Analytic approximations for double Higgs production at the LHC

TUM/MPP Collider Phenomenology Seminar (Munich), 2025

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Outline

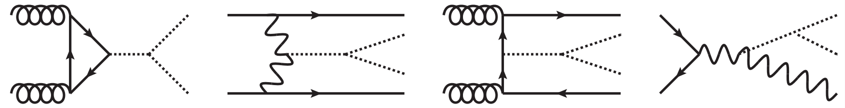
Introduction

Higgs Self Coupling

- Standard Model Higgs potential:

$$\boxed{V(H) = \frac{1}{2}m_H^2 H^2 + \lambda v H^3 + \frac{\lambda}{4}H^4}, \text{ where } \lambda = m_H^2/(2v^2) \approx 0.13.$$

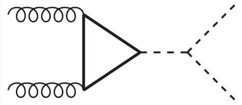
- Want to measure λ, to determine if $V(H)$ is consistent with nature.
  - Challenging! Cross-section $\approx 10^{-3} \times H$ prod.
  - $-1.24 < \lambda/\lambda_{SM} < 6.49$ [CMS '22] ; $-0.6 < \lambda/\lambda_{SM} < 6.6$ [Atlas '22]
- λ appears in various production channels:

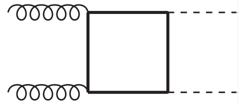


- Gluon fusion – dominant, 10x
- VBF
- $t\bar{t}$ associated production
- H-strahlung

Gluon Fusion

- Leading order (1 loop) partonic amplitude:





$$\boxed{\mathcal{M}^{\mu\nu} \sim \mathcal{A}_1^{\mu\nu}(\mathcal{F}_{tri} + \mathcal{F}_{box1}) + \mathcal{A}_2^{\mu\nu}(\mathcal{F}_{box2})}$$

  - $\mathcal{F}_{tri}$ contains the dependence on λ at LO
- Form factors:
  - LO: known exactly [Glover, van der Bij '88]
  - Beyond LO... no fully-exact (analytic) results to date
    - **QCD:** numerical evaluation, expansion in various kinematic limits
    - **EW:** heavy top expansion, high-energy expansion
      [Davies, Mishima, Schönwald, Steinhauser, Zhang '22]
    - see also Yuakwa corrections in (partial) HTL [Mühlleitner,Schlenk,Spira '22]
    - full (numerical) EW corrections [Bi, Huang, Huang, Ma, Yu '23]
    - numerical Yukawa- and Higgs self-coupling corrections
      [Heinrich, Jones, Kerner, Stone, Vestner '24]

$gg \to HH$ Beyond LO

NLO QCD:

- large-m_t [Dawson, Dittmaier, Spira '98] [Grigo, Hoff, Melnikov, Steinhauser '13]
- numeric [Borowka, Greiner, Heinrich, Jones, Kerner, Schlenk, Schubert, Zirke '16] [Baglio, Campanario, Glaus, Mühlleitner, Spira, Streicher '19]
- large-m_t + threshold exp. Padé [Gröber, Maier, Rauh '17]
- high-energy expansion [Davies, Mishima, Steinhauser, Wellmann '18,'19]
- small-p_T expansion [Bonciani, Degrassi, Giardino, Gröber '18]
  + high-energy expansion [Bagnaschi, Degrassi, Gröber '23]

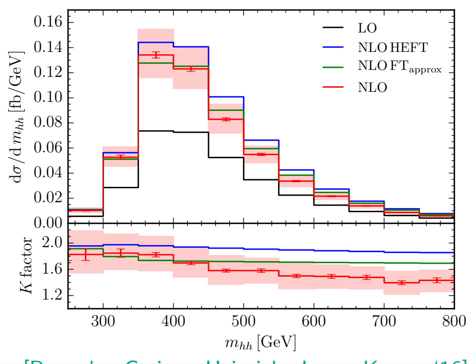
NNLO QCD:

- large-m_t virtuals [de Florian, Mazzitelli '13] [Grigo, Hoff, Steinhauser '15, Davies; Steinhauser '19]
- HTL+numeric real ("FTapprox") [Grazzini, Heinrich, Jones, Kallweit, Kerner, Lindert, Mazzitelli '18]
- large-m_t reals [Davies, Herren, Mishima, Steinhauser '19 '21]
- light fermion corrections at $p_T = 0$ [Davies, Schönwald, Steinhauser '23]

N3LO QCD:

- Wilson coefficient C_{HH} [Spira '16; Gerlach, Herren, Steinhauser '18]
- HTL [Chen, Li, Shao, Wang '19]

$gg \to HH$ Beyond LO



[Borowka, Greiner, Heinrich, Jones, Kerner '16]

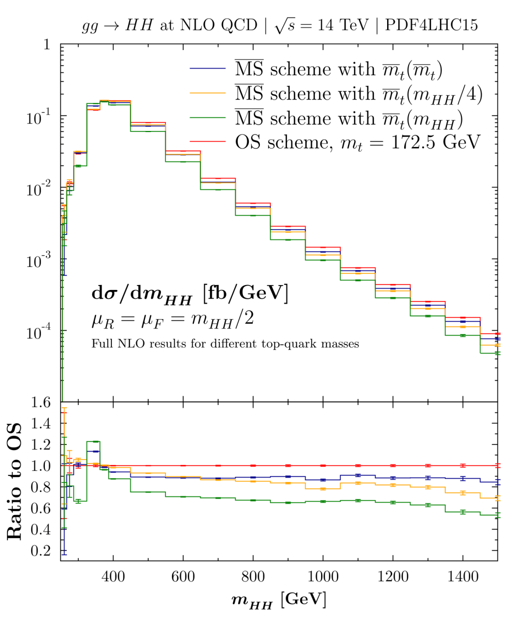
Total cross section (14TeV):

	σ_{LO}	σ_{NLO}	σ_{NNLO}
B-i HTL	–	$38.32^{+18.1\%}_{-14.9\%}$	$39.58^{+1.4\%}_{-4.7\%}$
FTapprox	–	$34.25^{+14.7\%}_{-13.2\%}$	$36.69^{+2.1\%}_{-4.9\%}$
Full	$19.85^{+27.6\%}_{-20.5\%}$	$32.88^{+13.5\%}_{-12.5\%}$	–

- Large uncertainties due to the m_t renormalization scheme.
- Can be reliably reduced with an NNLO calculation.
- Recent progress through resummation in the high-energy region.

[Jaskiewicz, Jones, Szafron, Ulrich '25]

More details in the talk by Stephen Jones.

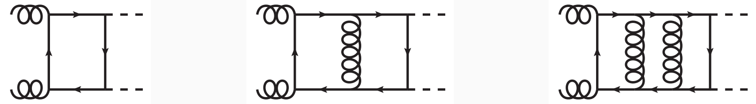


[Baglio, Campanario, Glaus, Mühlleitner, Ronca, Spira, Streicher '20]

QCD corrections to $gg \to HH$

QCD Corrections

Example diagrams at LO, NLO, NNLO:



Diagrams depend on ϵ, s, t, m_t, m_H:

- analytic result is very involved
- simplify by expanding in certain kinematic limits

We will consider:

- high-energy expansion: description for larger p_T vales $\quad s, |t| > m_t^2 > m_H^2$
- small-t expansion: description for smaller p_T values $\quad s, m_t^2 > |t|, m_H^2$

$\rightarrow$ The two expansions will cover the whole physically interesting phase space.

High-Energy Expansion

Seek an expansion where $s, |t| > m_t^2 > m_H^2$ [Davies, Mishima, Steinhauser, Wellmann '18-'19]

1. Form factors in terms of scalar Feynman integrals: $I(m_H^2, m_t^2, s, t, \epsilon)$
2. Taylor expand for $m_H^2 \to 0$ (with `LiteRed`): [Lee '14]

$$\boxed{I(m_H^2, m_t^2, s, t, \epsilon) = I(0, m_t^2, s, t, \epsilon) + m_H^2 I'(0, m_t^2, s, t, \epsilon) + \ldots}$$

3. IBP reduce to master integrals: $J(0, m_t^2 s, t, \epsilon)$ (`FIRE`, `Kira`) [Smirnov '15] [Klappert, Lange, Maierhöfer, Usovitsch '21]
4. Determine MIs as an expansion around $m_t \to 0$:

$$\boxed{J(0, m_t^2, s, t, \epsilon) = \sum_{i,j,k} C_{ijk}(s,t)\epsilon^i (m_t^2)^j \log(m_t^2)^k}$$

- Insert ansatz into differential equation $\to$ linear equations for c_{ijk}.
- Compute boundary conditions with expansion-by-regions.

High-Energy Expansion – Calculation of boundary conditions

- We start with the Schwinger parametrization of the integrals:

$$\boxed{I = \int_0^\infty \left(\prod_{i=1}^n d\alpha_i \frac{\alpha_i^{\delta_i}}{\Gamma(1+\delta_1)} \right) \mathcal{U}^{-d/2} e^{-\mathcal{F}/\mathcal{U}} \ ,}$$

  with the Symanzik polynomials $\mathcal{U}$ and $\mathcal{F}$.

- We use expansion-by-regions and reveal the different regions with `asy.m` [Pak, Smirnov '11] .
- In the high-energy limit: $\boxed{s, |t| \sim 1 \ , \ m_t^2 \sim \xi}$, we find 13 regions.
  - One hard region, where the master integrals are known. [Smirnov, Veretin '00; Bern, Sixon, Smirnov '05]
  - 12 'soft' regions, where the α parameters scale differently in ξ.
- We calculate the expansion using Mellin-Barnes techniques.

