The structure of quark mass corrections in the $gg \to HH$ amplitude at very high-energy

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[2501.00587]
w/ Heinrich, Kerner, Stone, Vestner
[2407.04653]

Outline

Higgs Precision Programme & Higgs Boson Pair Production

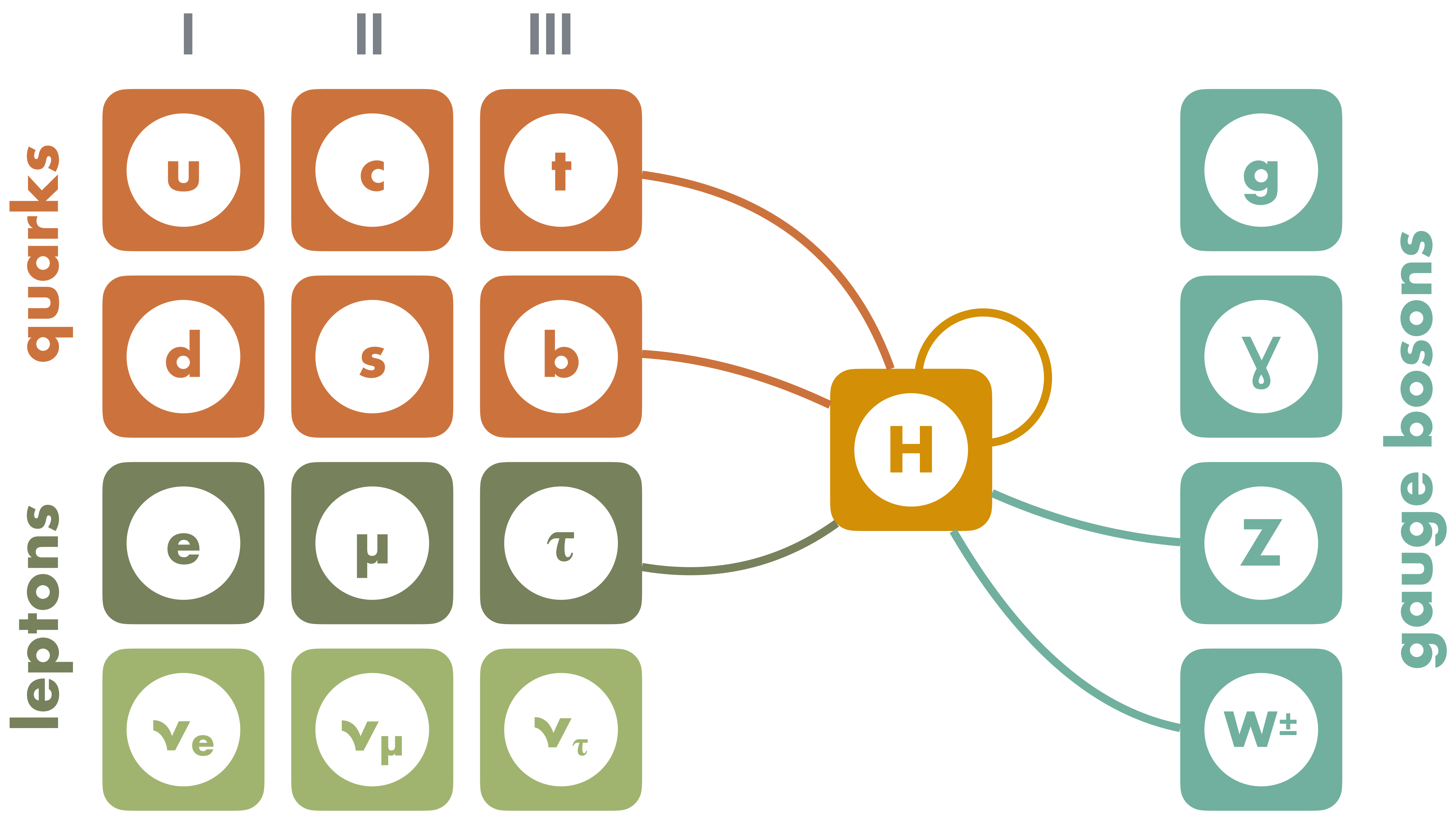
Higgs Pair Production State of the Art

- Overview
- Electroweak corrections
- Mass Scheme Uncertainty
- High Energy Limit

Method of Regions

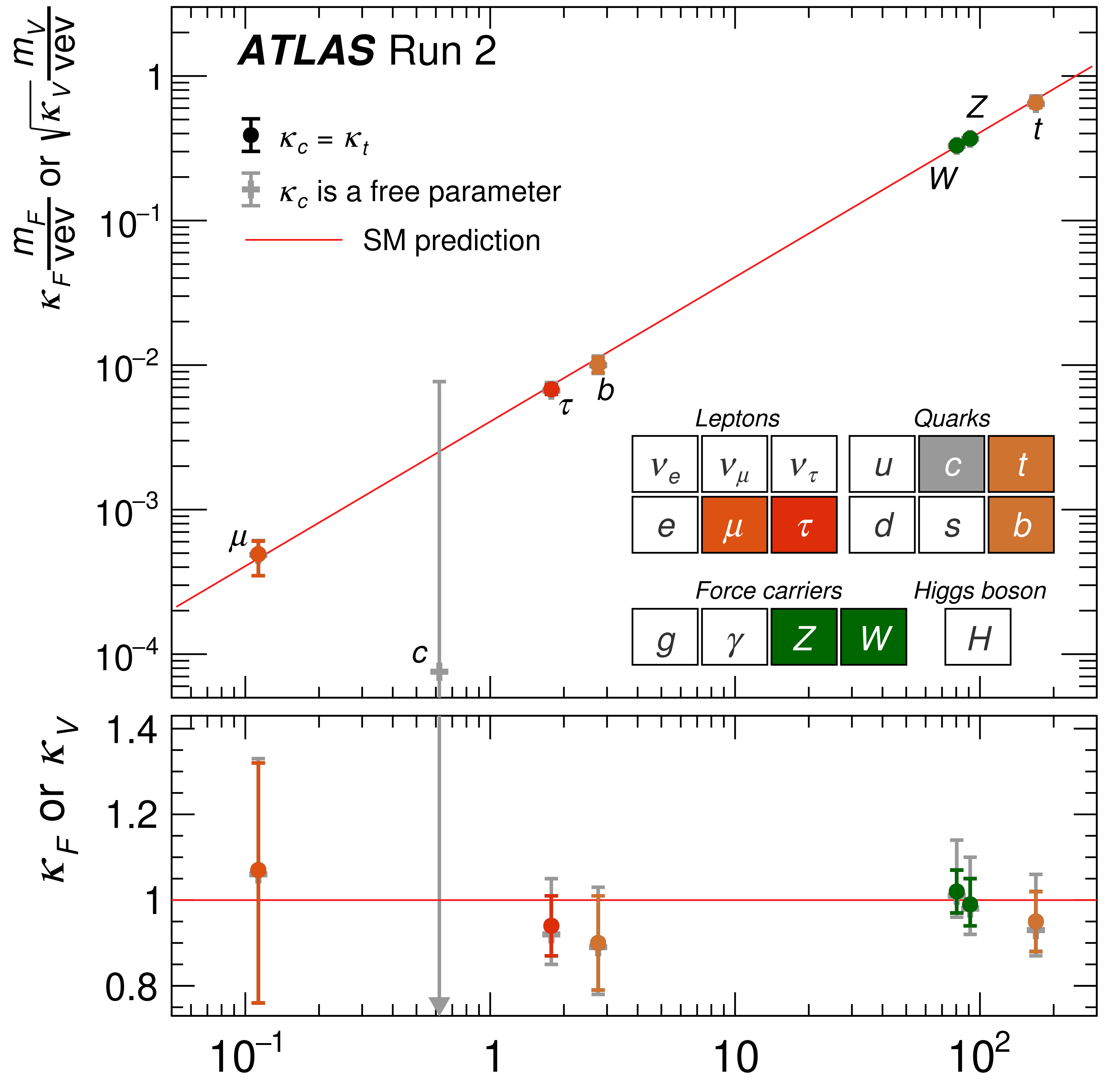
- Basics
- Application to $gg \to HH$
- SCET analysis of $gg \to HH$

Results & Outlook



Higgs Couplings

The Higgs sector continues to yield impressive fundamental discoveries



CERN-EP-2022-057

2018 ($t\bar{t}H$): First direct observation of top-quark Yukawa coupling

CMS 1804.02610/ ATLAS 1806.00425
CMS 2407.10896/ ATLAS 2407.10904

2020 ($H \to \mu\mu$): First direct evidence that Higgs field is responsible for mass of 2nd gen. leptons

CMS 2009.04363 / ATLAS 2007.07830

2022 ($H \to c\bar{c}$): First hints that Higgs field is responsible for mass of 2nd gen. quarks

CMS 2205.05550 / ATLAS 2201.11428
ATLAS 2410.19611

2024 (Sign HWW/HZZ): Exclude $\lambda_{WZ} = \kappa_W/\kappa_Z < 0$ using WH via VBF

CMS 2405.16566

HH: Why Measure it?

$$\mathcal{L} \supset -V(\phi), \quad V(\Phi) = -\mu^2(\Phi^\dagger\Phi) + \lambda(\Phi^\dagger\Phi)^2$$

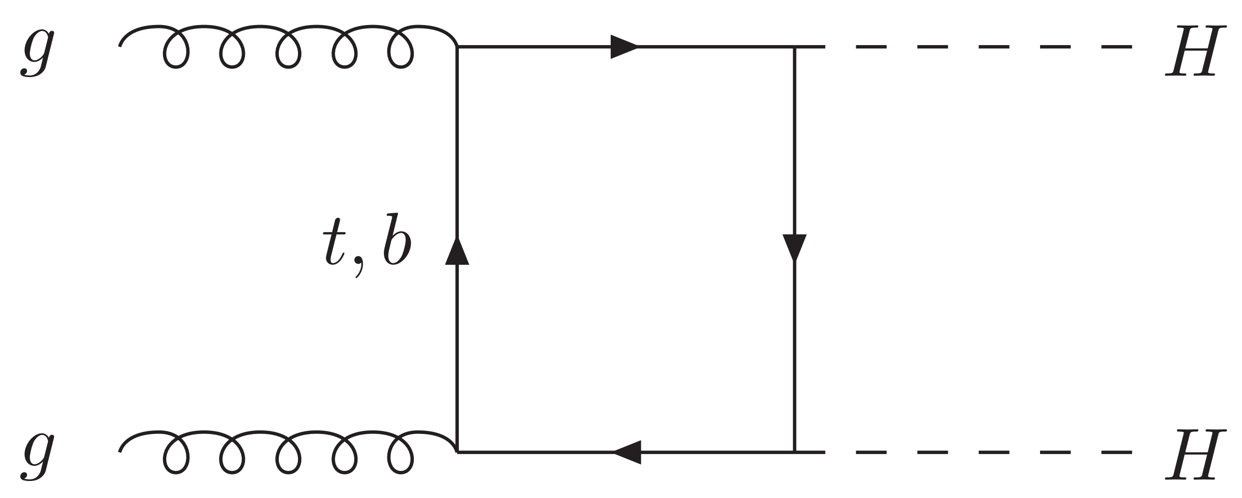
EW symmetry breaking

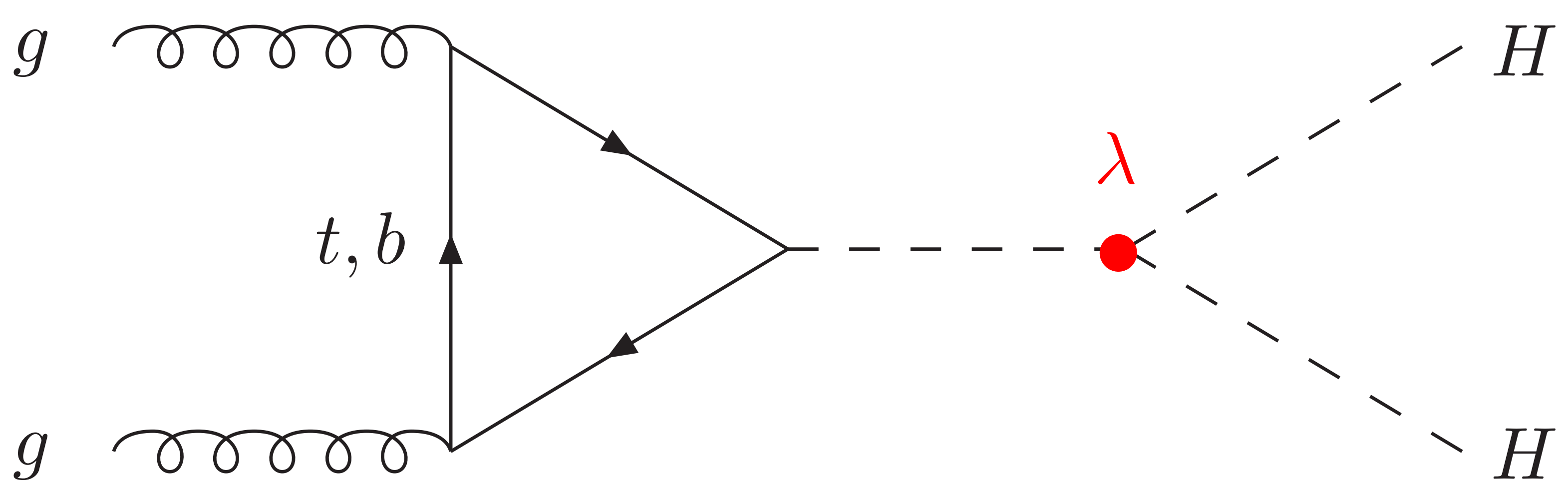
$$\mu^2 = \lambda v^2$$
$$m_H^2 = 2\lambda v^2$$

$$V(H) = \frac{1}{2}m_H^2 H^2 + \lambda v H^3 + \frac{\lambda}{4}H^4,$$

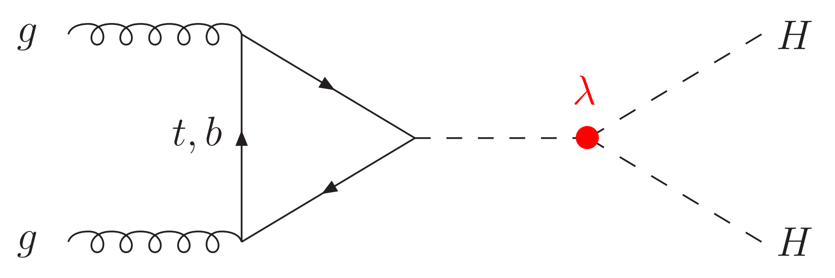
Standard model: self-couplings determined by m_H, v

