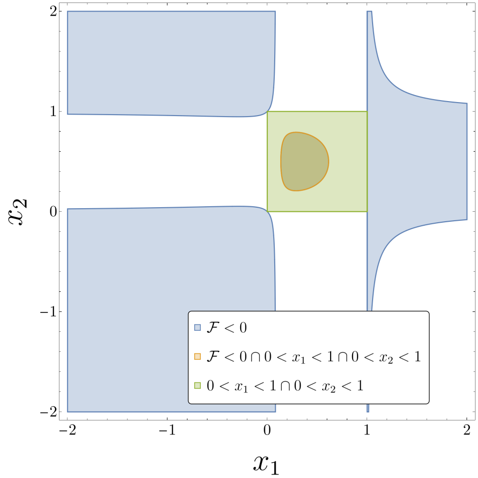
- Let's consider the variety of $\widetilde{\mathcal{F}}$

Massive Elliptic Sunrise



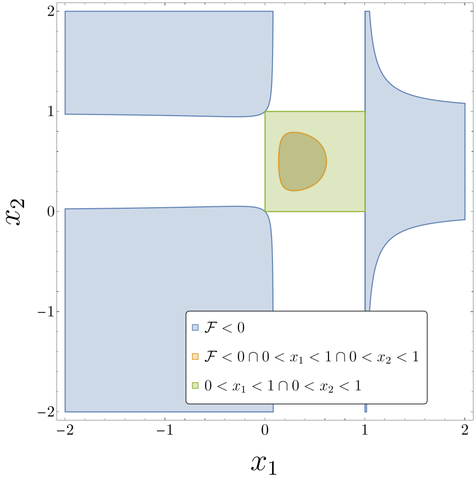
- Let's consider the variety of $\widetilde{\mathcal{F}}$
- Rescale integration domain to unit positive (hyper)quadrant

Massive Elliptic Sunrise



- Let's consider the variety of $\widetilde{\mathcal{F}}$
- Rescale integration domain to unit positive (hyper)quadrant
- *Entire* integrand is symmetric about $x_2 = \frac{1}{2}$
  $\Rightarrow$ let's use this!

Massive Elliptic Sunrise



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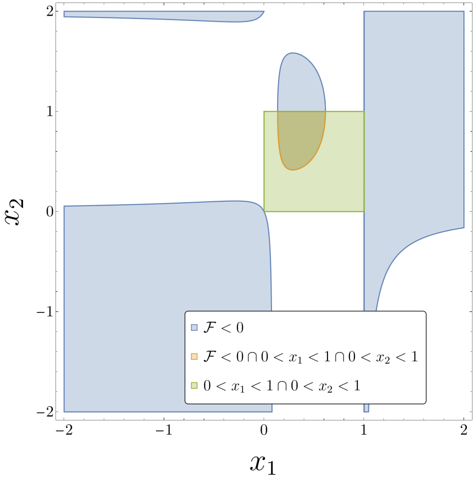

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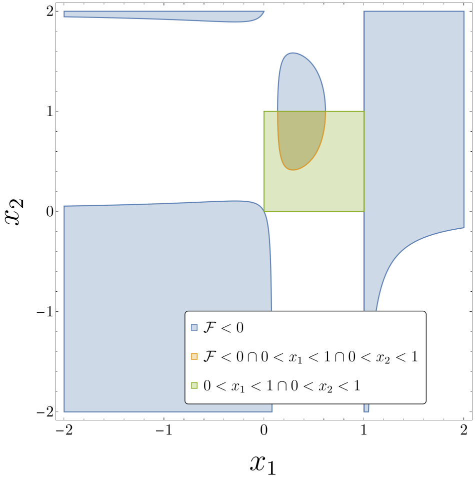
Massive Elliptic Sunrise