- E.g. we find:

$$\boxed{I_3 = m_t^{-4\epsilon+2} \int \frac{dz_1}{2\pi i} \frac{\Gamma\left[-z_1, z_1-\epsilon+2, -z_1+\epsilon-1, z_1+1, z_1+1, z_1+\epsilon\right]}{\Gamma\left[2-\epsilon, 2z_1+2\right]}}$$

- We use `MB` [Czakon '05] for the analytic continuation in ϵ:

$$I_3 = m_t^{-4\epsilon+2} e^{-2\epsilon\gamma_E} \left(-\frac{3}{2\epsilon^2} - \frac{9}{2\epsilon} - \frac{21}{2} - \frac{5\pi^2}{12} + I^{(MB)} + \mathcal{O}(\epsilon) \right)$$

with the remaining integral

$$I^{(MB)} = \int_{-1/7-i\infty}^{-1/7+i\infty} \frac{dz_1}{2\pi i} \frac{\Gamma\left[-z_1-1, -z_1, z_1, z_1+1, z_1+1, z_1+2\right]}{\Gamma\left[2z_1+2\right]}$$

- We can close the contour to the right and sim the residues at $\boxed{z_1 = 0, 1, 2, \ldots}$:

$$I^{(MB)} = \int\limits_{-1/7-i\infty}^{-1/7+i\infty} \frac{dz_1}{2\pi i} \frac{\Gamma\left[-z_1-1, -z_1, z_1, z_1+1, z_1+1, z_1+2\right]}{\Gamma\left[2z_1+2\right]}$$

$$= 4 + \frac{\pi^2}{6} + 2\sum_{k=0}^{\infty} \binom{2k+1}{k}^{-1} \frac{(4k^2+8k+3)\left[S_1(k) - S_1(2k)\right] - 4(k+1)}{(2k+1)(2k+2)(2k+3)^3}$$

- Summation over residues can be done analytically with `HarmonicSums` [Ablinger et al. '10-] , `Sigma` and `EvaluateMultiSums` [Schneider et al. '07-] .
- The (inverse) binomial sums we encounter sum to generalized iterated integrals and special constants, e.g.:

$$\sum_{k=0}^{\infty} x^k \binom{2k+1}{k}^{-1} \frac{1}{3+2k} = \frac{2}{x\sqrt{(4-x)x}} \int\limits_0^x dt \sqrt{(4-t)t} - 1 \overset{x \to 1}{=} \frac{4\pi}{3\sqrt{3}} - 2$$

Calculation of boundary conditions – Example

- Non-planar master integrals also have regions which scale in odd powers of m_t.
- Here elliptic constants appear:

$$
\begin{aligned}
c_Z &= \int_0^\infty \int_0^\infty \frac{\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2}{\sqrt{\alpha_1\,\alpha_2\,(\alpha_1+\alpha_2+1)\,(\alpha_2\alpha_1+\alpha_1+\alpha_2)}} \\
&= \sum_{k=0}^{\infty} \frac{\Gamma^4(k+1/2)}{\pi\Gamma^2(k+1)\Gamma(2k+1)} \Big(8\log(2)+6S_{-1}(2k)\Big) \\
&= 4\sqrt{3}\,K^2\Big(\frac{1}{2}-\frac{\sqrt{3}}{4}\Big) \\
&= 17.695031908454309764234228747255048751062059438637\ldots
\end{aligned}
$$

- We were able to determine these constants utilizing `HarmonicSums` and the `PSLQ` algorithm.

Calculation of boundary conditions – Results

- We obtain analytic results for 161 (QCD), 168(28) (electroweak) master integrals.
- The final result can be expressed via harmonic polylogarithms [Remiddi, Vermaseren '99] .

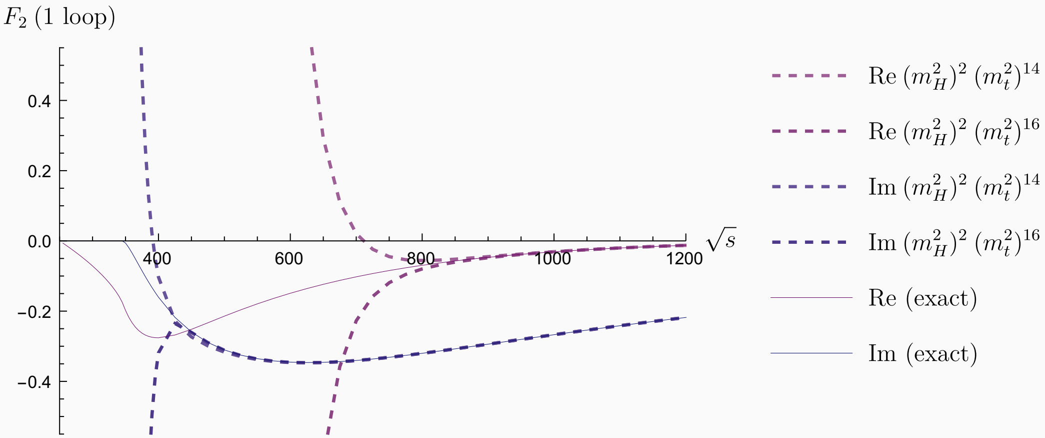
$$
\boxed{\begin{gathered} H_0(-t/s), H_1(-t/s), H_{0,1}(-t/s), H_{0,0,1}(-t/s), \\ H_{0,1,1}(-t/s), H_{0,0,0,1}(-t/s), H_{0,0,1,1}(-t/s), H_{0,1,1,1}(-t/s) \end{gathered}}
$$

and transcendental numbers

$$
\pi,\ \ln(3),\ \sqrt{3},\ \zeta_2,\ \zeta_3,\ \psi^{(1)}(1/3),\ \mathrm{Im}(\mathrm{Li}_3(i/\sqrt{3})),\ K^2\left(\frac{1}{2}-\frac{\sqrt{3}}{4}\right),\ E^2\left(\frac{1}{2}-\frac{\sqrt{3}}{4}\right)\ .
$$

- All master integrals are computed up to $\mathcal{O}(m_t^{120})$.

High-Energy Expansion: LO comparison



High-Energy Expansion: Padé approximants

The expansion diverges for $\sqrt{s} \lesssim 750\,\text{GeV}$.

The convergence can be improved by making use of Padé approximants:

- Approximate a function using a rational polynomial:

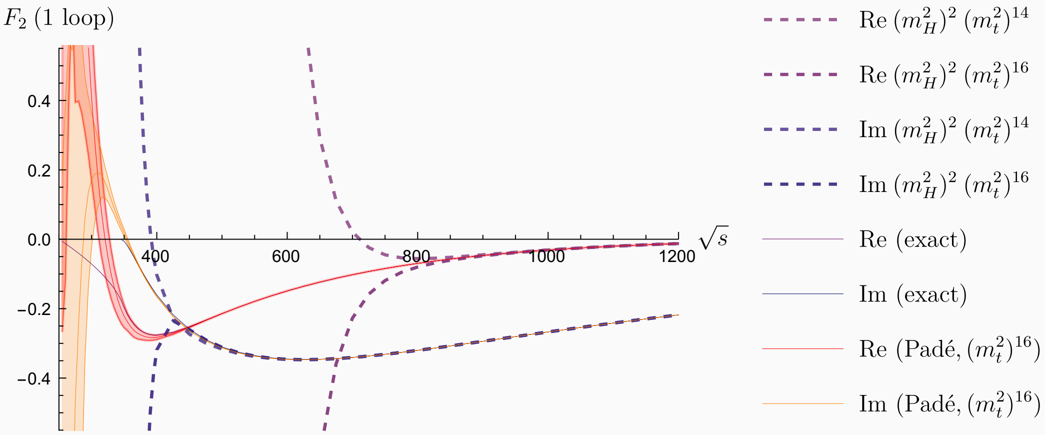
$$\boxed{f(x) \approx [n/m](x) = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n}{1 + b_1 x + b_2 x^2 + \cdots + b_m x^m}\,,}$$

  where the coefficients a_i, b_j are fixed by the series expansion of $f(x)$.

Compute a set of approximants (various choices of n, m):

- combine to give a central value and error estimate
- deeper expansions $\Rightarrow$ larger $n+m$ $\Rightarrow$ smaller error
- expansions to m_t^{120} allows for very high-order approximants

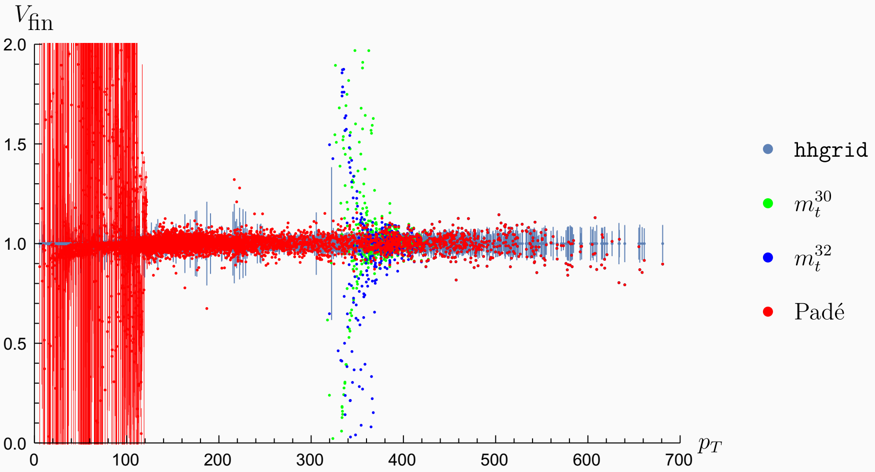
High-Energy Expansion: Padé approximants



High-Energy Expansion: V_{fin}

Comparison with `hhgrid`: [https://github.com/mppmu/hhgrid]

- interpolation grid of 6320 points evaluated numerically by `pySecDec`
- grid points normalized to `hhgrid` as function of p_T:

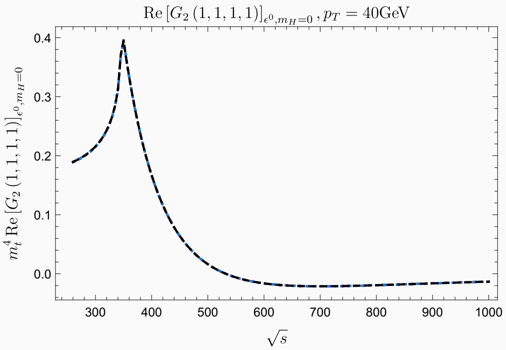


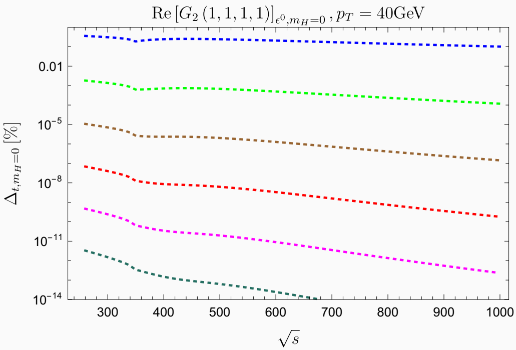
Small-t Expansion

As for high-energy expansion, first expand around $m_H \to 0$.