Need experimental measurements to confirm/refute this

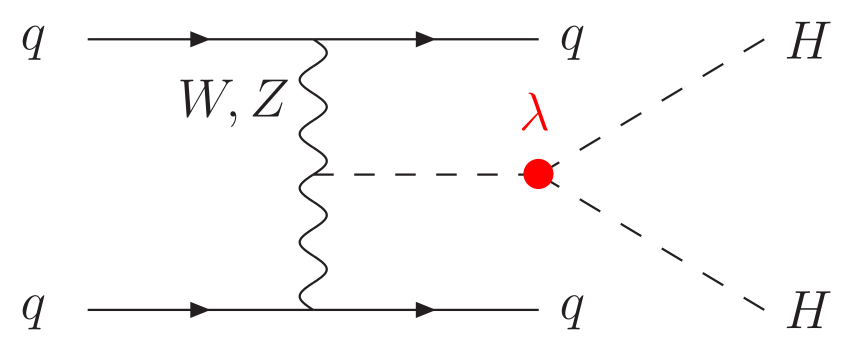




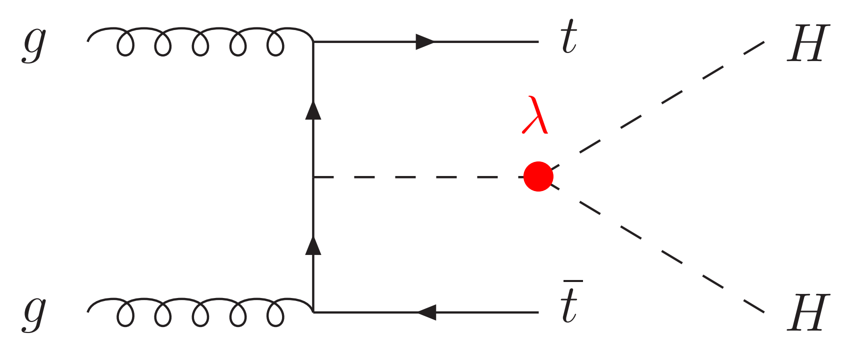
HH Production Channels at the LHC



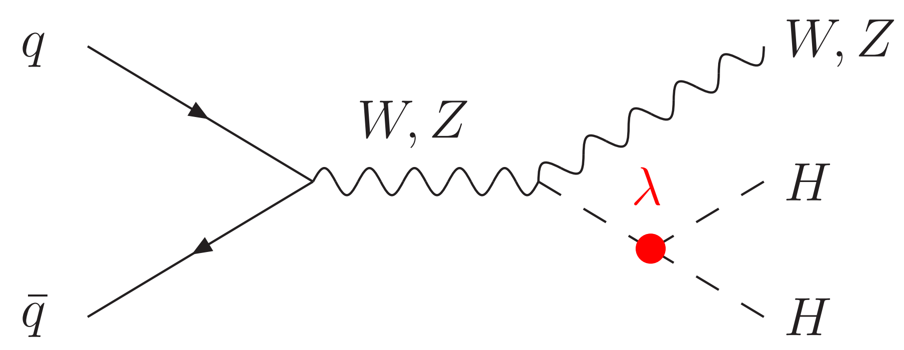
Gluon Fusion



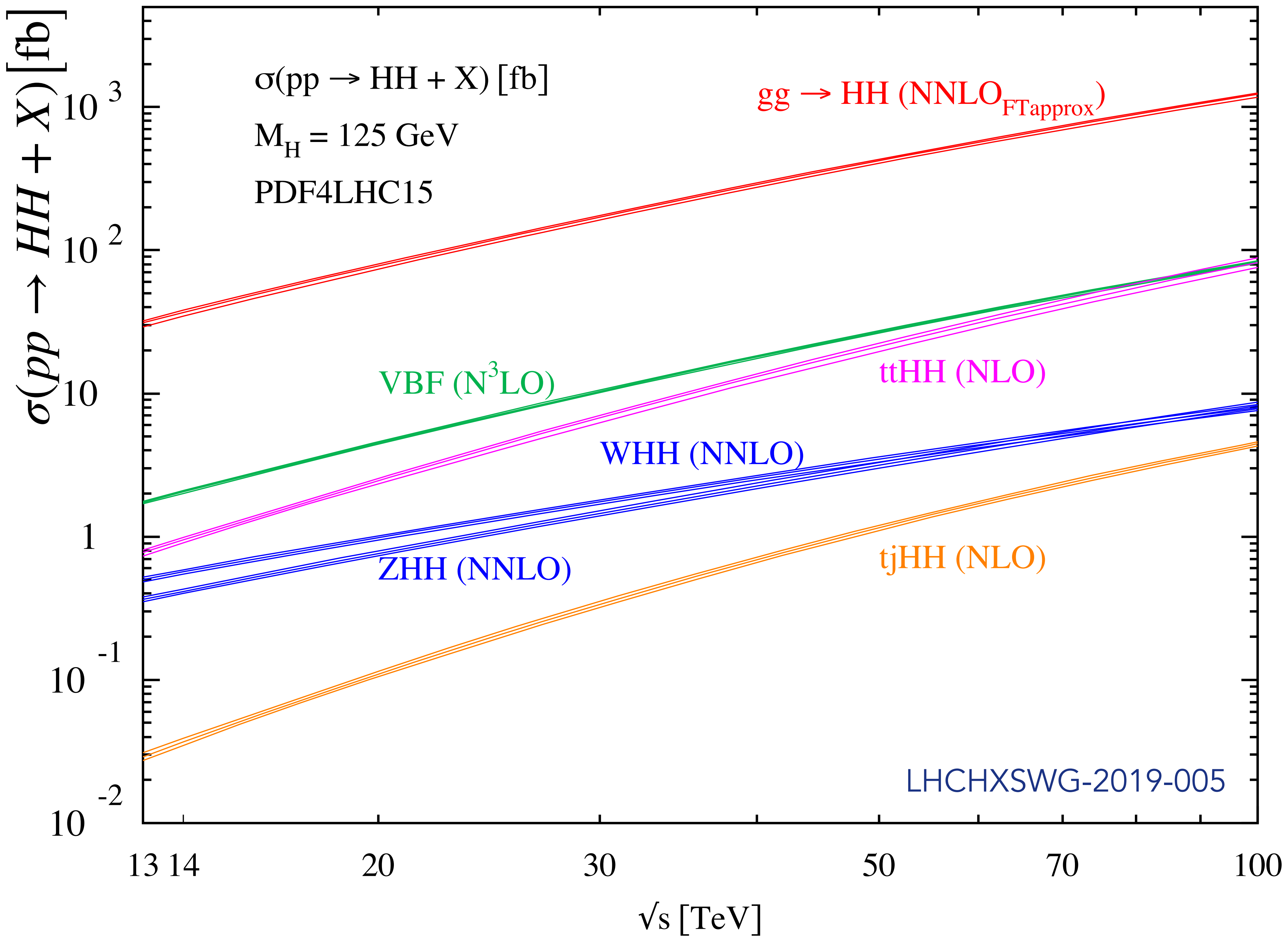
Vector Boson Fusion



Associated Production ($t\bar{t}$)



Associated Production (W,Z)



Production channels similar to H

A very important difference:

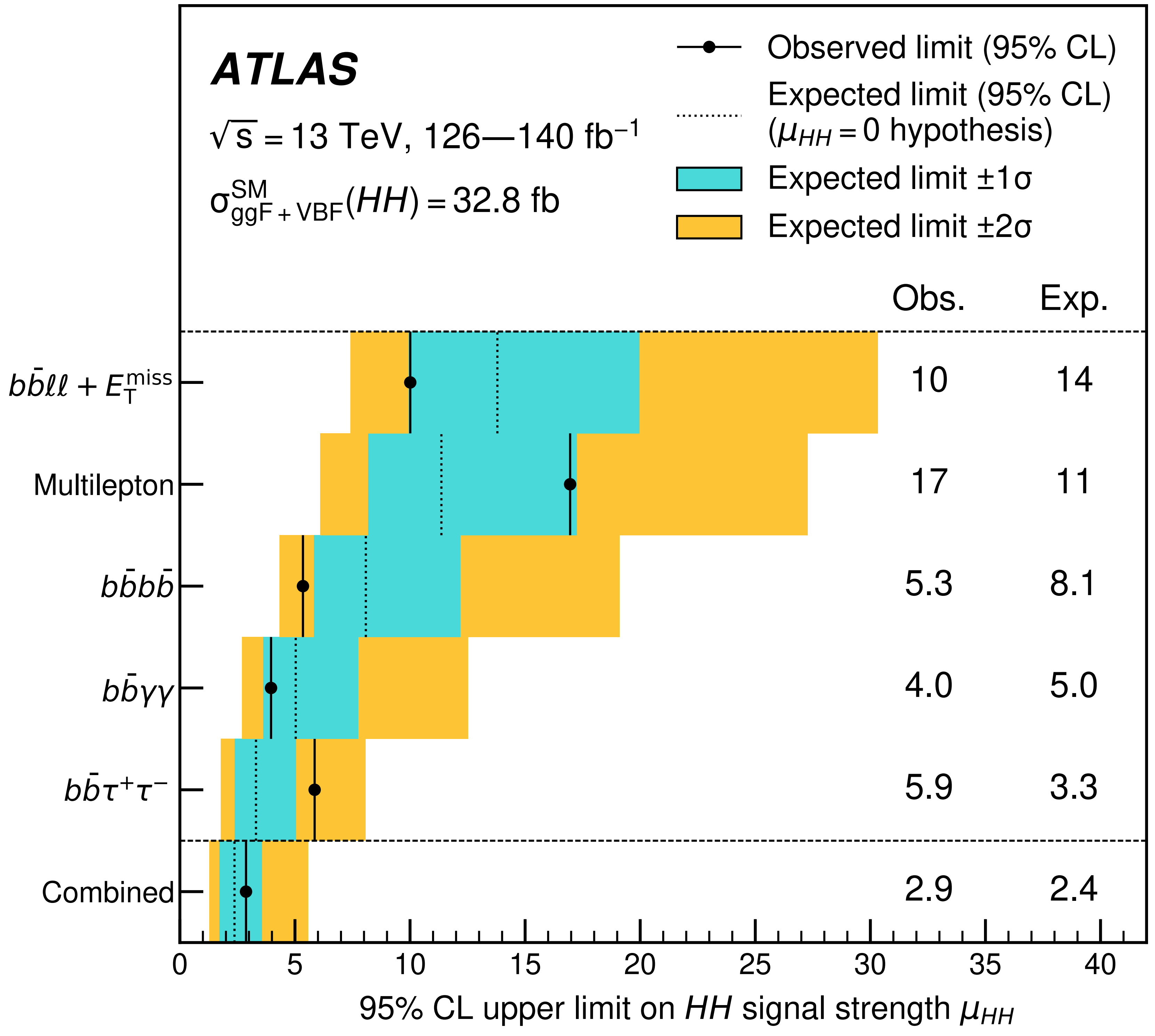
$$\sigma(pp \to HH) \sim \frac{\sigma(pp \to H)}{1000}$$

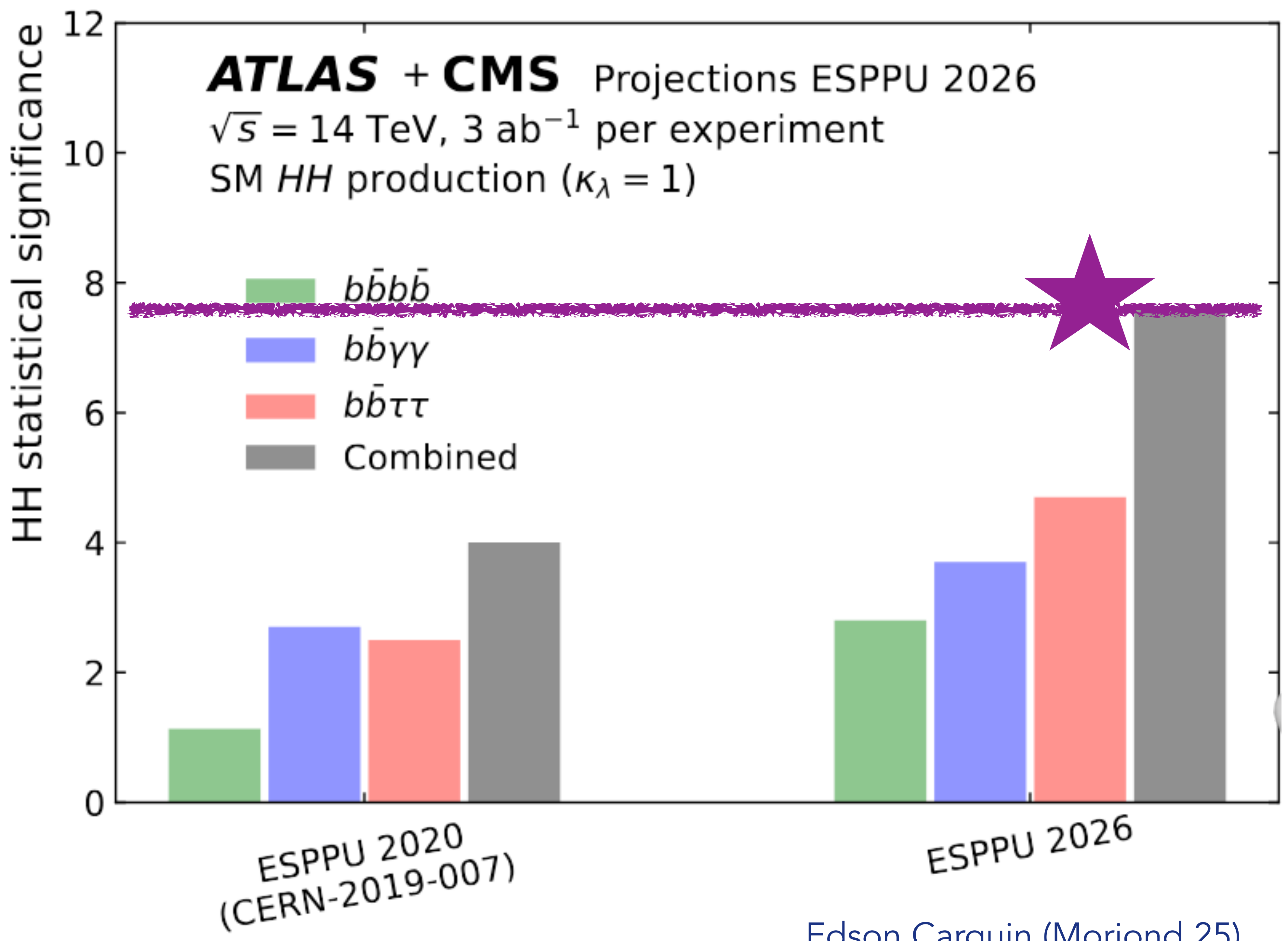
Experimental Limits

Current Experimental Limits (Run 2) @ 95% CL

ATLAS: $-1.2(-1.6) < \kappa_\lambda < 7.2(7.2)$ ATLAS 2406.09971

CMS: $-1.39(-1.02) < \kappa_\lambda < 7.02(7.19)$ CMS-PAS-HIG-20-011





Edson Carquin (Moriond 25)