$$J/2 = J_1^+ + J_2^+ + J_3^+ + (-1 - i\delta)^{1-2\epsilon} J_1^-$$

Massive Elliptic Sunrise



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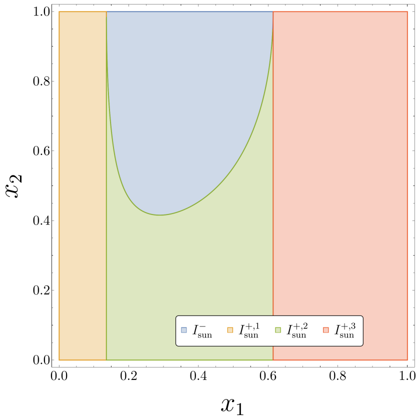
Massive Elliptic Sunrise

$$J/2 = J_1^+ + J_2^+ + J_3^+ + (-1 - i\delta)^{1-2\epsilon} J_1^-$$

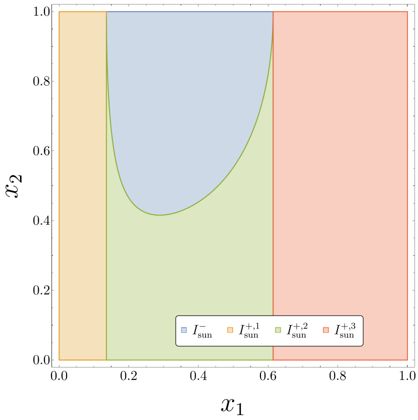
- Construct transformations which directly send the variety to the integration boundary

Massive Elliptic Sunrise

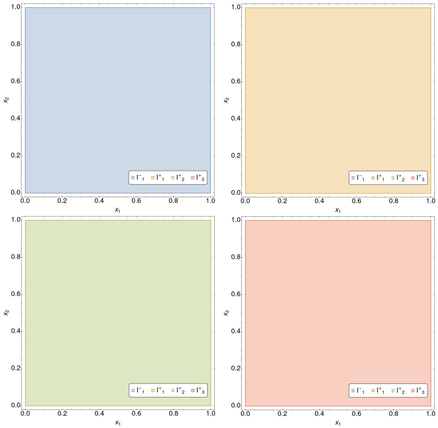
Massive Elliptic Sunrise



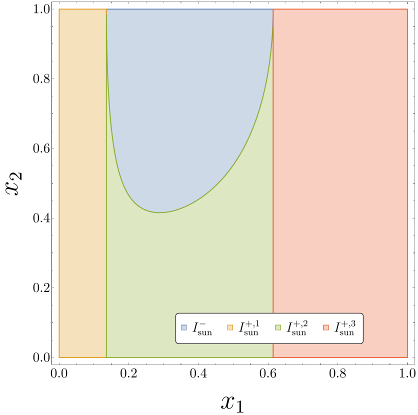
Massive Elliptic Sunrise



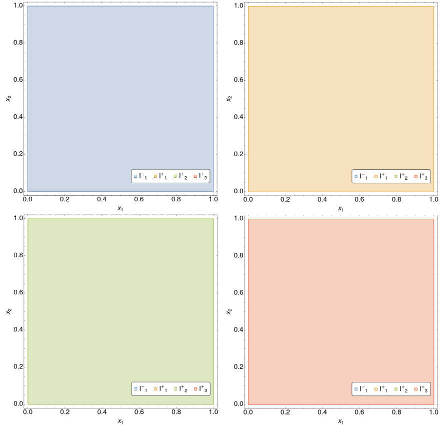
$\Rightarrow$



Massive Elliptic Sunrise



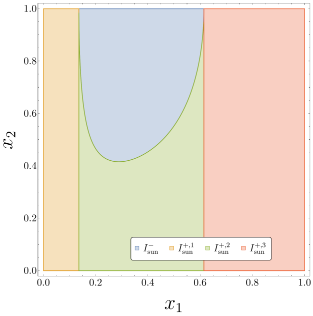
$\Rightarrow$



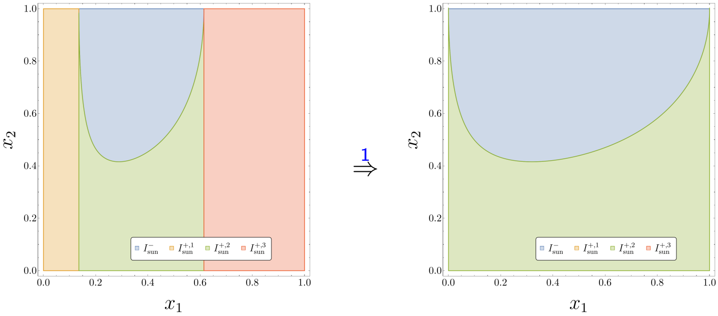
Let's consider the transformations for the negative contribution, I_1^-