Then two possible (and finally equivalent) approaches:

1. Take the IBP-reduced amplitude of the high-energy expansion:
   - expand the master integrals around $t \to 0$ instead of $m_t \to 0$
2. Expand the unreduced amplitude around $q_3 \to -q_1$ $(t \to 0)$:
   - IBP reduce to new master integrals which only depend on ϵ, s, m_t
   - this approach can be applied at NNLO, but only to restricted expansion depth





Small-t Expansion: Evaluation of the MIs

"Semi-analytic" determination of the $t \to 0$ MIs: [Fael, Lange, Schönwald, Steinhauser '21]

1. Establish system of DEs for the MIs, w.r.t. $\hat{s} = s/m_t^2$.
2. Expand around $\hat{s} = 0$:
   - insert ansatz into DE: $J(\epsilon, \hat{s} = 0) = \sum_{i,j} c_{ijk}\, \epsilon^i\, \hat{s}^j \ln^k(\hat{s})$
   - determine minimal set of c_{ijk} (`Kira`+`FireFly`)
   - evaluate minimal boundary constants analytically (in the large-mass expansion)
3. Expand around a new point $\hat{s} = \hat{s}_0$ (repeat the above, modify ansatz).
4. Match the expansions (numerically) at a point where they both converge.

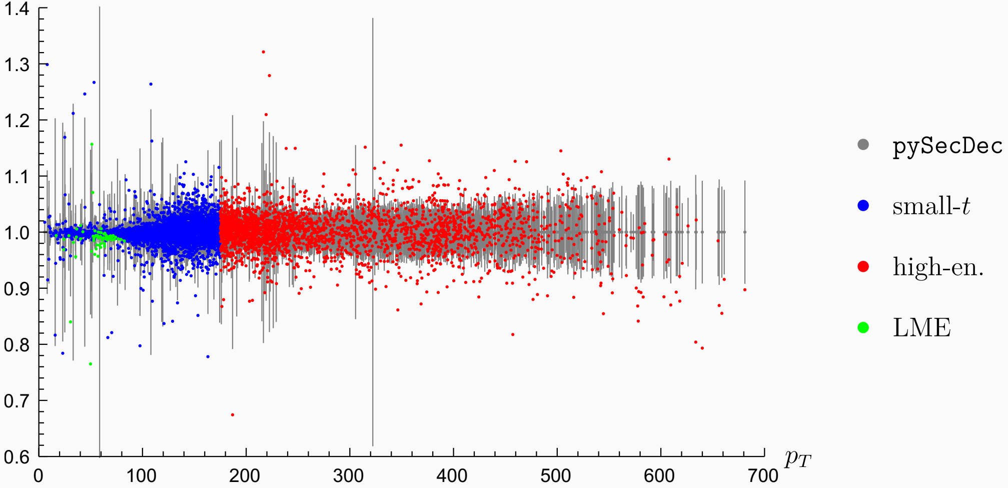
Here we have such "semi-analytic" expansions for the MIs at:

$$\hat{s} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 16, 20, 25, 30, 40, 50, \infty\}$$

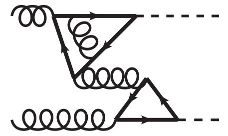
HE and $t \to 0$ Combination: "V_{fin}"

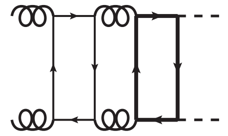
Comparison with `hhgrid`: [[https://github.com/mppmu/hhgrid]]

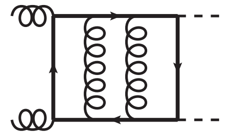
- merge both results, switch at $p_T = 175$ GeV.

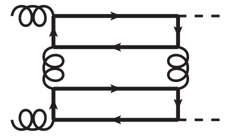


Split the amplitude into parts:









1PR

expand m_H,
rest exact

"$(gg \to H)^2$" w/
off-shell gluon

[Davies, Schönwald, Steinhauser, Vitti '24]

$n_l n_h \{C_A, C_F\}$

expand m_H,
small-t exp.

[Davies, Schönwald, Steinhauser '23]

$n_h \{C_A^2, C_A C_F, C_F^2\}$

expand m_H,
small-t exp.

N_c^2: [Davies, Schönwald, Steinhauser '25]

full result in progress

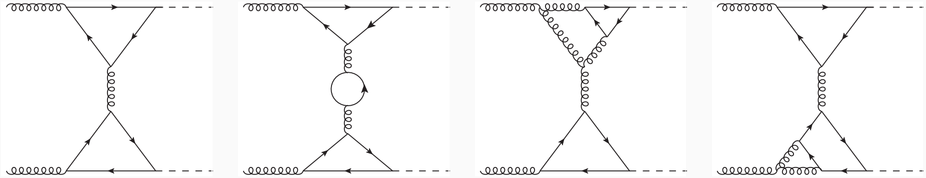
$n_h^2 \{C_A, C_F\}$

expand m_H,
small-t exp. (!)

in progress

massless t-channel cut

$gg \to HH$ at NNLO QCD: reducible contributions

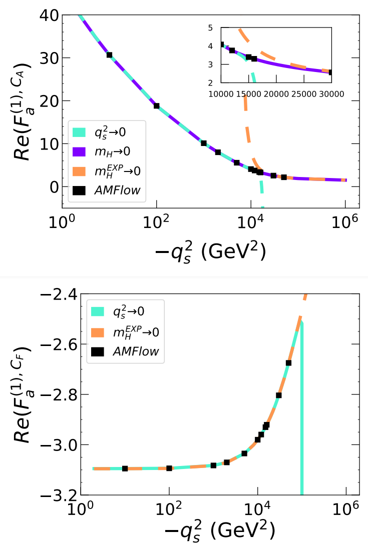


- Need to compute the of-shell $g(q_g)g^*(q_s)H(q_H)$ vertex up to 2 loops.
- Perform asymptotic expansions in:
  1. $m_H^2 \ll q_s^2, m_t^2$: hard region reduces to the same master integrals as the $t \to 0$ expansion
  2. $q_s^2 \ll m_H^2, m_t^2$: new analytic solutions for 2-loop master integrals in terms of HPLs

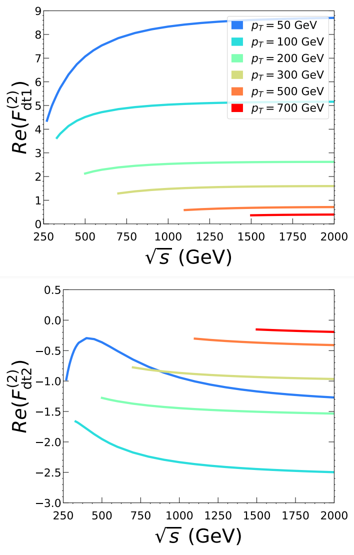
$\Rightarrow$ cover the whole phase space for $\{s, t; m_t, m_H\}$

$gg \to HH$ at NNLO QCD: reducible contributions

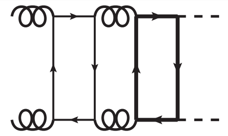
gg^*H form factor



$gg \to HH$ form factor



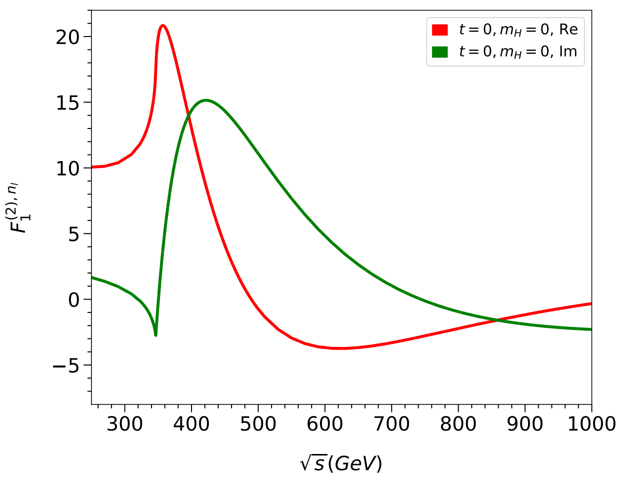
$gg \to HH$ at NNLO QCD: n_l part



$n_l n_h \{C_A, C_F\}$, leading expansion term $(m_H^0 t^0)$: [Davies, Schönwald, Steinhauser '23]

1. Expand $m_H \to 0$, $q_3 \to -q_1$ (FORM) [Ruijl, Ueda, Vermaseren '17]
2. Partial fraction decomposition (tapir, LIMIT) [Gerlach,Herren,Lang '23; Herren '20]
3. 60 integral topologies. 28 after common (sub-)sector identification
   - LiteRed, Feynson [Lee '14; Magerya '22]]
4. IBP (Kira) 85K→**176 MIs** (symm by Kira) [Klappert, Lange, Maierhöfer, Usovitsch '21]
   - (to compute $m_H^1 t^0 + m_H^0 t^1$: 4.5M integrals...)
5. Compute MIs with "expand and match".