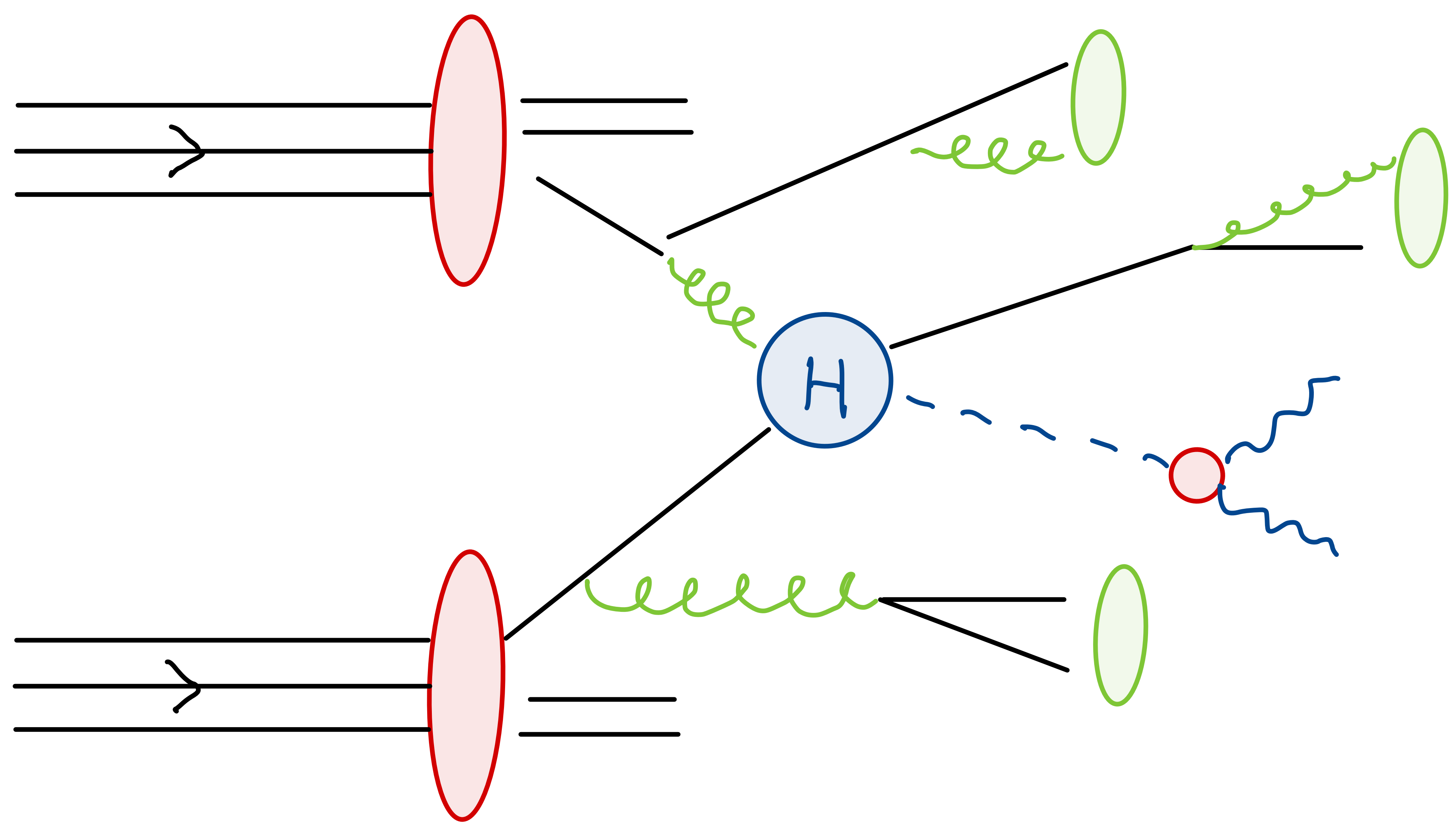
Projections for HL-LHC (European Strategy 2026)

Expected discovery significance ~3.8σ per experiment

Combined ATLAS+CMS projections ~7.6σ !

TH systematics approximately halved in projections

Improving Precision



- Parton Shower
- Resummation
- Hadronisation
- Underlying Event

$$\mathrm{d}\sigma = \int \mathrm{d}x_a \mathrm{d}x_b f_a(x_a) f_b(x_b) \mathrm{d}\hat{\sigma}_{ab}(x_a, x_b) F_J + \mathcal{O}\left((\Lambda/Q)^m\right)$$

Parton Distribution Functions (PDFs)

Hard Scattering Matrix Element

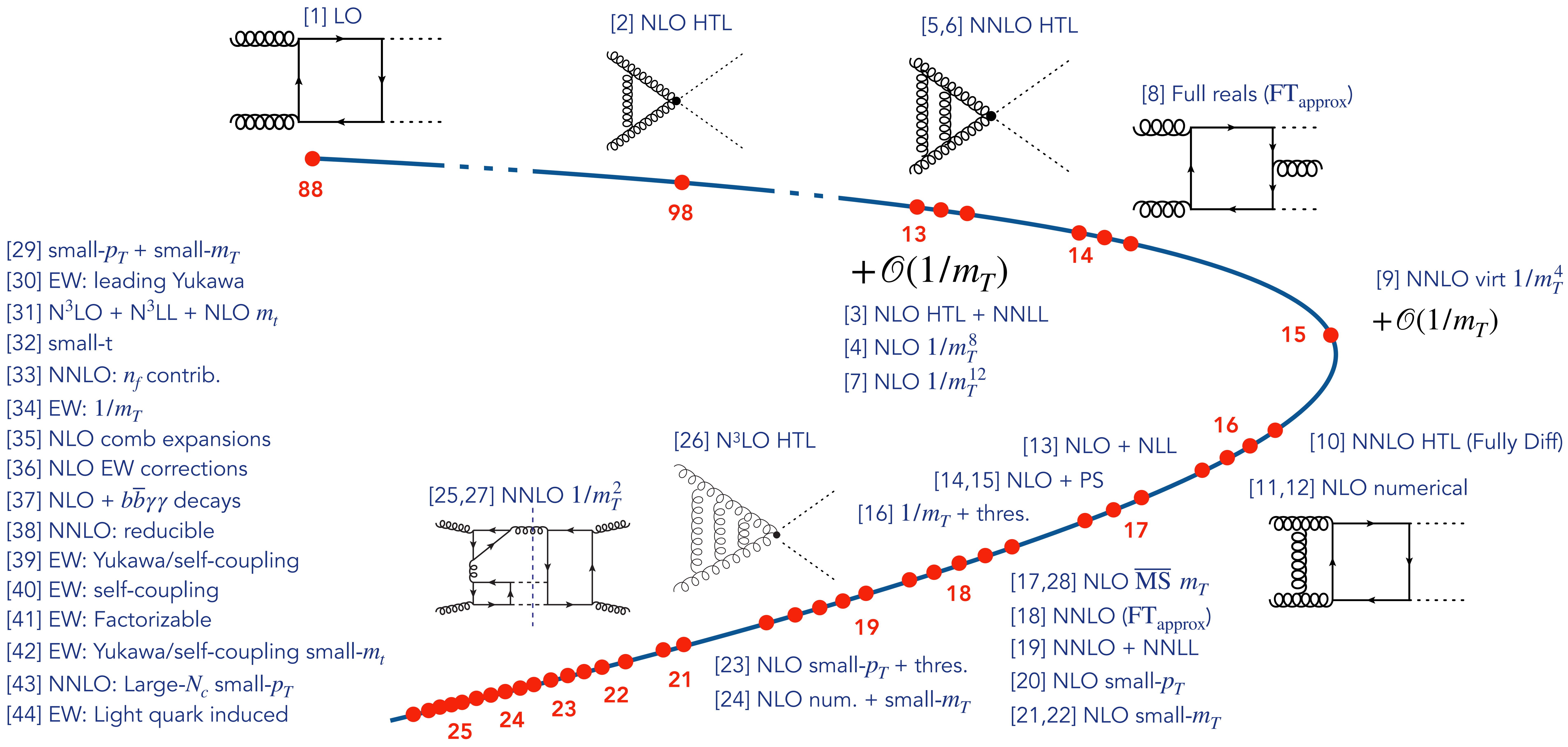
Non-perturbative effects ~ few %

With $\alpha_s \sim 0.1$, expect: NLO ~ 10% correction, NNLO ~ 1% correction

Higgs channels are important exceptions, receive much larger corrections!

Overview

(Slide design shamelessly stolen from G. Salam)



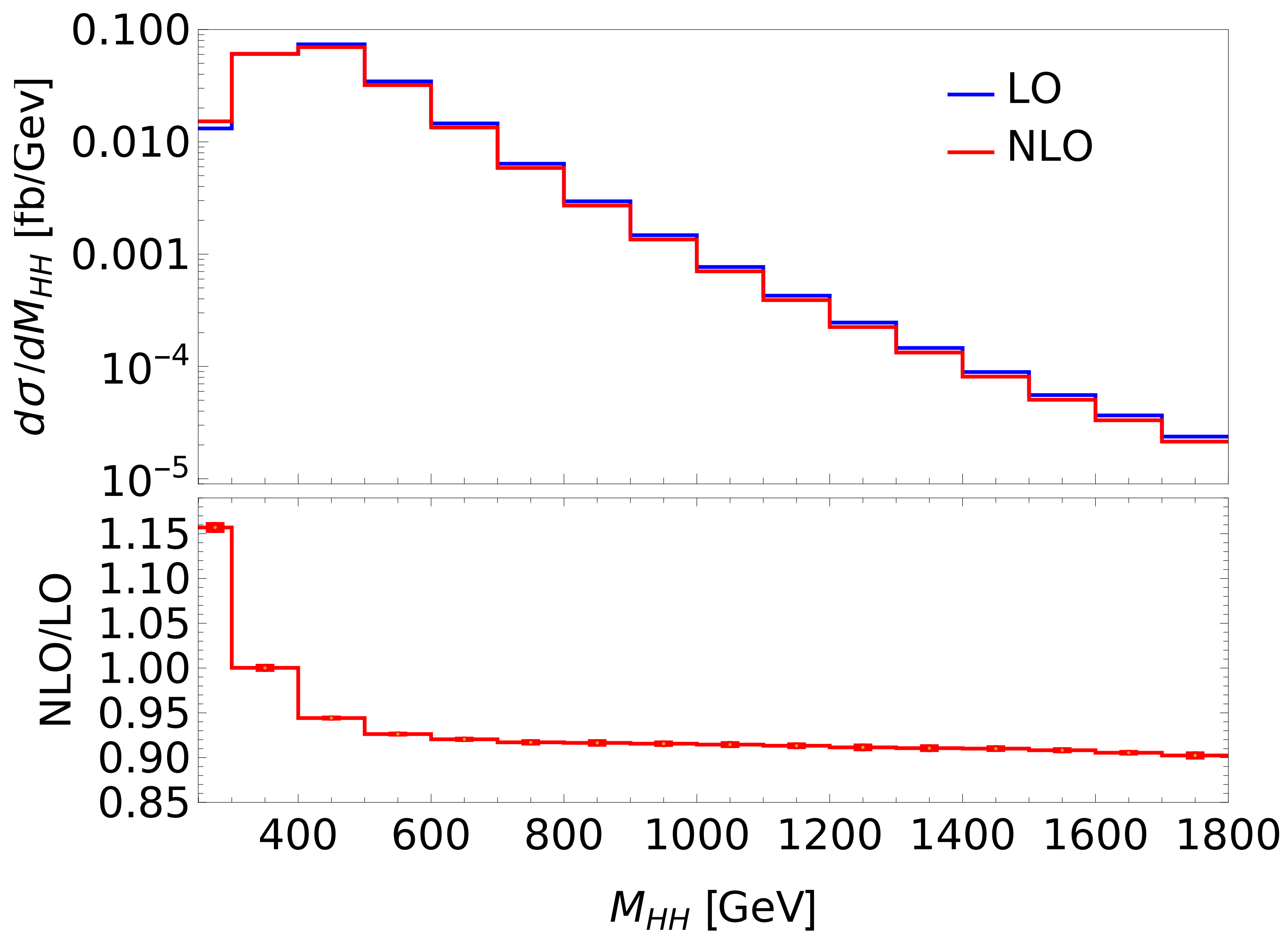
[1] Glover, van der Bij 88; [2] Dawson, Dittmaier, Spira 98; [3] Shao, Li, Li, Wang 13; [4] Grigo, Hoff, Melnikov, Steinhauser 13; [5] de Florian, Mazzitelli 13; [6] Grigo, Melnikov, Steinhauser 14; [7] Grigo, Hoff 14; [8] Maltoni, Vryonidou, Zaro 14; [9] Grigo, Hoff, Steinhauser 15; [10] de Florian, Grazzini, Hanga, Kallweit, Lindert, Maierhöfer, Mazzitelli, Rathlev 16; [11] Borowka, Greiner, Heinrich, SPJ, Kerner, Schlenk, Schubert, Zirke 16; [12] Borowka, Greiner, Heinrich, SPJ, Kerner, Schlenk, Zirke 16; [13] Ferrera, Pires 16; [14] Heinrich, SPJ, Kerner, Luisoni, Vryonidou 17; [15] SPJ, Kuttimalai 17; [16] Gröber, Maier, Rauh 17; [17] Baglio, Campanario, Glaus, Mühlleitner, Spira, Streicher 18; [18] Grazzini, Heinrich, SPJ, Kallweit, Kerner, Lindert, Mazzitelli 18; [19] de Florian, Mazzitelli 18; [20] Bonciani, Degrassi, Giardino, Gröber 18; [21] Davies, Mishima, Steinhauser, Wellmann 18, 18; [22] Mishima 18; [23] Gröber, Maier, Rauh 19; [24] Davies, Heinrich, SPJ, Kerner, Mishima, Steinhauser, David Wellmann 19; [25] Davies, Steinhauser 19; [26] Chen, Li, Shao, Wang 19, 19; [27] Davies, Herren, Mishima, Steinhauser 19, 21; [28] Baglio, Campanario, Glaus, Mühlleitner, Ronca, Spira 21; [29] Bellafronte, Degrassi, Giardino, Gröber, Vitti 22; [30] Davies, Mishima, Schönwald, Steinhauser, Zhang 22; [31] Ajjath, Shao 22; [32] Davies, Mishima, Schönwald, Steinhauser 23; [33] Davies, Schönwald, Steinhauser 23; [34] Davies, Schönwald, Steinhauser, Zhang 23; [35] Bagnaschi, Degrassi, Gröber 23; [36] Bi, Huang, Huang, Ma Yu 23 [37] Li, Si, Wang, Zhang, Zhao 24; [38] Davies, Schönwald, Steinhauser, Vitti 24; [39] Heinrich, SPJ, Kerner, Stone, Vestner [40] Li, Si, Wang, Zhang, Zhao 24; [41] Davies, Schönwald, Steinhauser, Zhang 24; [42] Davies, Schönwald, Steinhauser, Zhang 25; [43] Davies, Schönwald, Steinhauser 25; [44] Bonetti, Rendler, Bobadilla 25;