Massive Elliptic Sunrise (Negative Contribution)

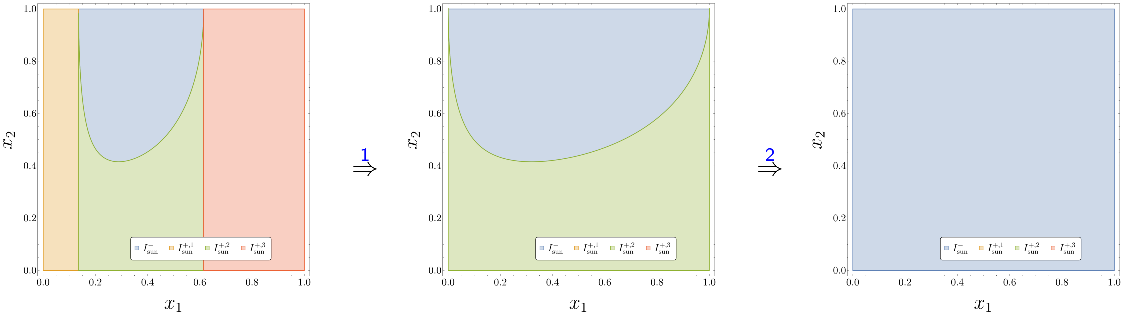
Massive Elliptic Sunrise (Negative Contribution)



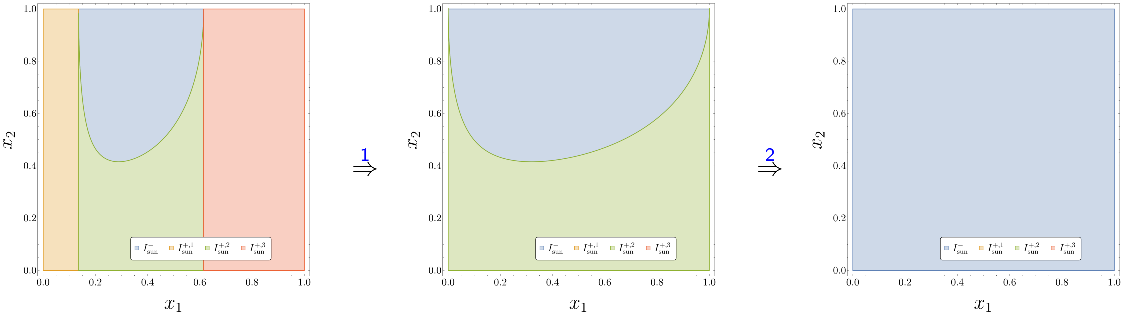
Massive Elliptic Sunrise (Negative Contribution)



Massive Elliptic Sunrise (Negative Contribution)

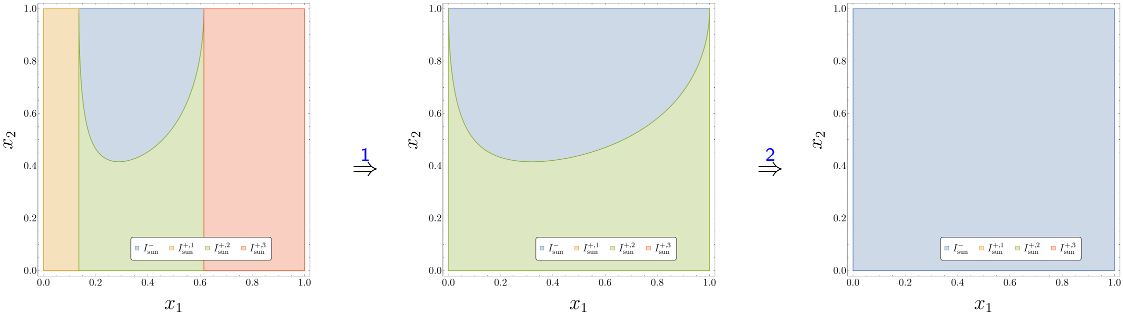


Massive Elliptic Sunrise (Negative Contribution)