$gg \to HH$ at NNLO QCD: n_l part



Sharp variation around $\sqrt{s} = 2m_t$ threshold:

- Leading behaviour $v \log^2 v$, cf. $v \log v$ at NLO ($v = \sqrt{1 - 4m_t^2/s}$).

F_2 **vanishes at** $m_H^0 t^0$**.**

$gg \to HH$ at NNLO QCD: n_h part

$n_h\{C_A^2, C_A C_F, C_F^2\}$, leading expansion term $(m_H^0 t^0)$:

1. Expand $m_H \to 0$, $q_3 \to -q_1$ (`FORM`)
2. Partial fraction decomposition (`LIMIT`)
3. 522 integral topologies. 203 after common (sub-)sector identification
   - `Feynson` (`LiteRed` is much too slow) [Magerya '22]
4. IBP (`Kira`) 2.6M→**33K MIs** across all topologies
   - Total: 330 days (16 core jobs)
   - Hardest single topology: 41 days, >2TB mem. Took several attempts:
     - master integral basis improvement, using `ImproveMasters.m`
     - change of momentum routings for smaller IBP relations

Cannot reduce master integrals between topologies with `Kira`:

- Symmetry finding and equation generation for each topology too slow.

$gg \to HH$ at NNLO QCD: n_h part, MI basis reduction

First step:

- Apply `FIRE`'s `FindRules` to MI list: 33K→**4313** [Smirnov, Chuharev '20]

Next:

- Apply `FindRules` to the 2.6M input integrals: 1.3M pairs
- Apply IBP tables to the pairs: 820K equations involving 4029 MIs
- Solve with `Kira`'s `user_defined_system`: 4313→**1647**

The basis is still not minimal.

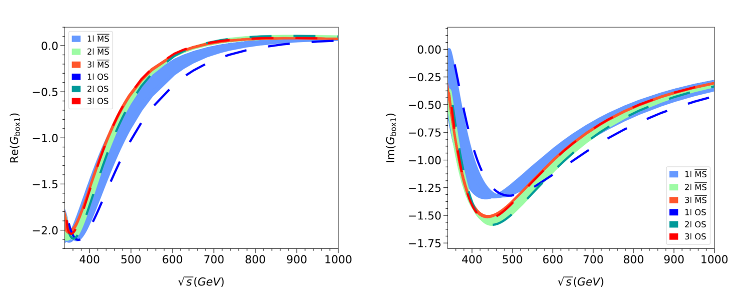
`FIRE` test reduction for all topologies (to a different basis):

- Repeat the above steps: 35K→1817→**1561** [**783** (leading N_c)]
    - Now the differential equations look better, and we can try to solve it.
    - (Probably, the basis is still not minimal)

$gg \to HH$ at NNLO QCD: top mass scheme

Use all available NNLO results to estimate top mass scheme uncertainties:

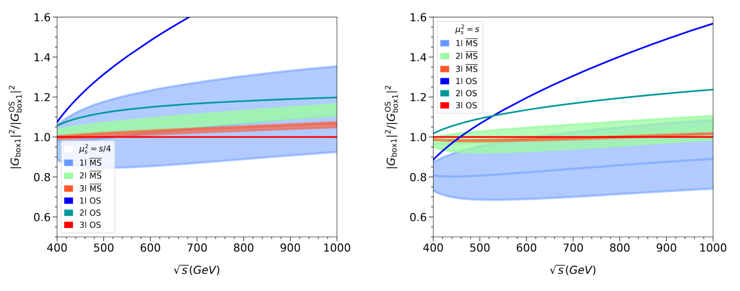
- Scheme dependence of the form factor F_1.
- $\overline{\text{MS}}$: envelope of $\mu_t^2 \in [s/16, s]$



$gg \to HH$ at NNLO QCD: top mass scheme

Use all available NNLO results to estimate top mass scheme uncertainties:

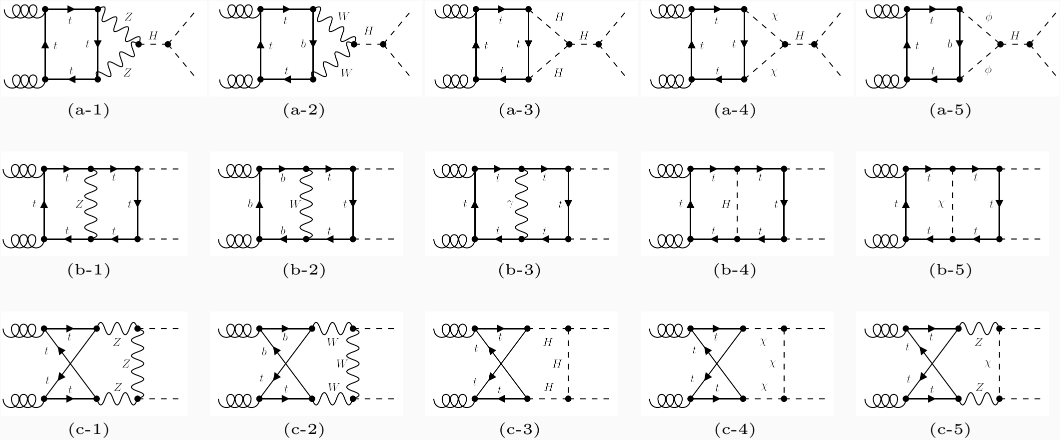
- Scheme dependence of the differential cross-section.
- $\overline{\text{MS}}$: envelope of $\mu_t^2 \in [s/16, s]$



Electroweak corrections to $gg \to HH$

Full Electroweak Corrections in the Large-m_t Expansion

- Sample Feynman diagrams involving:
  - SM fields: $\{t,\ b,\ H,\ \gamma,\ Z,\ W^{\pm},\ \chi,\ \phi^{\pm}\}$
  - ghosts: $\{u^{\gamma},\ u^{Z},\ u^{\pm}\}$



(a-1) (a-2) (a-3) (a-4) (a-5)

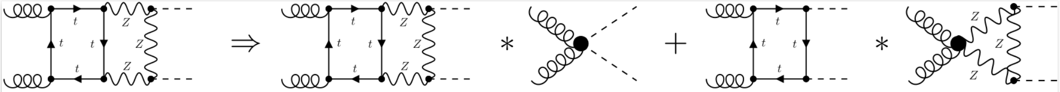
(b-1) (b-2) (b-3) (b-4) (b-5)

(c-1) (c-2) (c-3) (c-4) (c-5)

Goal: obtain analytic expressions in the large-m_t expansion

Large-m_t Expansion and Renormalization

- Expand and calculate in general R_ξ gauge with `qgraf` [Nogueira '93] , `tapir` [Gerlach, Herren, Lang '23] , `q2e&exp` [Harlander, Seidensticker, Steinhauser '97-'99] , `form` [Ruijl, Ueda, Vermaseren '17] , `LiteRed` [Lee '12] and `MATAD` [Steinhauser '01] .
- Expansion hierarchy: $m_t^2 \gg \xi_W m_W^2, \xi_Z m_Z^2 \gg s, t, m_H^2, m_W^2, m_Z^2$

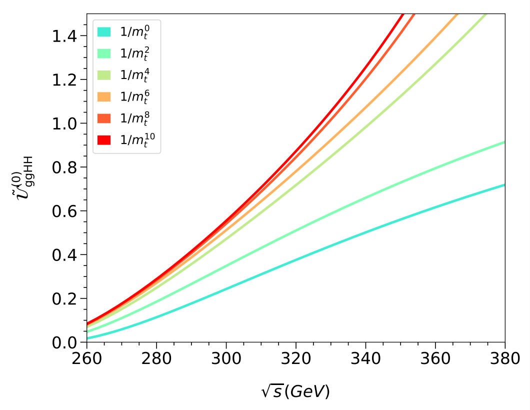


- We renormalize the input parameters $\{e,\ m_W,\ m_Z,\ m_t,\ m_H\}$ and the Higgs wave function on-shell and transform to the G_μ scheme.
  - $\xi_W,\ \xi_Z, \mu^2$ cancel analytically

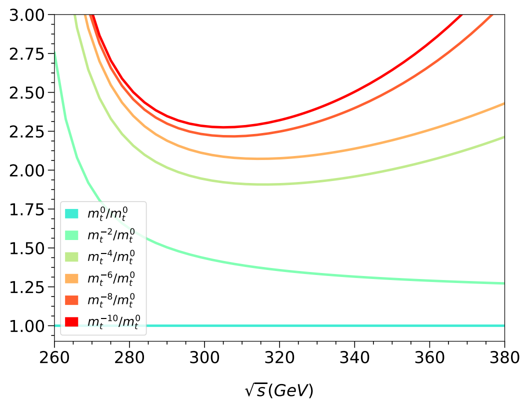
LO Matrix Elements for $gg \to HH$

$$\mathcal{M} = \frac{1}{8^2 2^2} \sum_{\text{col}} \sum_{\text{pol}} |\mathcal{A}|^2 = \frac{1}{16} \left(X_0^{ggHH} \right)^2 \tilde{U}_{ggHH}$$

$$\tilde{U}_{ggHH} = \tilde{U}^{(0)}_{ggHH} + \frac{\alpha}{\pi} \tilde{U}^{(0,1)}_{ggHH}$$



$\tilde{U}_{ggHH}$ up to different expansion orders in $1/m_t$.