EW Corrections

Interesting to explore the impact of EW corrections ($\pm 5\,\%$ for off-shell Higgs) Actis, Passarino, Sturm, Uccirati 08

Full EW Corrections

Computed using AMFlow



↪ -4% on total cross section

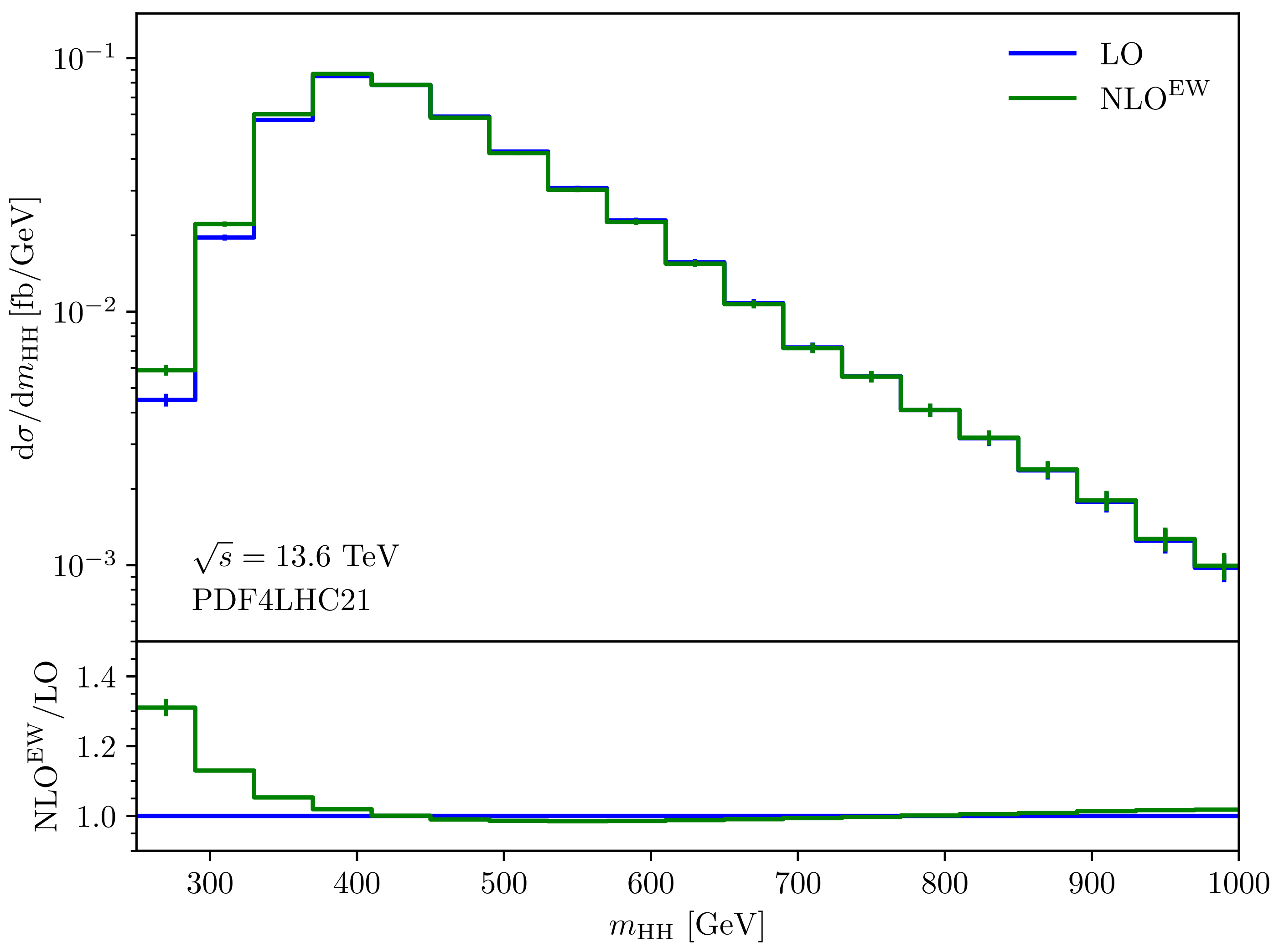
↪ +15% near production threshold

↪ -10% at high energy (Sudakov-like)

Bi, Huang, Huang, Ma, Yu 23

Partial EW Corrections ($y_t, \lambda_3, \lambda_4$)

Obtained using pySecDec



↪ +1% on total cross section

↪ +30% near production threshold

Can be adapted for EFT analyses

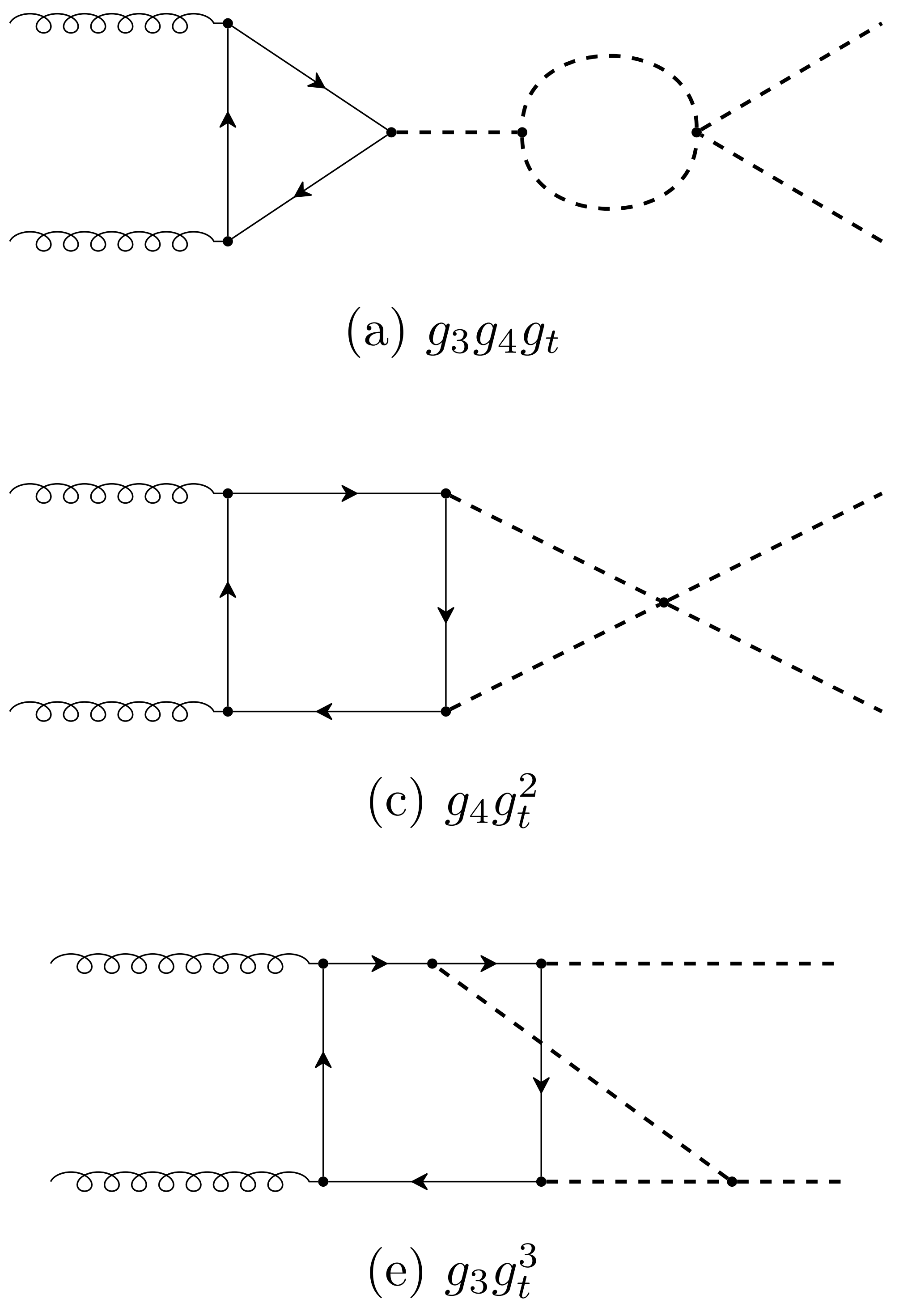
Heinrich, SPJ, Kerner, Stone, Vestner 24

EW corrections modify distributions and bounds in the SM & EFT frameworks

Partial EW Corrections

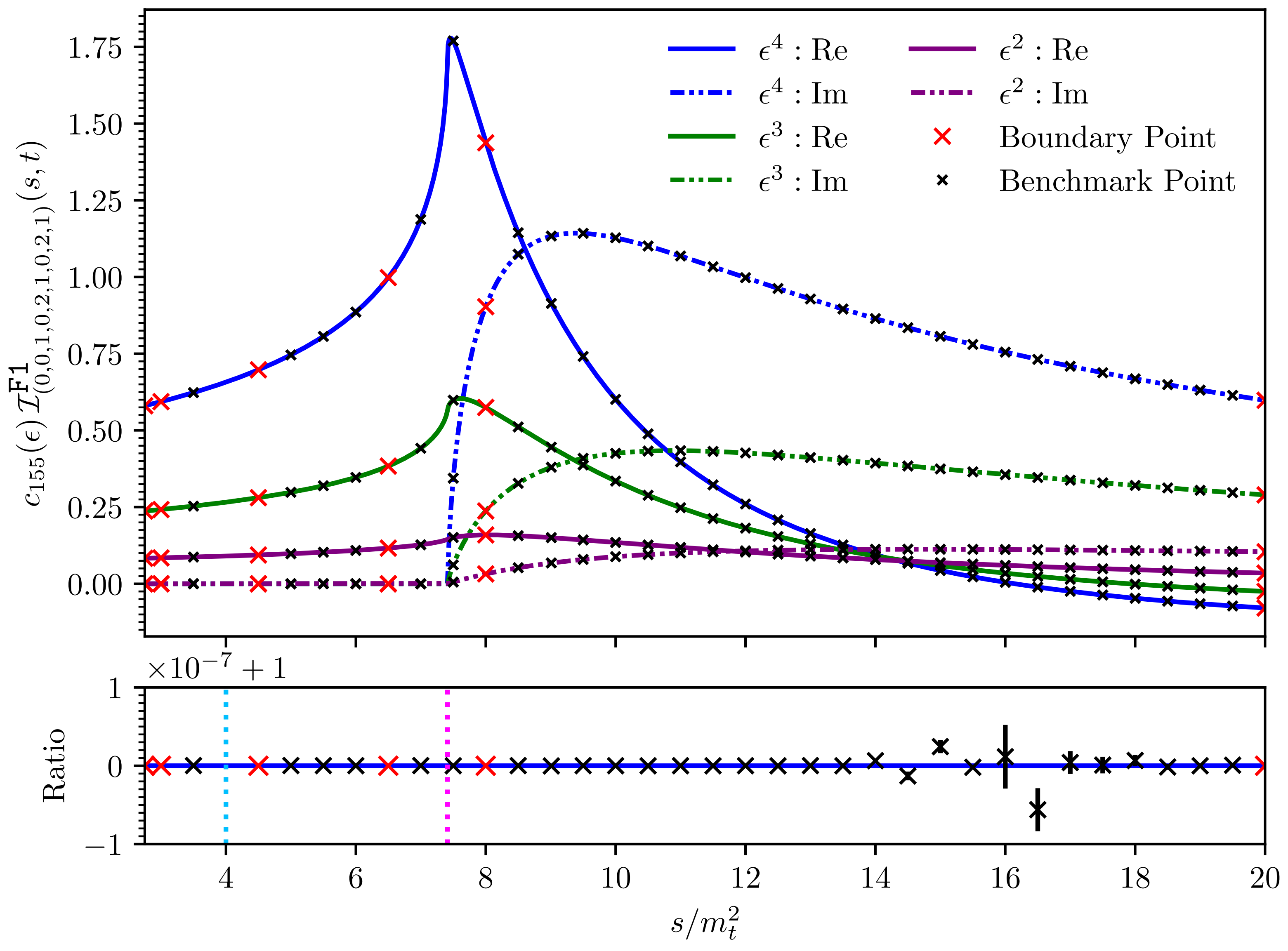
Fully symbolic (s, t, m_t, m_h) reduction to basis of 494 finite D-factorising master integrals obtained

Up to 11 master integrals within a single sector



(a) $g_3 g_4 g_t$ (b) $g_3^3 g_t$ (c) $g_4 g_t^2$ (d) $g_3^2 g_t^2$ (e) $g_3 g_t^3$ (f) g_t^4

Heinrich, SPJ, Kerner, Stone, Vestner 24



All integrals cross-checked with DiffExp/pySecDec

Hidding 21; Heinrich, SPJ, Kerner, Magerya, Olsson, Schlenk 24

Main results produced with pySecDec

Total Cross Section & Scale Uncertainty @ 14 TeV

Despite the significant theory progress Higgs WG recommendations have remained rather stable…

	σ_{LO} (fb)	σ_{NLO} (fb)	σ_{NNLO} (fb)	σ_{N3LO} (fb)
Basic HTL	$17.07^{+30.9\%}_{-22.2\%}$	$31.93^{+17.6\%}_{-15.2\%}$	$37.52^{+5.2\%}_{-7.6\%}$	$38.65^{+0.65\%}_{-2.7\%}$
B-i/proj HTL	$19.85^{+27.6\%}_{-20.5\%}$	$38.32^{+18.1\%}_{-14.9\%}$	$39.58^{+1.4\%}_{-4.7\%}$	$40.44^{+1.9\%}_{-4.7\%}$
FTapprox	$19.85^{+27.6\%}_{-20.5\%}$	$34.25^{+14.7\%}_{-13.2\%}$	$36.69^{+2.1\%}_{-4.9\%}$	–
Full Theory	$19.85^{+27.6\%}_{-20.5\%}$	$32.88^{+13.5\%}_{-12.5\%}$	–	–
NLO-i. HTL	–	$32.88^{+13.5\%}_{-12.5\%}$	$38.66^{+5.3\%}_{-7.7\%}$	$39.56^{+0.64\%}_{-2.7\%}$