1 Map sides of variety, $x_1^{\pm}(\beta)$, to $x_1 = 0$ and $x_1 = 1$

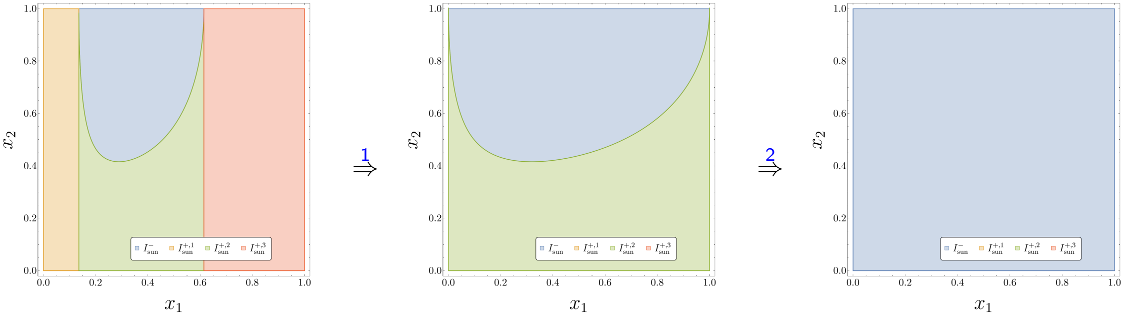
Massive Elliptic Sunrise (Negative Contribution)



1 Map sides of variety, $x_1^{\pm}(\beta)$, to $x_1 = 0$ and $x_1 = 1$

$$\hookrightarrow \left\{ x_1 \overset{!}{\to} x_1^{-}(\beta) + \left[x_1^{+}(\beta) - x_1^{-}(\beta) \right] x_1 , \quad x_2 \overset{!}{\to} x_2 \right\}$$

Massive Elliptic Sunrise (Negative Contribution)



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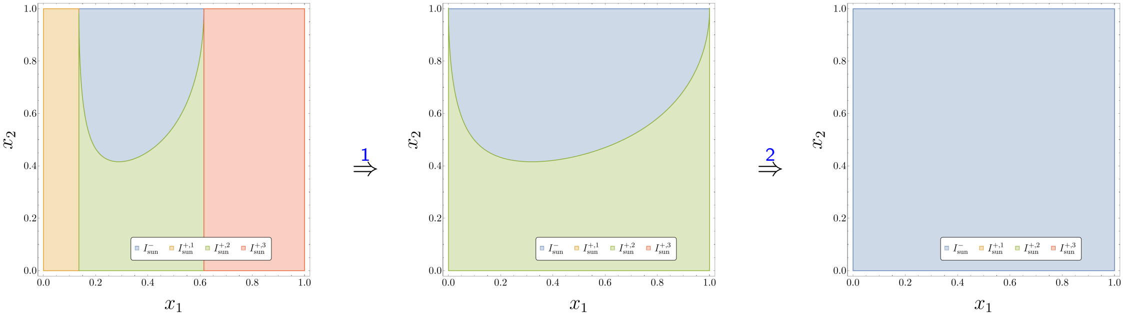
2. Solve variety for $x_2 = f(x_1)$ and map it to $x_2 = 0$ fixing $x_2 = 1$

Massive Elliptic Sunrise (Negative Contribution)



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2 Solve variety for $x_2 = f(x_1)$ and map it to $x_2 = 0$ fixing $x_2 = 1$

$$\hookrightarrow \left\{ x_1 \xrightarrow{!} x_1,\ \ x_2 \xrightarrow{!} f(x_1) + \left[1 - f(x_1)\right] x_2 \right\}$$