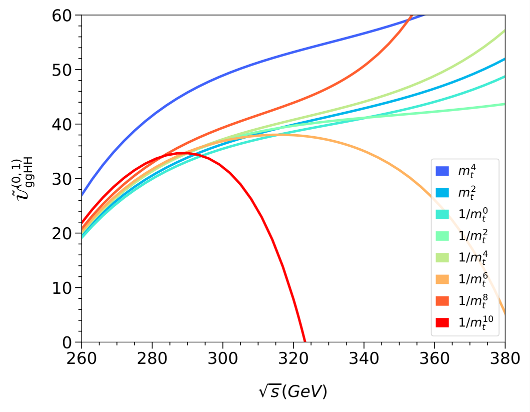
Different expansion orders normalized to m_t^0.

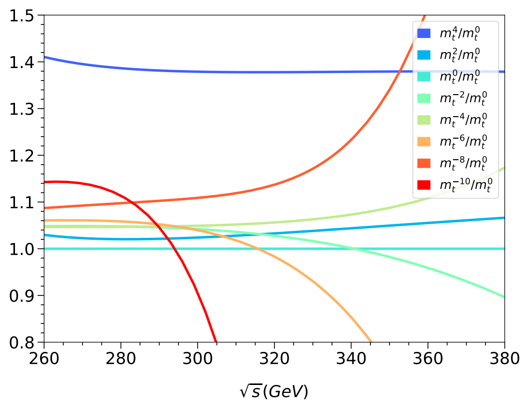
We see a nice convergence up to roughly $\sqrt{s} = 2m_t \approx 350$,GeV.

NLO Electroweak Matrix Elements for $gg \to HH$

$$\mathcal{M} = \frac{1}{8^2 2^2} \sum_{\text{col}} \sum_{\text{pol}} |\mathcal{A}|^2 = \frac{1}{16} \left(X_0^{ggHH} \right)^2 \tilde{U}_{ggHH}$$

$$\tilde{U}_{ggHH} = \tilde{U}^{(0)}_{ggHH} + \frac{\alpha}{\pi} \tilde{U}^{(0,1)}_{ggHH}$$





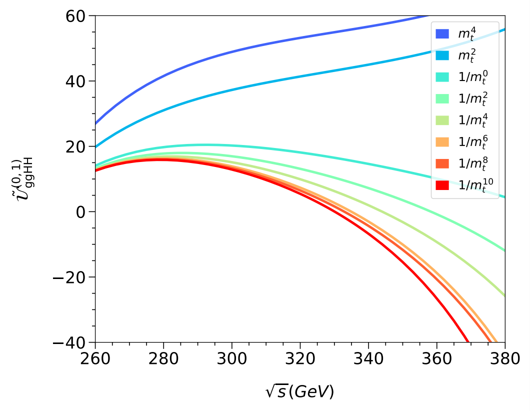
$\tilde{U}_{ggHH}$ up to different expansion orders in $1/m_t$.

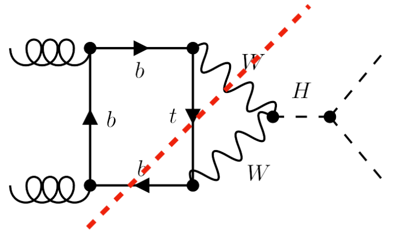
Different expansion orders normalized to m_t^0.

We do not see such a nice convergence at NLO.

NLO Electroweak Matrix Elements for $gg \to HH$

$$\mathcal{M} = \frac{1}{8^2 2^2} \sum_{\text{col}} \sum_{\text{pol}} |\mathcal{A}|^2 = \frac{1}{16} \left(X_0^{ggHH} \right)^2 \tilde{U}_{ggHH}$$





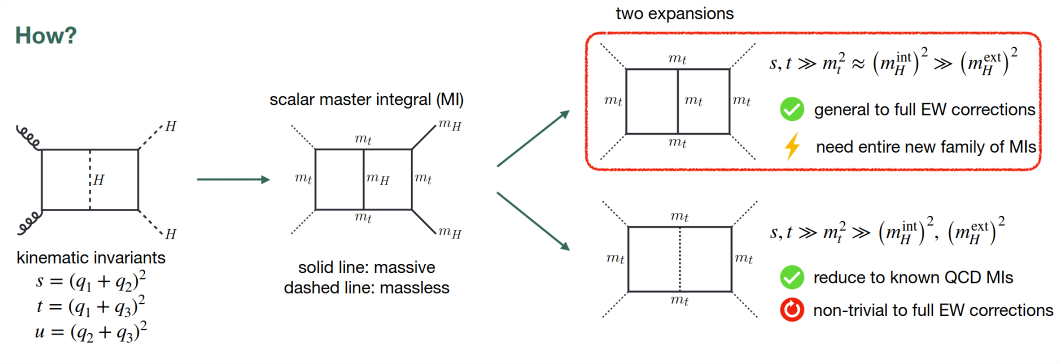
Cut through W-t-b affects convergence of the large-m_t expansion:

$$m_t + m_b + m_W \approx 250\,\text{GeV}$$

We can restore convergence by excluding diagrams with W-t-b cuts.

Beyond the Large-m_t Expansion – High-Energy Expansion

- Start with diagrams with internally propagating Higgs:
  - expansion parameter not small $\alpha_t = \alpha m_t^2/(2s_W^2 m_W^2) \sim \alpha_s/2$
  - only planar integrals contribute in this subset



High-Energy Expansion

Seek an expansion where $s, |t| > m_t^2 \sim (m_H^{int})^2 > (m_H^{ext})^2$

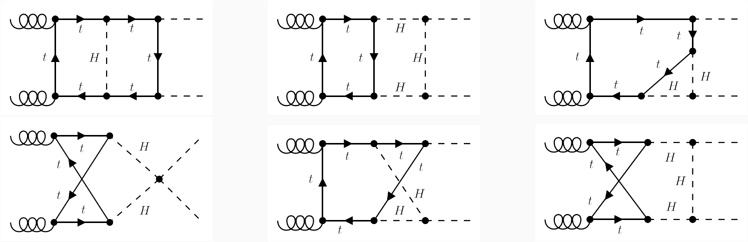
1. Form factors in terms of scalar Feynman integrals: $I(m_H^{ext}, m_H^{int}, m_t, s, t, \epsilon)$
2. Taylor expand for $m_H^{\mathrm{ext}} \to 0$ (with `LiteRed`): [Lee '14]

$$\boxed{I(m_H^{\mathrm{ext}}, m_H^{int}, m_t, s, t, \epsilon) = I(0, m_H^{int}, m_t, s, t, \epsilon) + (m_H^{ext})^2 I'(0, m_H^{int}, m_t, s, t, \epsilon) + \ldots}$$

3. Taylor expand $m_H^{\mathrm{int}} = m_t(1-\delta)$ around $\delta = 0$.
4. IBP reduce to master integrals: $J(0, m_t, m_t, s, t, \epsilon)$ (`FIRE`, `Kira`) [Smirnov '15]
   We find 140 planar and 28 non-planar master integrals.
   [Klappert, Lange, Maierhöfer, Usovitsch '21]
5. Determine MIs as an expansion around $m_t \to 0$:

$$\boxed{J(0, m_t, m_t, s, t, \epsilon) = \sum_{i,j,k} C_{ijk}(s,t)\epsilon^i (m_t)^j \log(m_t)^k}$$

- Insert ansatz into differential equation $\to$ linear equations for c_{ijk}.
- Compute boundary conditions with expansion-by-regions. [Beneke, Smirnov '98]

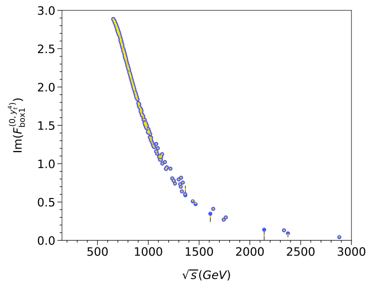


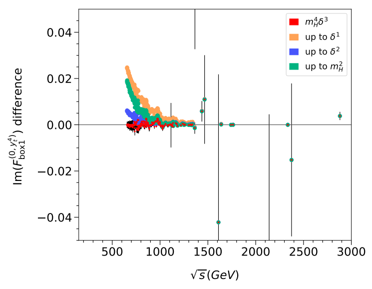
$$F_{\text{box}} = F_{\text{box}}^{(0)} + \frac{\alpha_s(\mu)}{\pi} F_{\text{box}}^{(1,0)} + \frac{\alpha_t}{\pi} F_{\text{box}}^{(0,y_t^4)} + \sqrt{\frac{\alpha_t}{\pi}\frac{\hat{\lambda}}{\pi}} F_{\text{box}}^{(0,y_t^3 g_3)} + \frac{\hat{\lambda}}{\pi} F_{\text{box}}^{(0,y_t^2 g_3^2)} + \frac{\hat{\lambda}}{\pi} F_{\text{box}}^{(0,y_t^2 g_4)} + \cdots$$

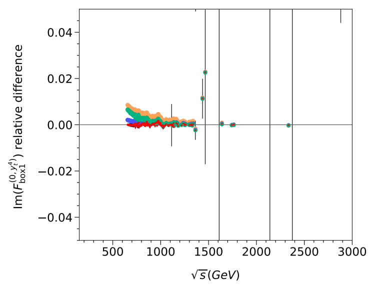
$$\hat{\lambda} = \frac{\lambda}{\pi} = \frac{1}{\pi}\frac{m_H^2}{2v^2}\,, \qquad \alpha_t = \frac{\alpha\, m_t^2}{2\,\sin^2\theta_W\, m_W^2}\,.$$

High-Energy Expansion – Results

$$F_{\text{box}} = F^{(0)}_{\text{box}} + \frac{\alpha_s(\mu)}{\pi} F^{(1,0)}_{\text{box}} + \frac{\alpha_t}{\pi} F^{(0,y_t^4)}_{\text{box}} + \sqrt{\frac{\alpha_t}{\pi}\frac{\hat{\lambda}}{\pi}} F^{(0,y_t^3 g_3)}_{\text{box}} + \frac{\hat{\lambda}}{\pi} F^{(0,y_t^2 g_3^2)}_{\text{box}} + \frac{\hat{\lambda}}{\pi} F^{(0,y_t^2 g_4)}_{\text{box}} + \cdots$$







Comparison with [Heinrich, Jones, Kerner, Stone, Vestner '24] .