$\sqrt{s}$ = 14 TeV

PDF4LHC15_nlo/nnlo

$m_H = 125$ GeV $\quad m_T = 173$ GeV

$\mu_R = \mu_F = \frac{m_{HH}}{2}$

$\mu \in \left[\frac{\mu_0}{2}, 2\mu_0\right]$ $\quad$ (7 – point)

Chen, Li, Shao, Wang 19, 19; Grazzini, Heinrich, SPJ, Kallweit, Kerner, Lindert, Mazzitelli 18;
de Florian, Grazzini, Hanga, Kallweit, Lindert, Maierhöfer, Mazzitelli, Rathlev 16; Maltoni, Vryonidou, Zaro 14;
Borowka, Greiner, Heinrich, SPJ, Kerner, Schlenk, Schubert, Zirke 16; Dawson, Dittmaier, Spira 98; Glover, van der Bij 88

If we trust the NLO + N^{m}LO HTL combinations

Scale: $+2.1\,\%$ / $-4.9\,\%$ $\qquad$ PDF+α_s: $\pm 2.2\,\%$ $\qquad$ m_T approx: $\pm 2.7\,\%$

m_T scheme: $+4.0\,\%$ / $-18.0\,\%$

HWG HH Twiki

Mass Scheme Uncertainty

Converting the top quark mass to the $\overline{\text{MS}}$ scheme

$$\frac{m(\mu)}{M} = \frac{Z_m^{\text{OS}}}{Z_m^{\overline{\text{MS}}}} \equiv \sum_{n\geq 0} \left(\frac{\alpha_s(\mu)}{2\pi}\right)^n \left(z_m^n(M) + z_m^{n,\log}(\mu)\right)$$

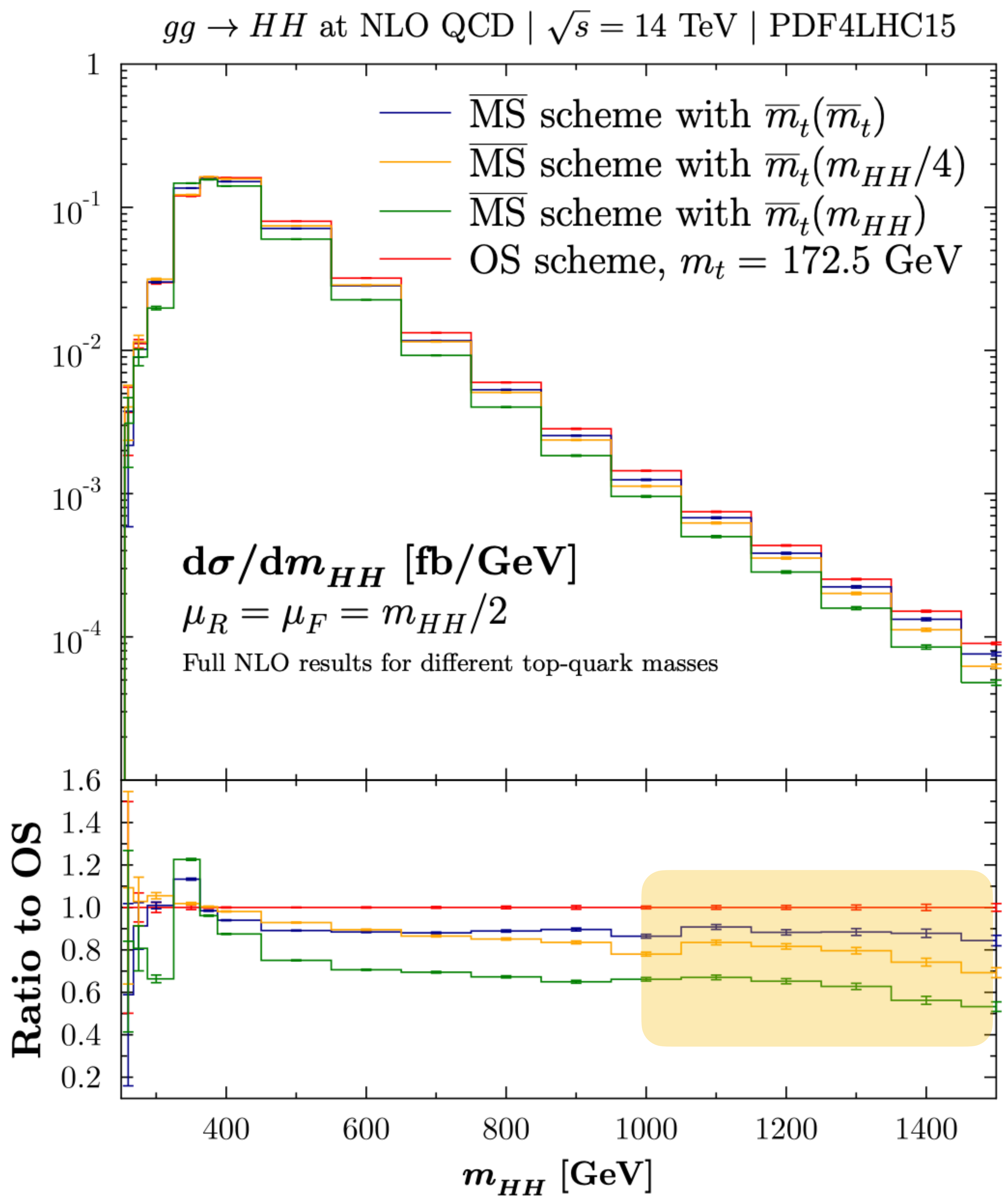
$$z_m(\mu) = 1 + \frac{\alpha_s(\mu)}{2\pi}\left(-2C_F - \frac{3}{2}C_F \ln\frac{\mu^2}{M^2}\right) + \mathcal{O}(\alpha_s^2)$$

4-loop: Marquard, Smirnov, Smirnov, Steinhauser, Wellmann 16

Top quark mass scheme uncertainty

$$\left.\frac{d\sigma(gg\to HH)}{dQ}\right|_{Q=300\text{ GeV}} = 0.0312(5)^{+9\%}_{-23\%}\,\text{fb/GeV},$$
$$\left.\frac{d\sigma(gg\to HH)}{dQ}\right|_{Q=400\text{ GeV}} = 0.1609(4)^{+7\%}_{-7\%}\,\text{fb/GeV},$$
$$\left.\frac{d\sigma(gg\to HH)}{dQ}\right|_{Q=600\text{ GeV}} = 0.03204(9)^{+0\%}_{-26\%}\,\text{fb/GeV},$$
$$\left.\frac{d\sigma(gg\to HH)}{dQ}\right|_{Q=1200\text{ GeV}} = 0.000435(4)^{+0\%}_{-30\%}\,\text{fb/GeV},$$

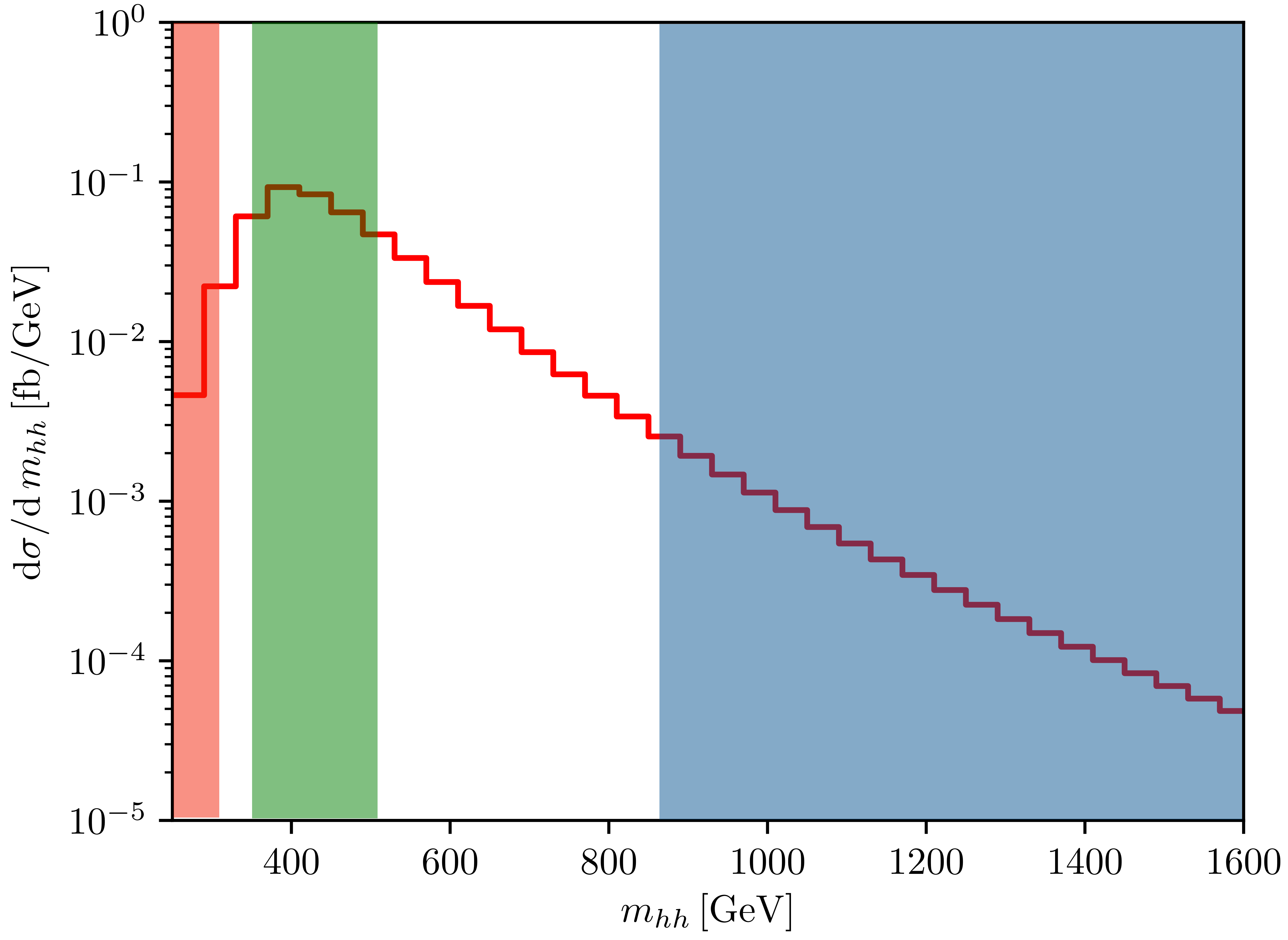
Large uncertainty comparing OS with $\overline{\text{MS}}$ mass



Baglio, Campanario, Glaus, Mühlleitner, Ronca, Spira, Streicher 18, 20, 20

Tackling Mass Scheme Uncertainties

A full/expanded NNLO calculation is motivated to reduce the mass scheme uncertainty
Expansions can help us study/understand the origin of this uncertainty



Low invariant mass:

expand in $1/m_t^2$

known to NNLO

Grigo, Hoff, Steinhauser 15;

Around Peak:

Threshold expansion

known at NLO

Gröber, Maier, Rauh 17

High energy:

small-m_t expansion

known at NLO

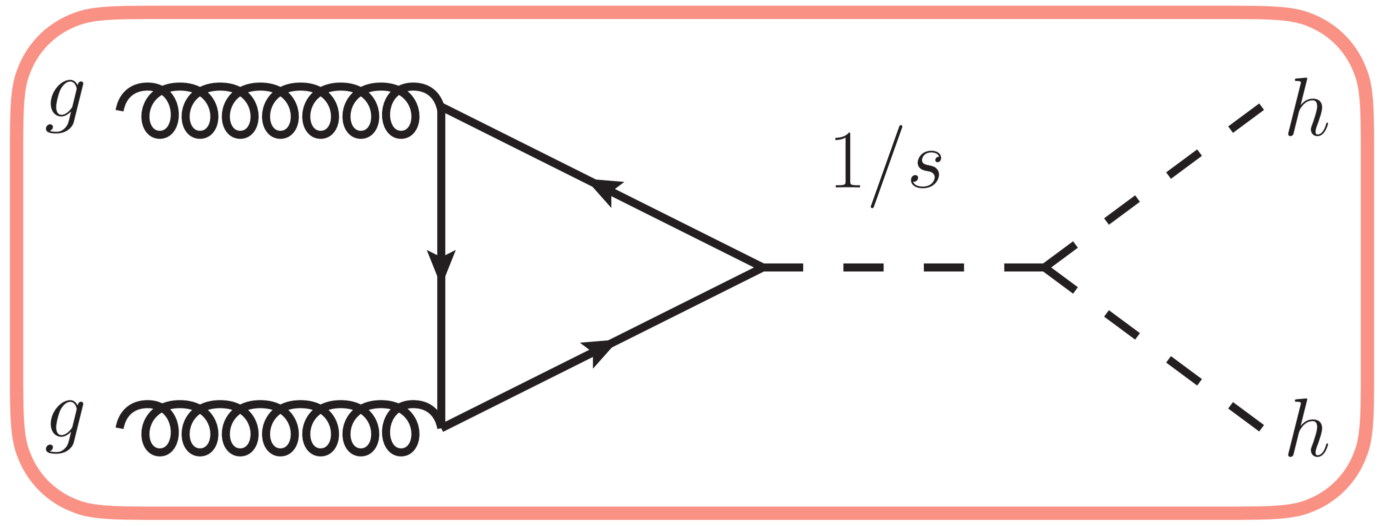
n_f, Large N_c and reducible pieces at NNLO

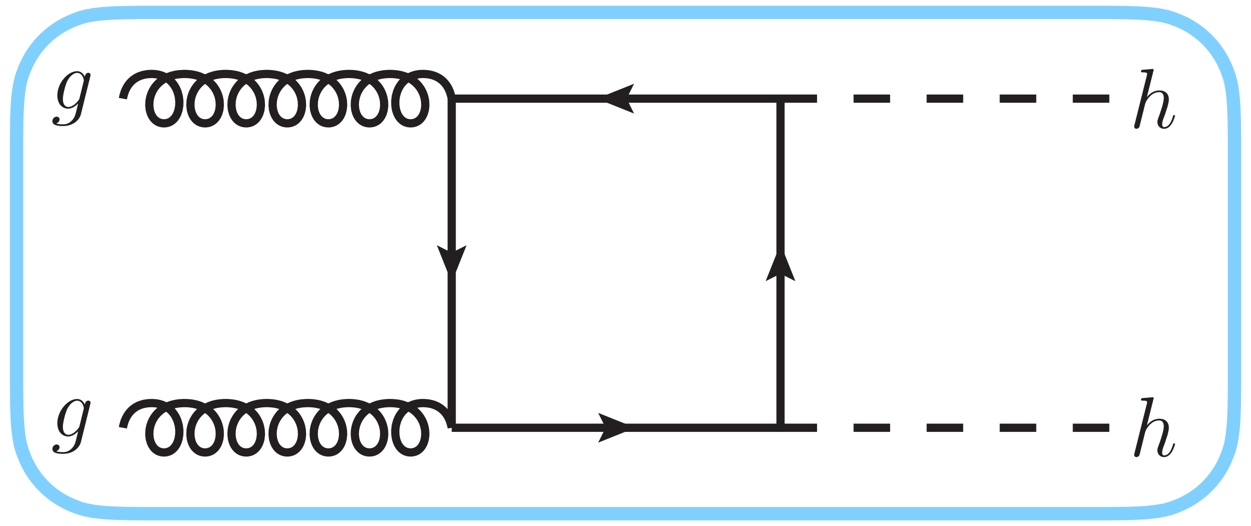
Davies, Mishima, Steinhauser, Wellmann 18, 19; Davies, Schönwald, Steinhauser 23; Davies, Schönwald, Steinhauser, Vitti 24; Davies, Schönwald, Steinhauser 25