Massive 3-Loop Banana

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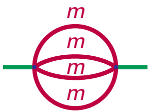

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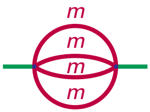

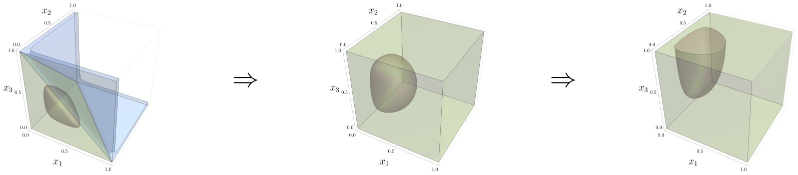

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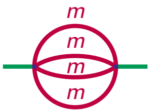
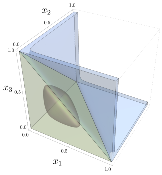
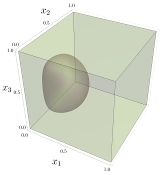




$\Rightarrow$



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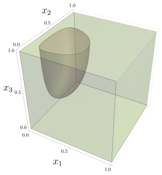
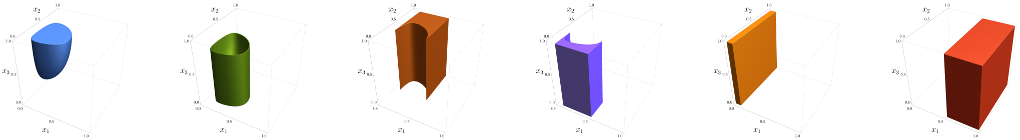
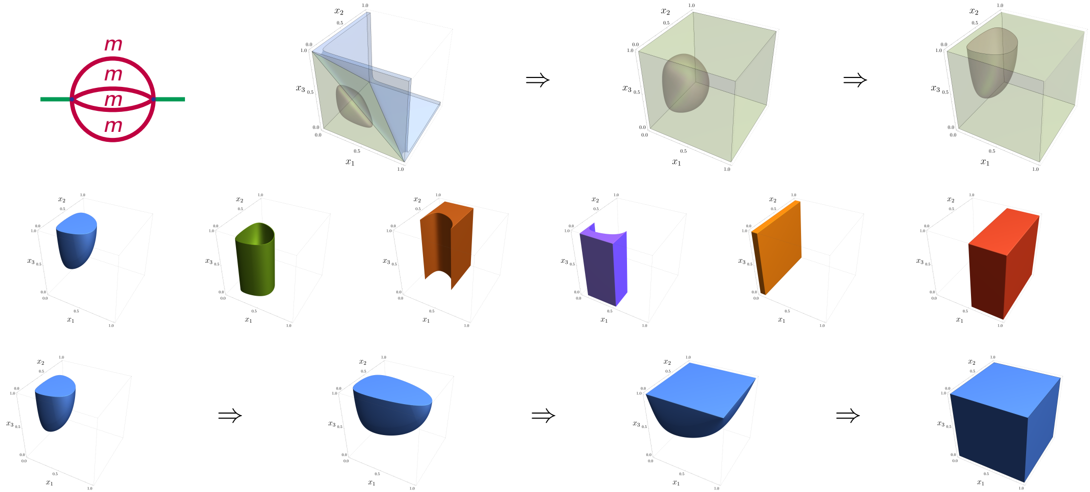




Massive 3-Loop Banana



Outline

Generic Cylindrical Algebraic Decomposition [see also Jones, Loop Summit 2025]

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Can we **always** perform this procedure?

Generic Cylindrical Algebraic Decomposition [see also Jones, Loop Summit 2025]

Can we **always** perform this procedure?