High-Energy Expansion – Results

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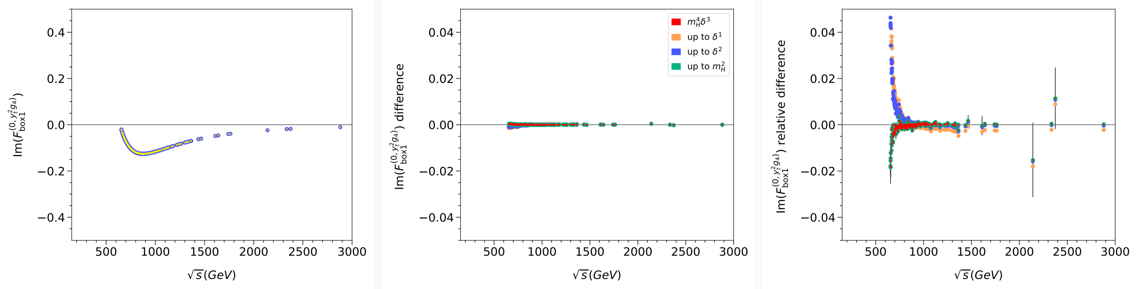

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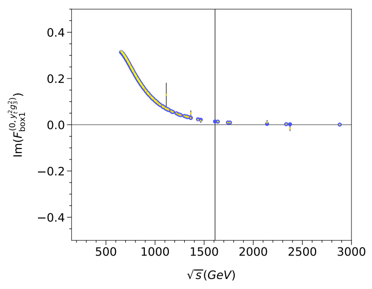

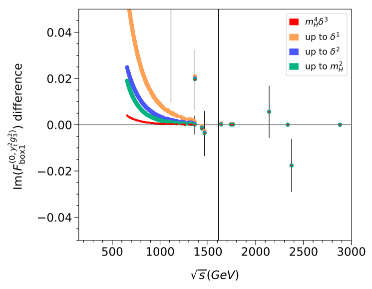

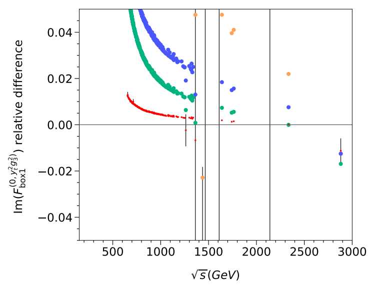

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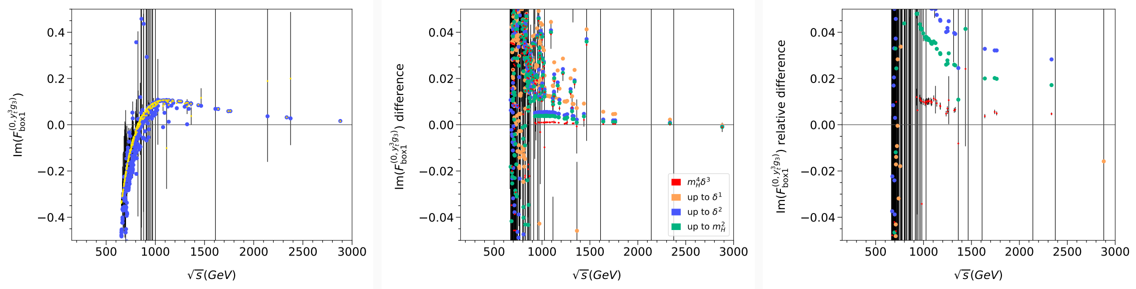


Comparison with [Heinrich, Jones, Kerner, Stone, Vestner '24] .

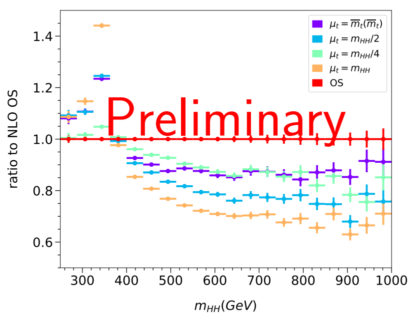
ggxy: numerical evaluation of $gg \to HH$ at NLO

ggxy:

- Right now only provided results for evaluation in `mathematica`.
- `ggxy` is a c++ library which combines, the evaluation of:
  - the LO matrix element,
  - the NLO virtual corrections in the small-t and high-energy expansion,
  - the NLO real radiation corrections,
  - the phase-space integration,

  for the top mass in on-shell or $\overline{\text{MS}}$ scheme.
- Flexible framework to include further processes in the future.

0.5h runtime for a chosen scheme!



Conclusions and Outlook

Conclusions and Outlook

<u>**Conslusions**</u>**:**

- Multi-scale, multi-loop integrals are hard to evaluate:
  $\rightarrow$ Reduce complexity by expanding in physically relevant regions.
- Expansions give a good description for $gg \rightarrow HH$ at NLO QCD.
- We made first steps toward NNLO and see a reduction in the top mass scheme uncertainty.
- We have calculated full NLO electroweak corrections to $gg \rightarrow HH$ and $gg \rightarrow gH$ in the large-m_t expansion.
  - The convergence of these expansions is hindered by W-t-b cuts.
- We have calculated parts of the electroweak corrections in the high-energy region and see a good convergence of our approach.
  - All ingredients for the full electroweak corrections in the high-energy expansion are now available.

Outlook:

- Calculate full NNLO QCD corrections.
    - to come: remaining diagrams to leading expansion order
    - are deeper expansion orders possible? (very challenging IBP reduction)
- Calculate the full EW corrections in the
    1. high-energy expansion.
    2. small-t expansion.
- Provide a numerical program, which can be incorporated into Monte-Carlo studies.

Backup

Padé-Improved High-Energy Expansion

The master integrals for both methods are computed as an expansion in $m_t \ll s, |t|$.

The expansions diverge for $\sqrt{s} \sim 750\text{GeV}$ ("A"), $\sqrt{s} \sim 1000\text{GeV}$ ("B").

The situation can be improved using Padé Approximants:

- Approximate a function using a rational polynomial

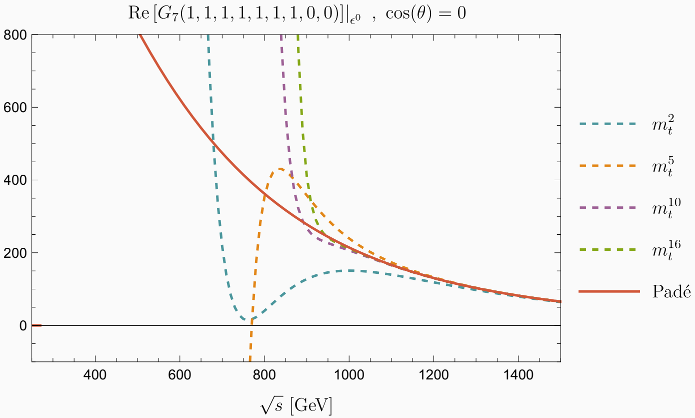
$$\boxed{f(x) \approx \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n}{1 + b_1 x + b_2 x^2 + \cdots + b_m x^m}},$$

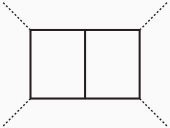
  where a_i, b_j coefficients are fixed by the series coefficients of $f(x)$.

We compute a set of various Padé Approximants:

- combine to give a central value and error estimates
- a deeper input expansion $\Rightarrow$ larger $n+m$ $\Rightarrow$ smaller error
- here, m_t^{120} expansion allows for very high-order Padé Approximants

Master Integrals Results

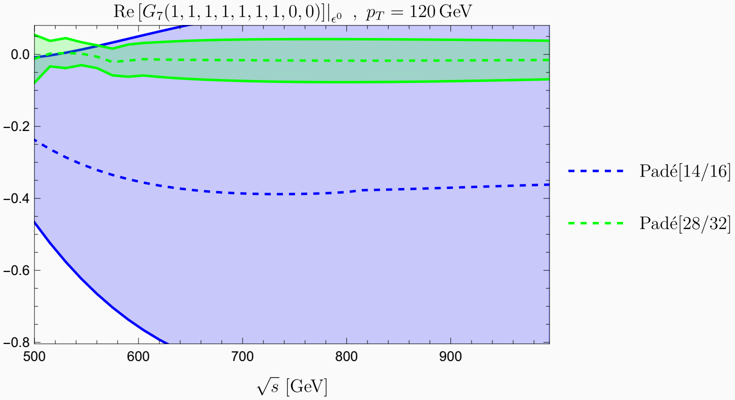




$$\cos(\theta) = \frac{s + 2t - 2m_h^2}{s\sqrt{1 - 4m_h^2/s}}$$

- Fixed order m_t expansions diverge at $\sqrt{s} \sim 1000\,\text{GeV}$.
- The Padé approximation extends the range of validity.