Amplitude Structure

Structure of QCD corrections in the amplitude

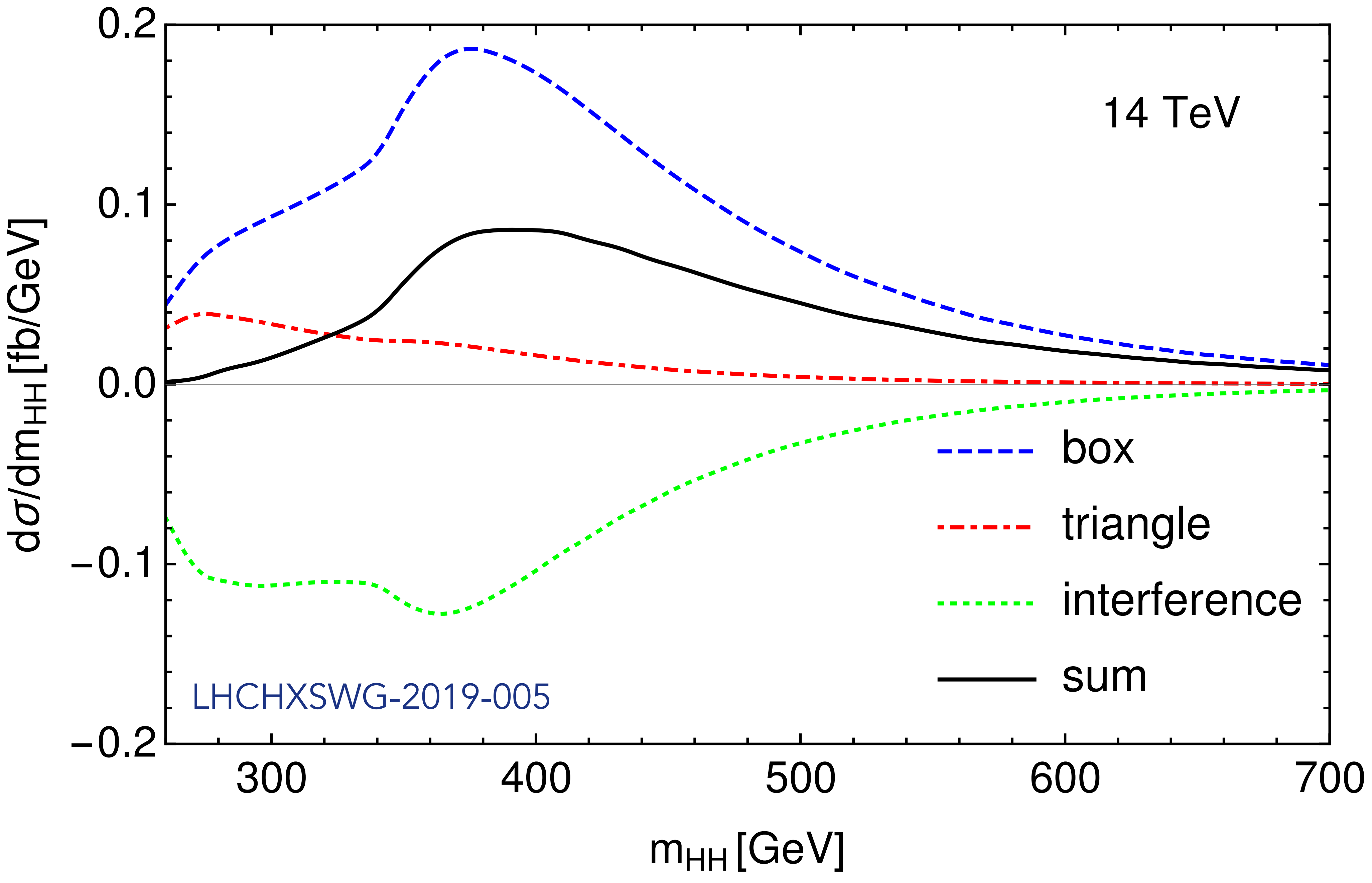
$$\mathcal{M} = \varepsilon_{1,\mu}\,\varepsilon_{2,\nu}\,\delta^{AB}\left(A_1 P_1^{\mu\nu} + A_2 P_2^{\mu\nu}\right)$$





$$A_1 = T_F \frac{G_F}{\sqrt{2}} \frac{\alpha_s}{2\pi} s \left[\frac{3m_H^2}{s - m_H^2} A_{1,y_t\lambda} + A_{1,y_t^2} \right]$$

$$A_2 = T_F \frac{G_F}{\sqrt{2}} \frac{\alpha_s}{2\pi} s \left[A_{2,y_t^2} \right]$$



Triangle type amplitudes studied in literature

Liu, Penin 17, 18; Anastasiou, Penin 20; Liu, Modi, Penin 22; Liu, Neubert, Schnubel, Wang 22

We will study the box type amplitudes in the high-energy or small quark mass limit

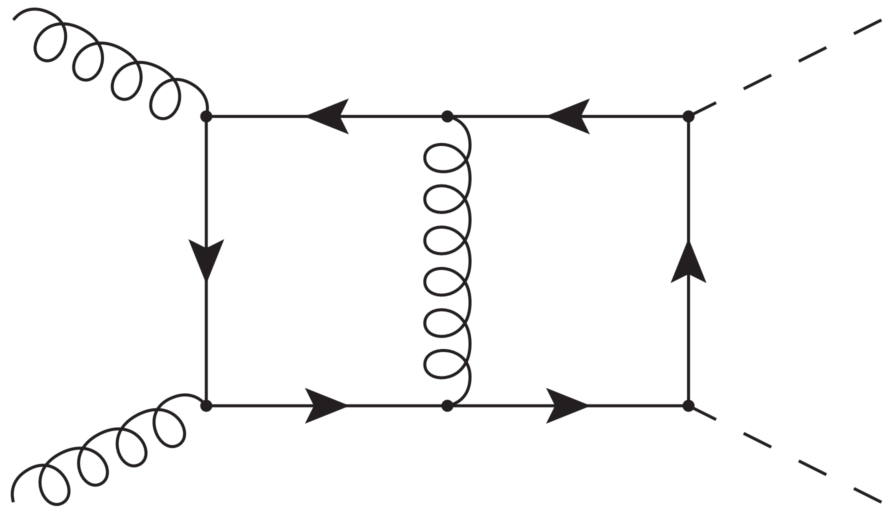
$$s, |t|, |u| \gg m_t^2 \gg m_H^2$$

High-energy limit

Expanding amplitude perturbatively $A_i^{\text{fin}} = \frac{\alpha_s}{2\pi} A_i^{(0),\text{fin}} + \left(\frac{\alpha_s}{2\pi}\right)^2 A_i^{(1),\text{fin}} + \mathcal{O}(\alpha_s^3)$ and around $m_t \sim 0$

$gg \to HH$

Davies, Mishima, Steinhauser, Wellmann 18; Baglio, Campanario, Glaus, Mühlleitner, Ronca, Spira, Streicher 20



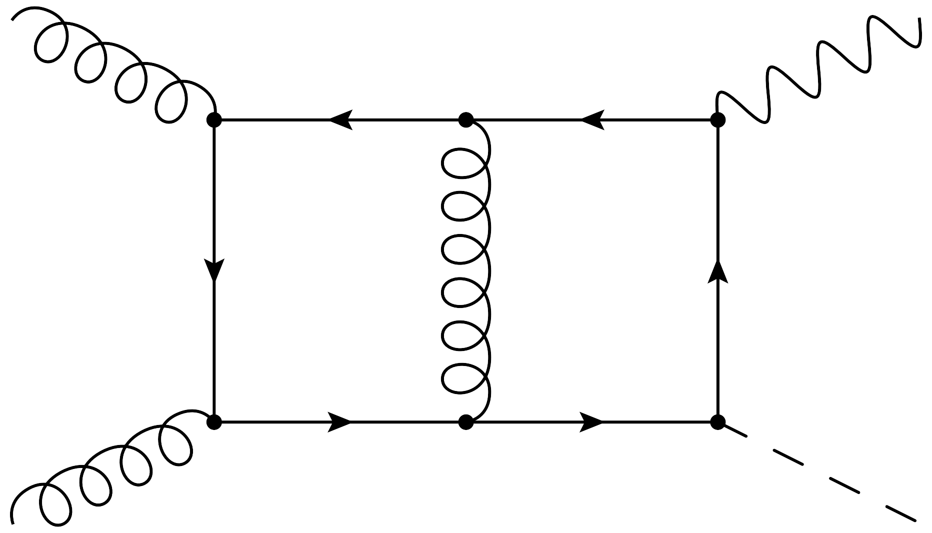
$$A_{i,y_t^2}^{(0)} \sim y_t^2 f_i(s,t) + y_t^2 \mathcal{O}(m_t^2)$$

$$A_{i,y_t^2}^{(1)} \sim 3 C_F A_i^{(0)} \log\left[\frac{m_t^2}{s}\right] + y_t^2 \mathcal{O}(m_t^2)$$

Leading $\log(m_t^2)$ from mass counter term, converting to $\overline{\text{MS}}$ gives $\log\left[\mu_t^2/s\right]$ → scale choice of $\mu_t^2 \sim s$

$gg \to ZH$

Davies, Mishima, Steinhauser 20; Chen, Davies, Heinrich, SPJ, Kerner, Mishima, Schlenk, Steinhauser 22



$$A_i^{(0)} \sim y_t m_t f_i(s,t) \, \log^2\left[\frac{m_t^2}{s}\right]$$

$$A_i^{(1)} \sim \frac{(C_A - C_F)}{12} A_i^{(0)} \, \log^2\left[\frac{m_t^2}{s}\right]$$

Leading $\log(m_t^2)$ not coming from mass counter term ($C_A - C_F$ structure)

Goal: How does the simple structure in $gg \to HH$ arise? Does it generalise to all orders in α_s? Can we resum these logarithms?

Method of Regions

Method of Regions

Consider expanding an integral about some limit:
e.g. $p_i^2 \sim \lambda Q^2$, $p_i \cdot p_j \to \lambda Q^2$ or $m^2 \sim \lambda Q^2$ for $\lambda \to 0$

Integration and series expansion do not necessarily commute

Method of Regions

$$I(\mathbf{s}) = \sum_R I^{(R)}(\mathbf{s}) = \sum_R T_{\mathbf{t}}^{(R)} I(\mathbf{s})$$

1. Split integrand up into regions (R)
2. Series expand each region in λ
3. Integrate each expansion over the whole integration domain
4. Discard scaleless integrals (= 0 in dimensional regularisation)
5. Sum over all regions

Smirnov 91; Beneke, Smirnov 97; Smirnov, Rakhmetov 99; Pak, Smirnov 11; Jantzen 2011; …

Parameter Space

Exchange integrals over loop momenta for integrals over parameters

Lee-Pomeransky Parametrisation

Lee, Pomeransky 13

$$I(s) = \frac{\Gamma(D/2)}{\Gamma\left((L+1)D/2-\nu\right)\prod_{e\in G}\Gamma(\nu_e)}\int_0^\infty [dx]\, x^{\nu}\left(\mathscr{G}(\mathbf{x},s)\right)^{-D/2}$$

$$\mathscr{G}(\mathbf{x};s) = \mathscr{U}(\mathbf{x}) + \mathscr{F}(\mathbf{x};s)$$

$\mathscr{U}, \mathscr{F}$ homogeneous polynomials of degree L and $L+1$

Finding Regions

Assuming all c_i have the same sign, rescale $\boldsymbol{s} \to \lambda^{\boldsymbol{\omega}}\boldsymbol{s}$

$$I(\mathbf{s}) \sim \int_{\mathbb{R}^N_{>0}} [\mathrm{d}\boldsymbol{x}]\, \boldsymbol{x}^{\nu}\left(c_i\, \boldsymbol{x}^{\mathbf{r}_i}\right)^t \to \int_{\mathbb{R}^N_{>0}} [\mathrm{d}\boldsymbol{x}]\, \boldsymbol{x}^{\nu}\left(c_i\, \boldsymbol{x}^{\mathbf{r}_i}\lambda^{r_{i,N+1}}\right)^t \to \mathscr{N}^{N+1}$$