Sign-invariant decomposition of

$$\{\mathcal{F}(\mathbf{x}, \mathbf{s}) \gtrless 0\} \wedge \{\mathbf{x} > 0\} \wedge \mathbf{s}_R$$

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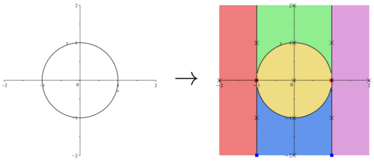
General solution: **GCAD**



Lee, del Río, Rahkooy 25

$$\begin{array}{ccc}
\mathcal{F}_n & & \mathrm{CAD}(\mathcal{F}_n) \\
\big\downarrow \text{proj} & & \big\uparrow \text{Lift} \\
\mathcal{F}_{n-1} & & \mathrm{CAD}(\mathcal{F}_{n-1}) \\
\big\downarrow \text{proj} & & \big\uparrow \text{Lift} \\
\vdots & & \vdots \\
\big\downarrow \text{proj} & & \big\uparrow \text{Lift} \\
\mathcal{F}_2 & & \mathrm{CAD}(\mathcal{F}_2) \\
\big\downarrow \text{proj} & & \big\uparrow \text{Lift} \\
\mathcal{F}_1 & \xrightarrow[\text{split into cells}]{\text{find roots}} & \mathrm{CAD}(\mathcal{F}_1)
\end{array}$$

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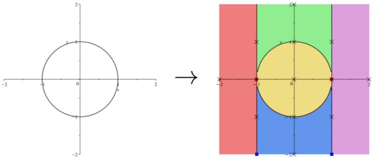
✓ PROS: constructive algorithm, guaranteed to work



Lee, del Río, Rahkooy 25

$$\begin{tikzcd}
\mathcal{F}_n \arrow[d, "\mathrm{proj}"] & \mathrm{CAD}(\mathcal{F}_n) \\
\mathcal{F}_{n-1} \arrow[d, "\mathrm{proj}"] & \mathrm{CAD}(\mathcal{F}_{n-1}) \arrow[u, "\mathrm{Lift}"'] \\
\vdots \arrow[d, "\mathrm{proj}"] & \vdots \arrow[u, "\mathrm{Lift}"'] \\
\mathcal{F}_2 \arrow[d, "\mathrm{proj}"] & \mathrm{CAD}(\mathcal{F}_2) \arrow[u, "\mathrm{Lift}"'] \\
\mathcal{F}_1 \arrow[r, "\text{find roots}", "\text{split into cells}"'] & \mathrm{CAD}(\mathcal{F}_1) \arrow[u, "\mathrm{Lift}"']
\end{tikzcd}$$

Generic Cylindrical Algebraic Decomposition [see also Jones, Loop Summit 2025]

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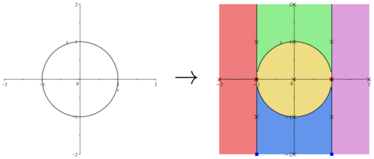

Lee, del Río, Rahkooy 25

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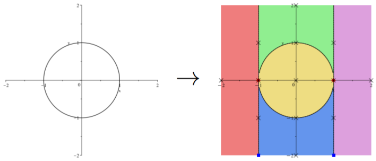

Lee, del Río, Rahkooy 25

General solution: **GCAD**

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Leverage physics insight? [WIP]:
$\mathcal{F}$ multilinear/quadratic in $\mathbf{x}$, homogeneous

$$\begin{tikzcd}
\mathcal{F}_n \arrow[d, "\text{proj}"] & \mathrm{CAD}(\mathcal{F}_n) \\
\mathcal{F}_{n-1} \arrow[d, "\text{proj}"] & \mathrm{CAD}(\mathcal{F}_{n-1}) \arrow[u, "\text{Lift}"] \\
\vdots \arrow[d, "\text{proj}"] & \vdots \arrow[u, "\text{Lift}"] \\
\mathcal{F}_2 \arrow[d, "\text{proj}"] & \mathrm{CAD}(\mathcal{F}_2) \arrow[u, "\text{Lift}"] \\
\mathcal{F}_1 \arrow[r, "\text{find roots}", "\text{split into cells}"'] & \mathrm{CAD}(\mathcal{F}_1) \arrow[u, "\text{Lift}"]
\end{tikzcd}$$