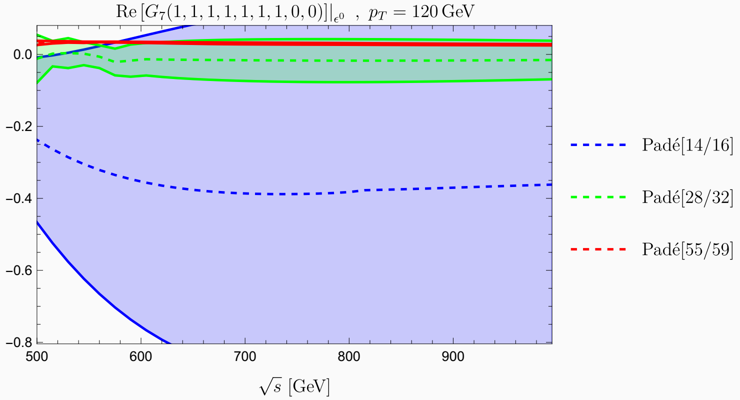
Padé Improvement



$$p_T^2 = \frac{tu - m_h^4}{s}$$

- Lower order Padé approximantions cannot reach low values of p_T.
- For QCD corrections expansions up to m_t^{32} were available: $p_T \gtrsim 150\,\mathrm{GeV}$
- With expansions up to m_t^{120} we reach: $p_T \gtrsim 120\,\mathrm{GeV}$.
- Error estimate from Padé approximations is reliable.

Padé Improvement



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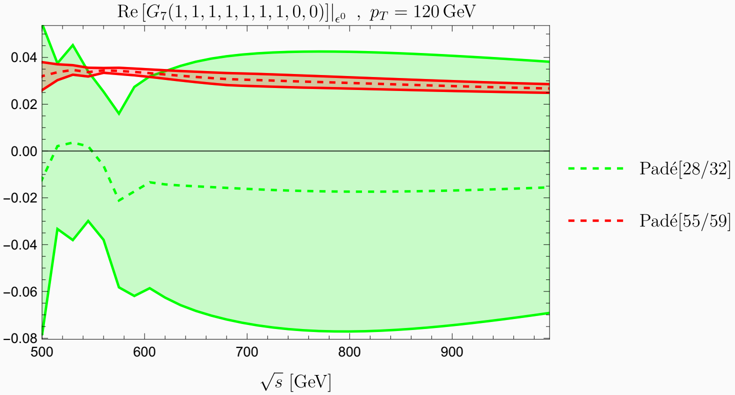

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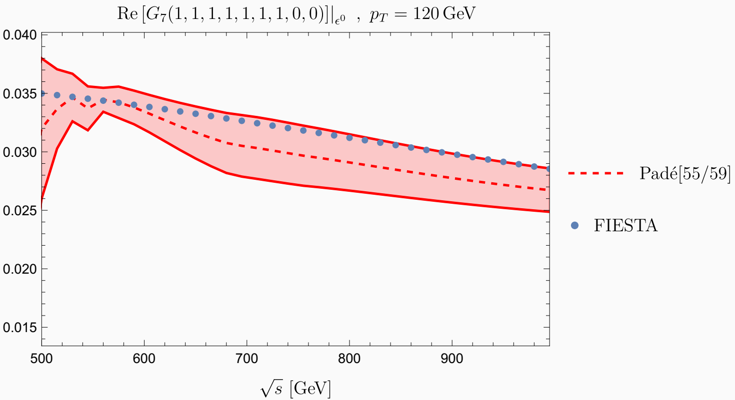

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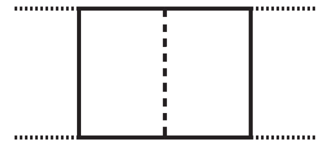
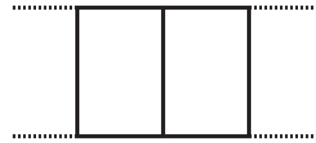
Comparison to the $m_H \to 0$ Expansion



Approach A:

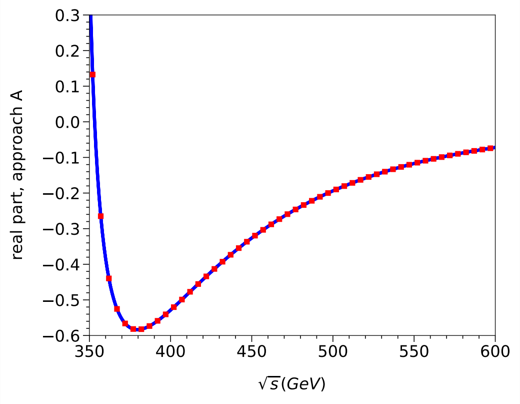
- middle line massless $m_H^{\text{int}} \approx 0$
- calculated in the context of QCD corrections [Davies, Mishima, Steinhauser, Wellmann '18, '19]

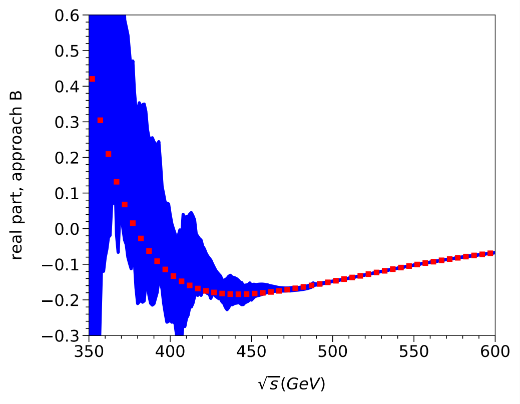


Approach B:

- middle line massive $m_H^{\text{int}} \approx m_t$

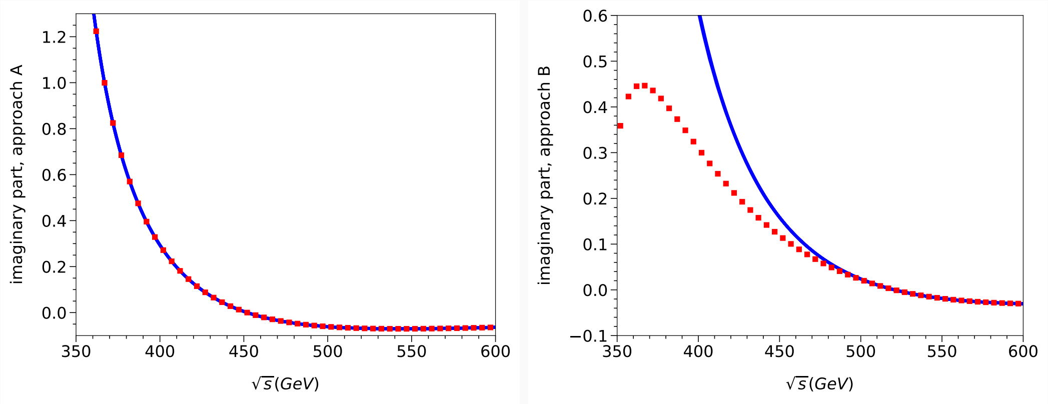
Comparison with Approach A





Approach A: threshold at $\sqrt{s} = 2m_t = 346\,\text{GeV}$ Approach B: threshold at $\sqrt{s} = 3m_t = 519\,\text{GeV}$

Comparison with Approach A



Approach A: threshold at $\sqrt{s} = 2m_t = 346\,\text{GeV}$ Approach B: threshold at $\sqrt{s} = 3m_t = 519\,\text{GeV}$

Beyond the Large-m_t Expansion – High-Energy Expansion

Analytic high-energy expansion:

- Expansion hierarchy: $s, t \gg m_t^2 \approx (m_H^2)^{int} \gg (m_H^2)^{ext}$
- We get a system of differential equations for 140 master integrals

$$\boxed{\frac{\partial}{\partial m_t^2}\vec{I} = M(s,t,m_t^2,\epsilon)\cdot\vec{I}}, \quad \text{with } \vec{I} = (I_1,\ldots,I_{140})$$

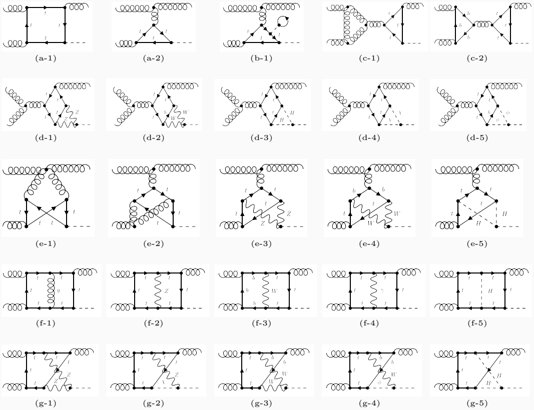
- Plug in power-log ansatz for each master integral

$$\boxed{I_n = \sum_{i=-2}^{0}\sum_{j=-1}^{60}\sum_{k=0}^{i+4} c_n^{ijk}(s,t)\epsilon^i (m_t^2)^j \ln^k(m_t^2)}$$

- Solve the system of linear equations for a small set of boundary constants with `Kira` and `FireFly` [Klappert, Lange, Maierhöfer, Usovitsch '21] .
- Solve boundary master integrals in the asymptotic limit $m_t \to 0$ with Mellin-Barnes methods and symbolic summation using `Asy` [Pak, Smirnov '11] , `MB.m` [Czakon '05] , `HarmonicSums` [Ablinger '10] and `Sigma` [Schneider '07] .

NLO Electroweak Matrix Elements for $gg \to Hg$

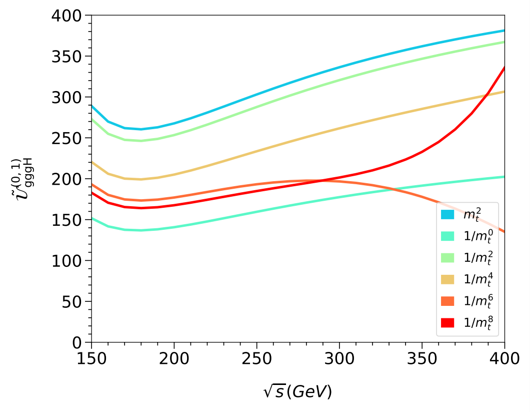
$$\mathcal{M} = \frac{1}{8^2 2^2} \sum_{\text{col}} \sum_{\text{pol}} |\mathcal{A}|^2 = \frac{3}{32} \left(X_0^{gggH} \right)^2 s \, \tilde{U}_{gggH}$$



Graphs contributing to $gg \to Hg$.