Newton Polytope

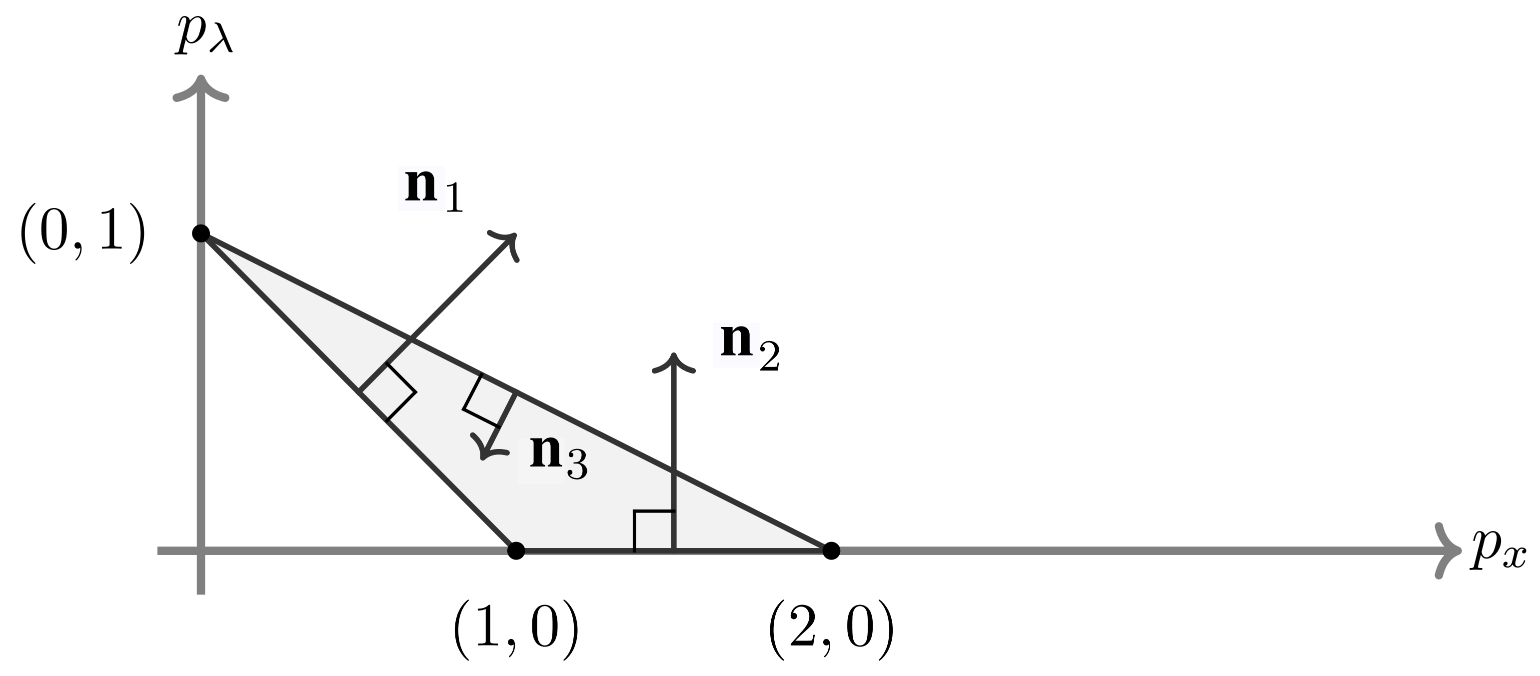
Normal vectors w/ positive λ component define change of variables $\mathbf{n}_f = (v_1, \ldots, v_N)$

$$\boldsymbol{x} = \lambda^{\mathbf{n}_f}\mathbf{y}$$

Pak, Smirnov 10; Semenova, A. Smirnov, V. Smirnov 18

Example

$p(x,\lambda) = \lambda + x + x^2$

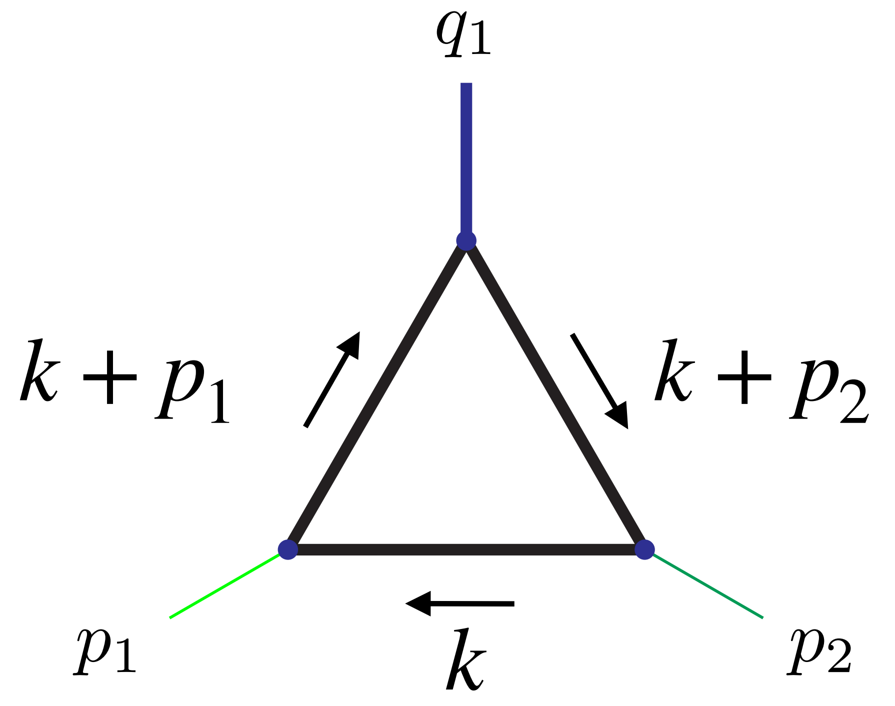


$1,2 \in F^+$

$3 \notin F^+$

Original integral I approximated as $I = \sum_{f\in F^+} I^{(f)} + \ldots$

Feynman Integral Example



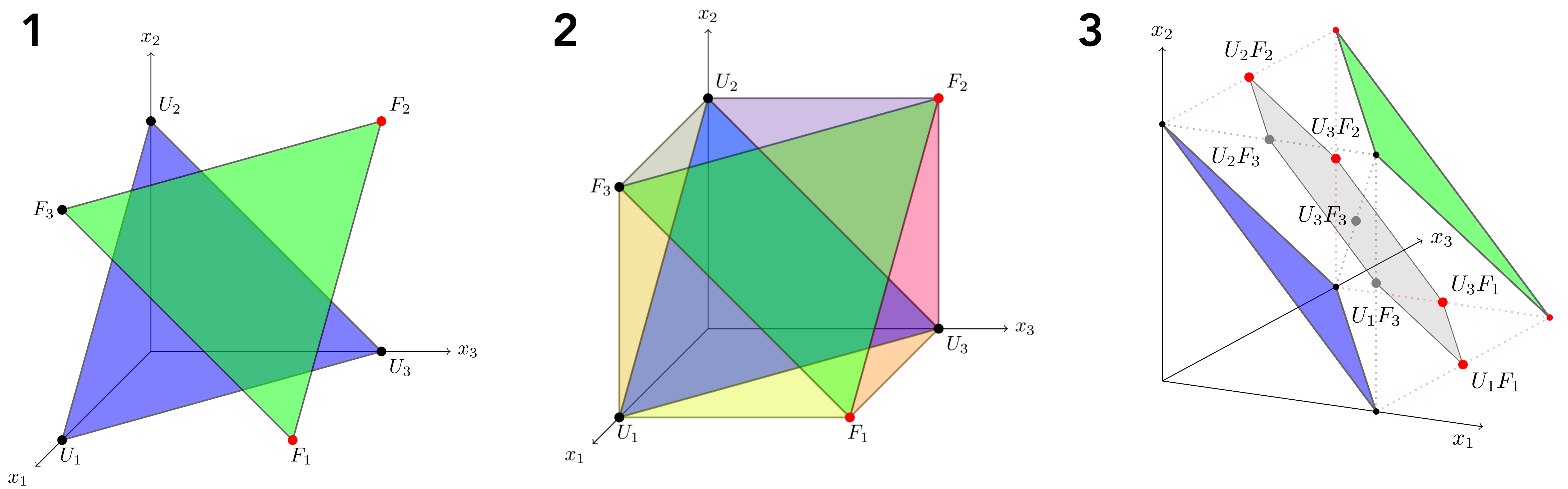
$$I = i\pi^{D/2}\mu^{4-D}\int \mathrm{d}^D k \frac{1}{(k+p_1)^2(k+p_2)^2(k^2)}$$

$$\mathscr{U}(\mathbf{x}) = x_1 + x_2 + x_3$$

$$\mathscr{F}(\mathbf{x},\mathbf{s}) = (-p_1^2)\,\lambda x_1 x_3 + (-p_2)^2\,\lambda x_2 x_3 + (-q_1)^2 x_1 x_2$$

On-shell limit of a 1-loop triangle

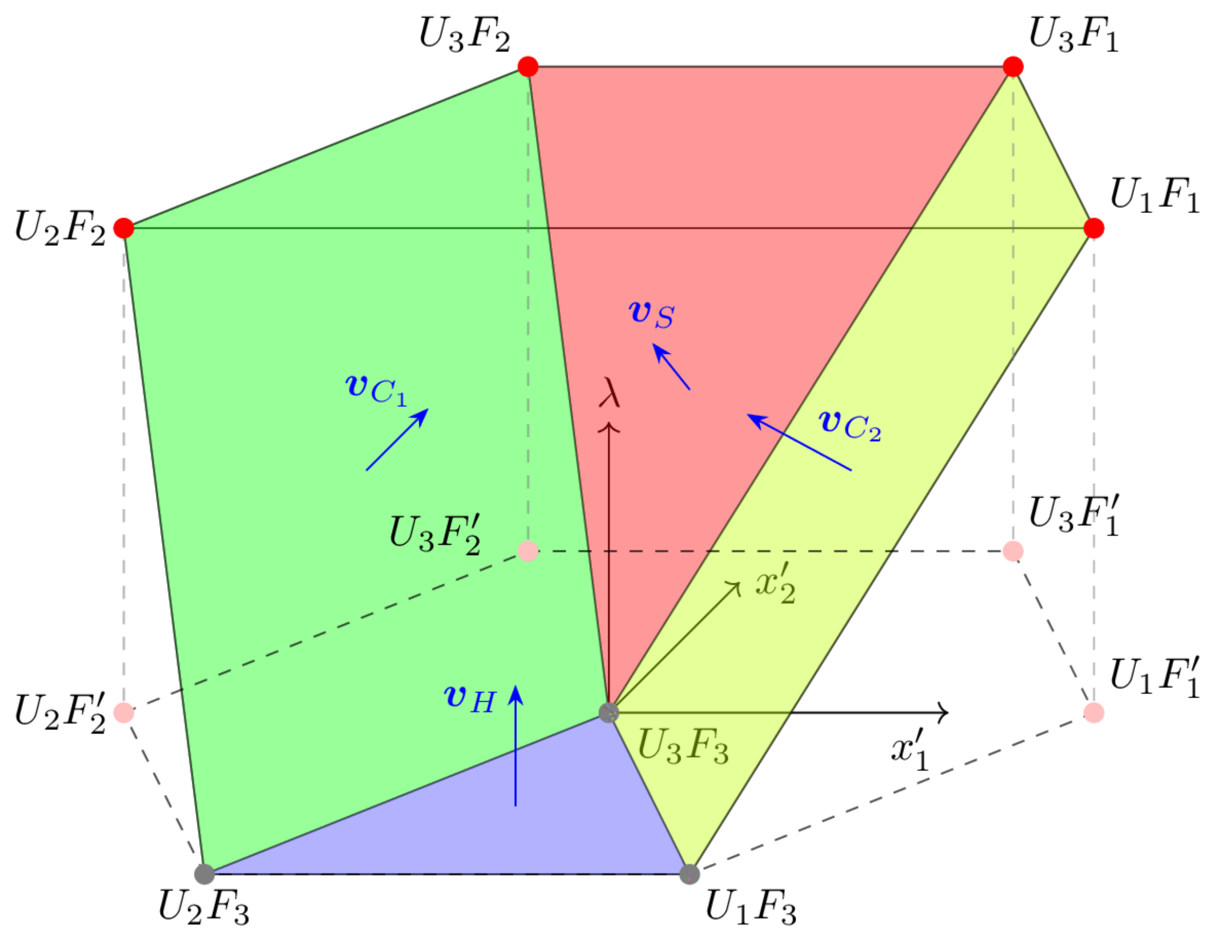
$p_1^2 \sim p_2^2 \sim \lambda q_1^2$ for $\lambda \to 0$



$\mathscr{U} + \mathscr{F}$ → $\mathscr{U}\mathscr{F}$

Cayley trick

Region vectors ↔ facets



Building Bridges

The scaling of propagators and Lee-Pomeransky parameters are related, using Schwinger parameters $\tilde{x}_e$

$$\frac{1}{D_n^{\nu_e}} = \frac{1}{\Gamma(\nu_e)} \int_0^\infty \frac{\mathrm{d}\tilde{x}_e}{\tilde{x}_e} \, \tilde{x}_e^{\nu_e} \, e^{-\tilde{x}_e D_e} \text{, with } x_e \propto \tilde{x}_e$$

$$(D_1^{-1}, \ldots, D_N^{-1}) \sim (\tilde{x}_1, \ldots, \tilde{x}_N) \sim (x_1, \ldots, x_N)$$

Triangle example

$$\begin{aligned}
\text{Hard}: \quad & (D_1^{-1}, D_2^{-1}, D_3^{-1}) \sim (\lambda^0, \lambda^0, \lambda^0), & (x_1, x_2, x_3) \sim (\lambda^0, \lambda^0, \lambda^0) \\
\text{Collinear to } \mathrm{p}_1: \quad & (D_1^{-1}, D_2^{-1}, D_3^{-1}) \sim (\lambda^{-1}, \lambda^0, \lambda^{-1}), & (x_1, x_2, x_3) \sim (\lambda^{-1}, \lambda^0, \lambda^{-1}) \\
\text{Collinear to } \mathrm{p}_2: \quad & (D_1^{-1}, D_2^{-1}, D_3^{-1}) \sim (\lambda^0, \lambda^{-1}, \lambda^{-1}), & (x_1, x_2, x_3) \sim (\lambda^0, \lambda^{-1}, \lambda^{-1}) \\
\text{Soft}: \quad & (D_1^{-1}, D_2^{-1}, D_3^{-1}) \sim (\lambda^{-1}, \lambda^{-1}, \lambda^{-2}), & (x_1, x_2, x_3) \sim (\lambda^{-1}, \lambda^{-1}, \lambda^{-2})
\end{aligned}$$

Feynman Integral Example

Regions in parameter space can be matched to specific scalings (modes) of the loop momenta

$p_1 = (p_1^+, p_1^-, p_1^\perp) \sim Q(\lambda, 1, \lambda^{\frac{1}{2}}),\ p_2 \sim Q(1, \lambda, \lambda^{\frac{1}{2}})$

$$\text{Hard}: k_H^\mu \sim (1,1,1)\,Q$$
$$\text{Collinear to } \mathrm{p}_1: k_{J_1}^\mu \sim (\lambda, 1, \lambda^{\frac{1}{2}})\,Q$$
$$\text{Collinear to } \mathrm{p}_2: k_{J_2}^\mu \sim (1, \lambda, \lambda^{\frac{1}{2}})\,Q$$
$$\text{Soft}: k_S^\mu \sim (\lambda, \lambda, \lambda)\,Q$$

Translation from scaling of propagators to loop momentum modes is a one-to-many map

Also depends on the momentum routing

$$I_H = i\pi^{d/2}\mu^{4-D}\int \mathrm{d}^D k \frac{1}{(k^2 + 2k^+ \cdot p_1^-)(k^2 + 2k^- \cdot p_2^+)(k^2)}$$
$$I_{C_1} = i\pi^{d/2}\mu^{4-D}\int \mathrm{d}^D k \frac{1}{(k + p_1)^2(2k^- \cdot p_2^+)(k^2)}$$
$$I_{C_2} = i\pi^{d/2}\mu^{4-D}\int \mathrm{d}^D k \frac{1}{(2k^- \cdot p_1^+)(k + p_2)^2(k^2)}$$
$$I_S = i\pi^{d/2}\mu^{4-D}\int \mathrm{d}^D k \frac{1}{(2k^+ \cdot p_1^- + p_1^2)(2k^- \cdot p_2^+ + p_2^2)(k^2)}$$