[PRELIMINARY] ggHH 2-Loop Box

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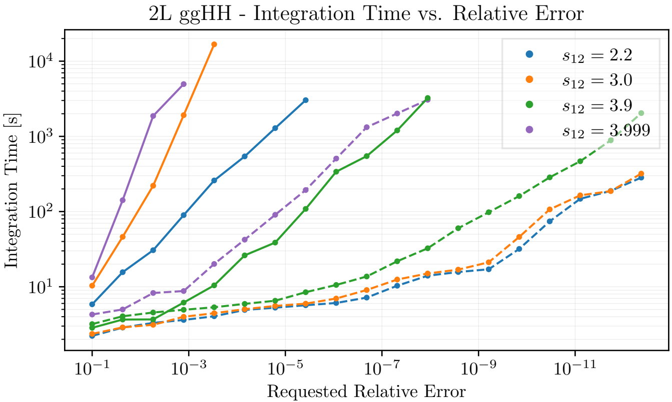

pySecDec

CD

NOCD

Outlook

Outlook

UB integrals $\rightarrow$ problem *solved*

Outlook

UB integrals $\longrightarrow$ problem *solved* $\longrightarrow$ implement in public codes

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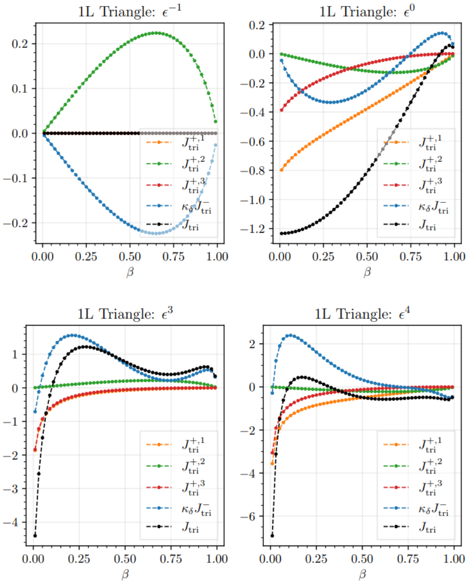
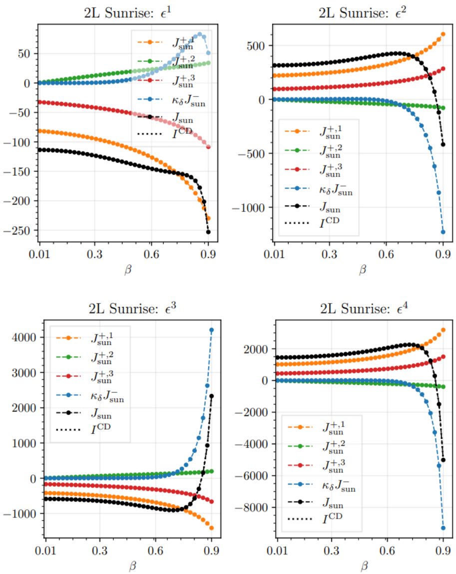
Thank you for listening!

Outline



1L Triangle: ϵ^{-1}
1L Triangle: ϵ^{0}
1L Triangle: ϵ^{3}
1L Triangle: ϵ^{4}
$J_{\rm tri}^{+,1}$
$J_{\rm tri}^{+,2}$
$J_{\rm tri}^{+,3}$
$\kappa_\delta J_{\rm tri}^{-}$
$J_{\rm tri}$
β


2L Sunrise: ϵ^1
$J_{\rm sun}^{+,1}$
$J_{\rm sun}^{+,2}$
$J_{\rm sun}^{+,3}$
$\kappa_\delta J_{\rm sun}^{-}$
$J_{\rm sun}$
$I^{\rm CD}$
50
0
−50
−100
−150
−200
−250
0.01
0.3
0.6
0.9
β
2L Sunrise: ϵ^2
500
0
−500
−1000
β
2L Sunrise: ϵ^3
4000
3000
2000
1000
0
−1000
β
2L Sunrise: ϵ^4
2000
0
−2000
−4000
−6000
−8000
β