We observe a nice convergence at NLO.

$$\tilde{U}_{gggH} = \tilde{U}^{(0)}_{gggH} + \frac{\alpha}{\pi} \tilde{U}^{(0,1)}_{gggH}$$

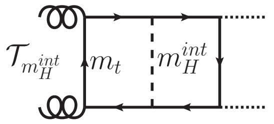


Different expansion orders in $1/m_t$.

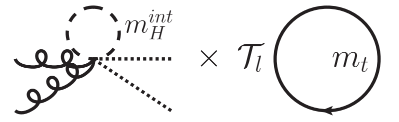
High-Energy Expansion: Option A

Option A: asymptotic expansion around $m_H^{int} = 0$. Expnsion-by-subgraphs:

- two sub-graphs:



$+$



The two-loop subgraph is a Taylor expansion of the Higgs propagator:

- results in integrals with a massless internal line. Scales: s, t, m_t.
- IBP reduce with `FIRE` and `Kira` [Smirnov '15; Klappert, Lange, Maierhöfer, Usovitsch '21]
- these coincide with the QCD master integrals – reuse the old results

[Davies, Mishima, Steinhauser, Wellmann '18,'19]

The massive tadpoles are easily computed by `MATAD`. [Steinhauser '00]

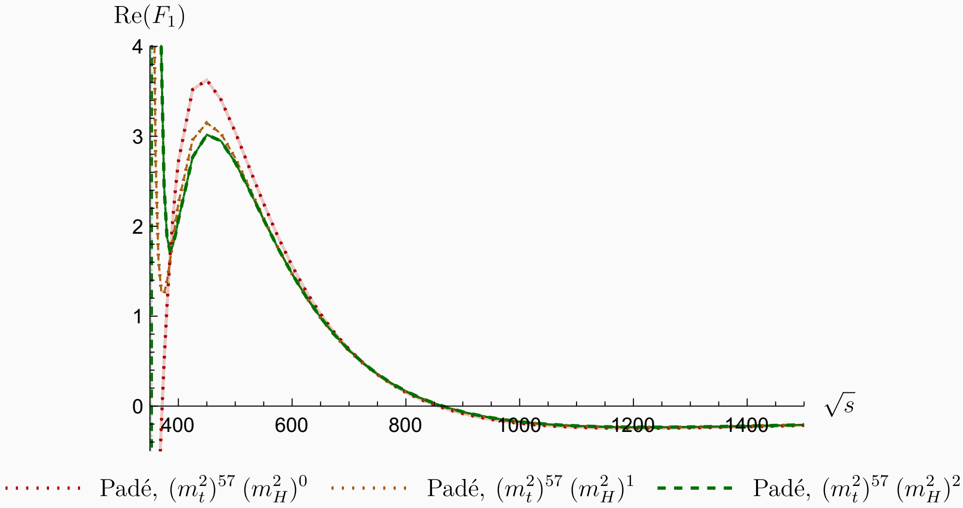
The asymptotic expansion procedure is done by `exp` and `FORM`. [Harlander, Seidelsticker, Steinhauser '97] [Ruijl, Ueda, Vermaseren '17]

We expand to quartic order: $(m_H^{int})^a\,(m_H^{ext})^b,\ \ 0 \leq (a+b) \leq 4$.

High-Energy Expansion "A": convergence

Re(F_{box1}), fixed $\cos\theta = 0$, expansion "A" Padé (to $(m_H^2)^{\{0,1,2\}}$):

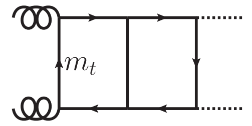
- $(m_H^2)^1$ and $(m_H^2)^2$ terms differ by at most 5% for $\sqrt{s} \geq 400$GeV



High-energy Expansion “B”

Option B: expand around $m_H^{int} \approx m_t$,

- simple Taylor expansion, easy to implement



Write Higgs propagator as: $\frac{1}{p^2 - m_H^2} = \frac{1}{p^2 - m_t^2(1 - [2-\delta]\delta)}$

- expand around $\delta \to 0$ where $\delta = 1 - m_H/m_t \approx 0.28$.

This yields new integral families compared to the QCD computation:

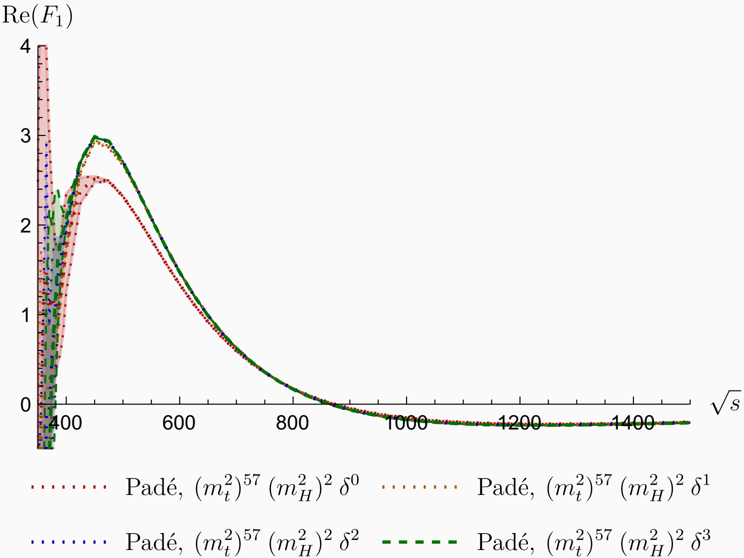
- All lines have the mass m_t.
- IBP reduce and compute the master integrals (140) in the high-energy limit.

Expand to $(m_H^{ext})^4$ and δ^3.

High-energy Expansion "B": convergence

$\mathrm{Re}(F_{box1})$, fixed $\cos\theta = 0$, expansion "B" Padé (to $(m_H^2)^2\delta^{\{0,1,2,3\}}$):

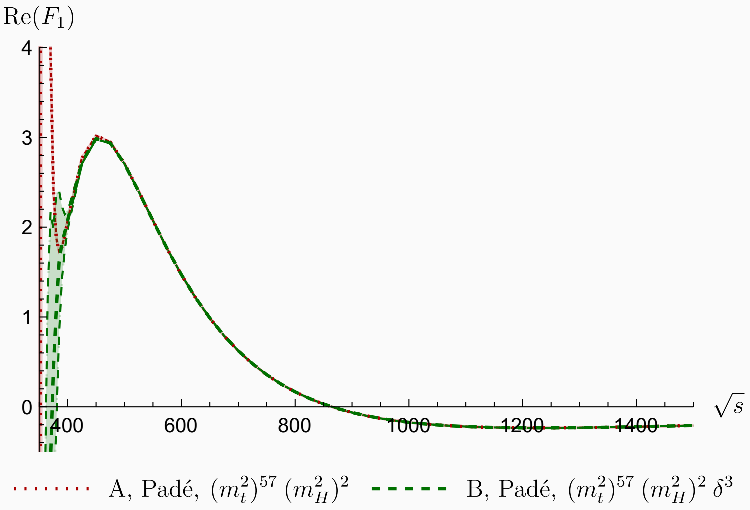
- δ^2 and δ^3 terms differ by at most 0.5% for $\sqrt{s} \geq 400$GeV



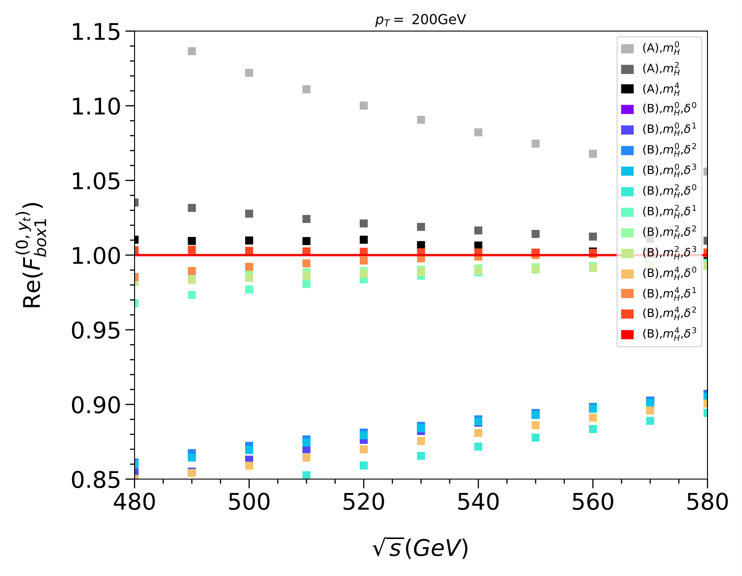
High-energy Expansion: “A”, “B” comparison

$\mathrm{Re}(F_{box1})$, fixed $\cos\theta = 0$, best “A” and “B” Padé

- “A”, “B” differ by at most 2% for $\sqrt{s} \geq 400\mathrm{GeV}$,
- 0.1% for $\sqrt{s} \geq 500\mathrm{GeV}$



Beyond the Large-m_t Expansion – High-Energy Expansion



$$\mathcal{A}^{\mu\nu} = T_1^{\mu\nu} F_{box1} + T_2^{\mu\nu} F_{box2}$$

- We benchmark against the expansion to $O(m_H^4, \delta^3, m_t^{116})$, with $\delta = 1 - m_H/m_t$.
- Convergence of different expansion orders at fixed $p_T = 200\,\text{GeV}$.
- Verified agreement with the `pySecDec` group.