$\longrightarrow$

$$I_H = \frac{\Gamma(1+\epsilon)}{Q^2}\left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln\frac{\mu^2}{Q^2} + \frac{1}{2}\ln^2\frac{\mu^2}{Q^2} - \frac{\pi^2}{6} + \mathcal{O}(\lambda)\right)$$
$$I_{C_1} = \frac{\Gamma(1+\epsilon)}{Q^2}\left(-\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{\mu^2}{P_1^2} - \frac{1}{2}\ln^2\frac{\mu^2}{P_1^2} + \frac{\pi^2}{6} + \mathcal{O}(\lambda)\right)$$
$$I_{C_2} = \frac{\Gamma(1+\epsilon)}{Q^2}\left(-\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{\mu^2}{P_2^2} - \frac{1}{2}\ln^2\frac{\mu^2}{P_2^2} + \frac{\pi^2}{6} + \mathcal{O}(\lambda)\right)$$
$$I_S = \frac{\Gamma(1+\epsilon)}{Q^2}\left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln\frac{\mu^2 Q^2}{P_2^2 P_1^2} + \frac{1}{2}\ln^2\frac{\mu^2 Q^2}{P_2^2 P_1^2} + \frac{\pi^2}{6} + \mathcal{O}(\lambda)\right)$$
$$I = I_H + I_{C_1} + I_{C_2} + I_S = \frac{1}{Q^2}\left(\ln\frac{Q^2}{P_2^2}\ln\frac{Q^2}{P_1^2} + \frac{\pi^2}{3} + \mathcal{O}(\lambda)\right)$$

Becher, Broggio, Ferroglia 14

Additional Regulators

MoR subdivides $\mathcal{N}(I) \to \{\mathcal{N}(I^R)\} \implies$ new internal facets F^{int}

New facets can introduce spurious singularities not regulated by dimensional regularisation

$$\mathcal{N}(I^{(R)}) = \bigcap_{f \in F} \left\{ \mathbf{m} \in \mathbb{R}^N \mid \langle \mathbf{m}, \mathbf{n}_f \rangle + a_f \geq 0 \right\}$$

$$I \sim \sum_{\sigma \in \Delta^T_{\mathcal{N}}} |\sigma| \int_{\mathbb{R}^N_{>0}} \left[\mathrm{d}\mathbf{y}_f \right] \prod_{f \in \sigma} y_f^{\langle \mathbf{n}_f, \boldsymbol{\nu} \rangle + \frac{D}{2} a_f} \left(c_i \prod_{f \in \sigma} y_f^{\langle \mathbf{n}_f, \mathbf{r}_i \rangle + a_f} \right)^{-\frac{D}{2}}$$

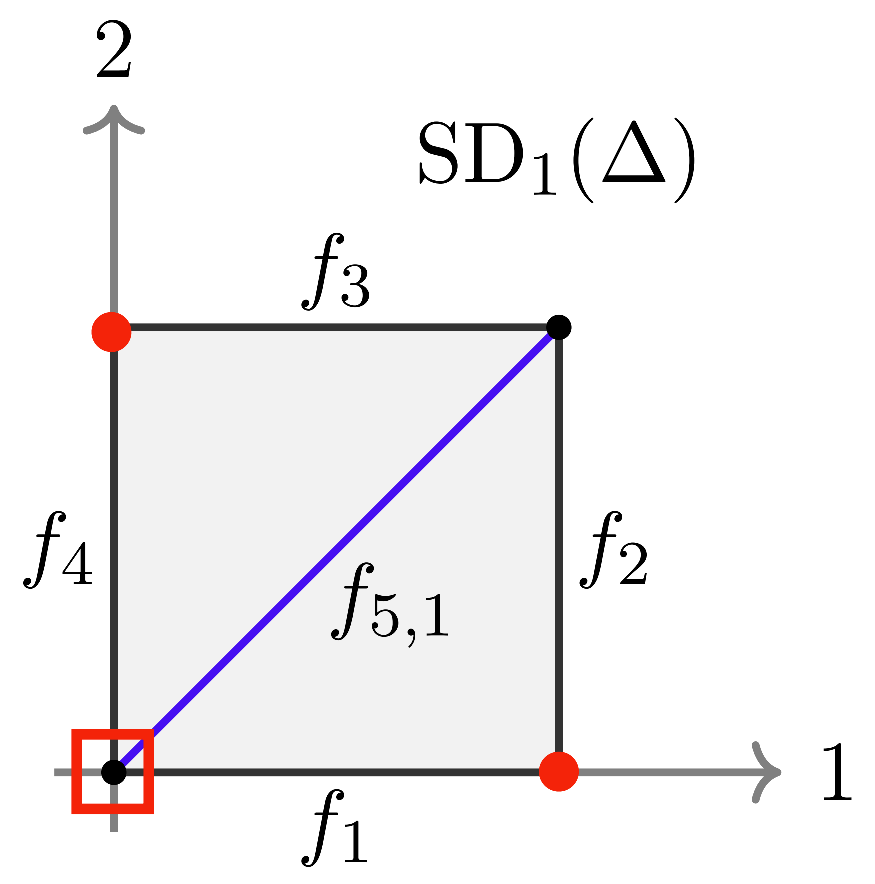
If $f \in F^{\text{int}}$ have $a_f = 0$ need analytic regulators $\boldsymbol{\nu} \to \boldsymbol{\nu} + \boldsymbol{\delta\nu}$

Situation can be automatically detected in parameter space

Heinrich, Jahn, SJ, Kerner, Langer, Magerya, Põldaru, Schlenk, Villa 21; Schlenk 16

Example

$$P_1(x, \lambda) = 1 + \lambda x_1 + x_1 x_2 + \lambda x_2$$



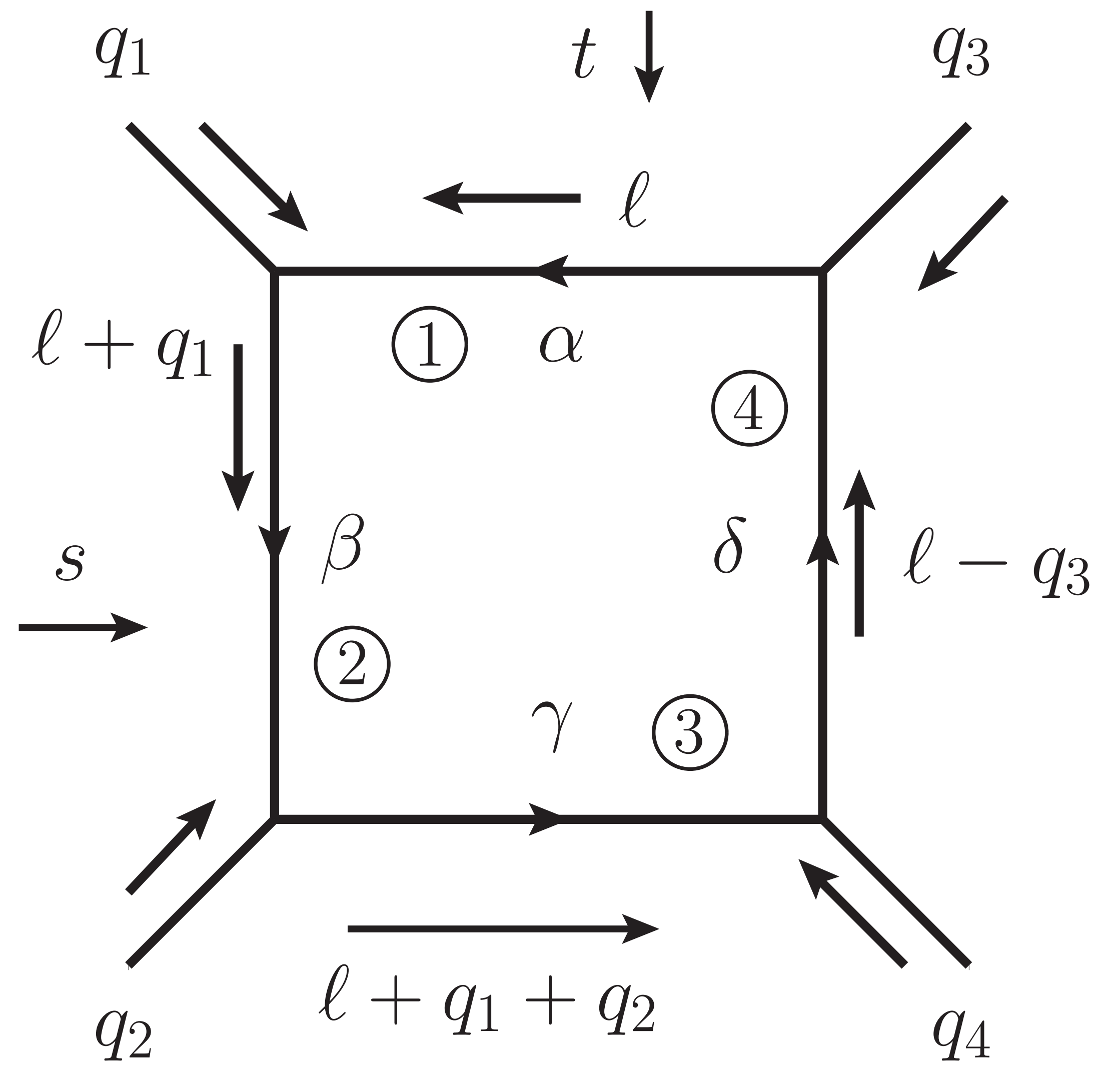
The need for additional regulators is sometimes called a **collinear anomaly** Becher, Neubert 11;

Analytic regulators must be set to zero before ϵ, can give rise to logarithms of the parameter being expanded

Application to $gg \to HH$:
Scalar Integral Level

High-Energy Expansion of $gg \to HH$ @ 1-loop

Limit: $s, |t|, |u| \gg m_t^2 \gg m_H^2$, $m_H^2 \to 0$ and $\lambda \sim m_t/Q$



$$\int \frac{d^d\ell}{(2\pi)^d} \frac{1}{\ell^2 - m_t^2} \frac{1}{(\ell + q_1)^2 - m_t^2} \frac{1}{(\ell + q_1 + q_2)^2 - m_t^2} \frac{1}{(\ell - q_3)^2 - m_t^2}$$

$\downarrow$

Hard region $\mathbf{u}^{(0)} = (0,0,0,0)$

$\downarrow$

Every propagator scales as λ^0

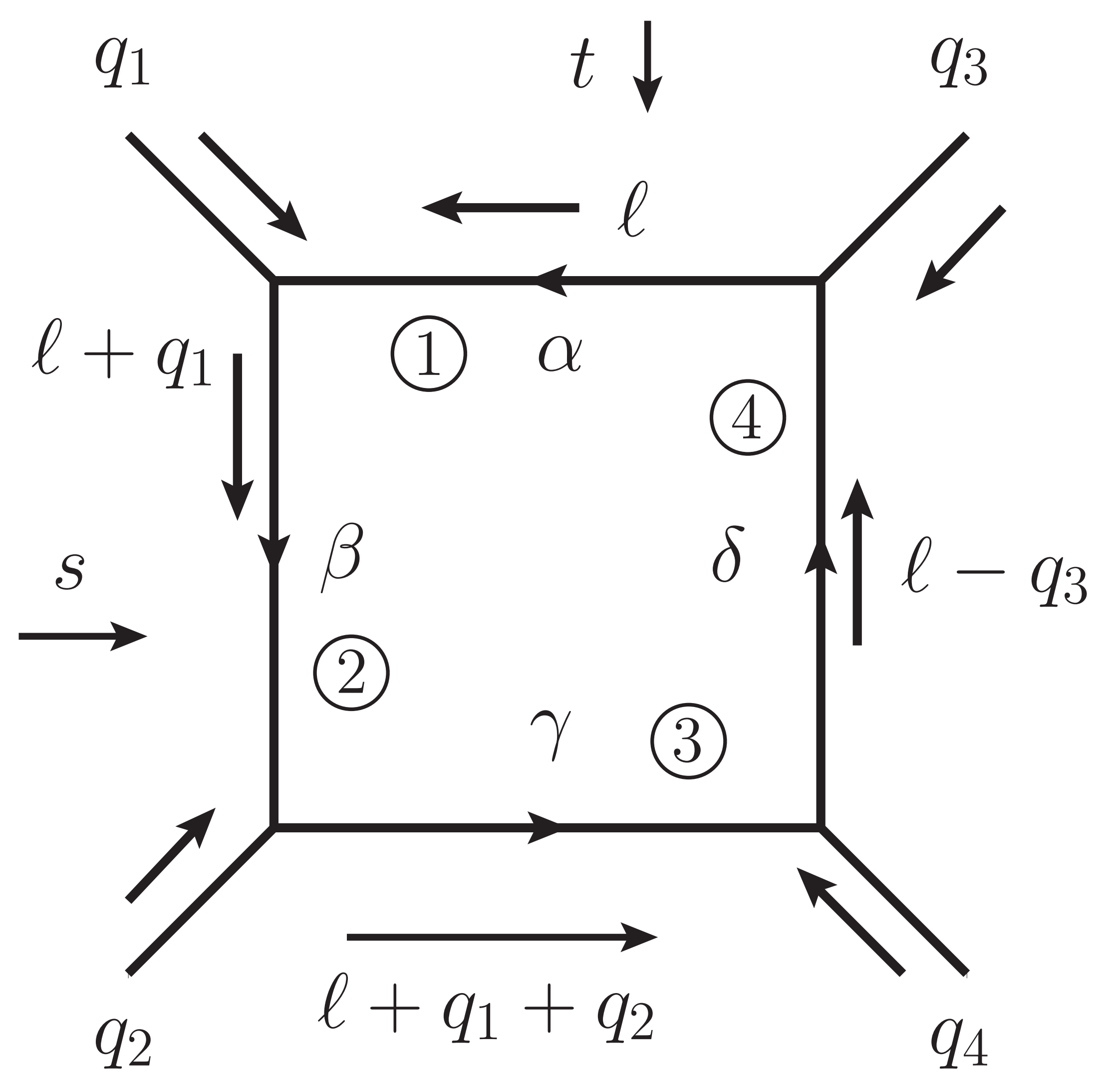
Achieved by **hard scaling** of the loop momenta $\ell^\mu = Q(1,1,1)$

$\downarrow$

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{1}{\ell^2} \frac{1}{(\ell + q_1)^2} \frac{1}{(\ell + q_1 + q_2)^2} \frac{1}{(\ell - q_3)^2} \sim \lambda^0$$

High-Energy Expansion of $gg \to HH$ @ 1-loop

Limit: $s, |t|, |u| \gg m_t^2 \gg m_H^2$, $m_H^2 \to 0$ and $\lambda \sim m_t/Q$



Automatically find remaining regions in parameter space

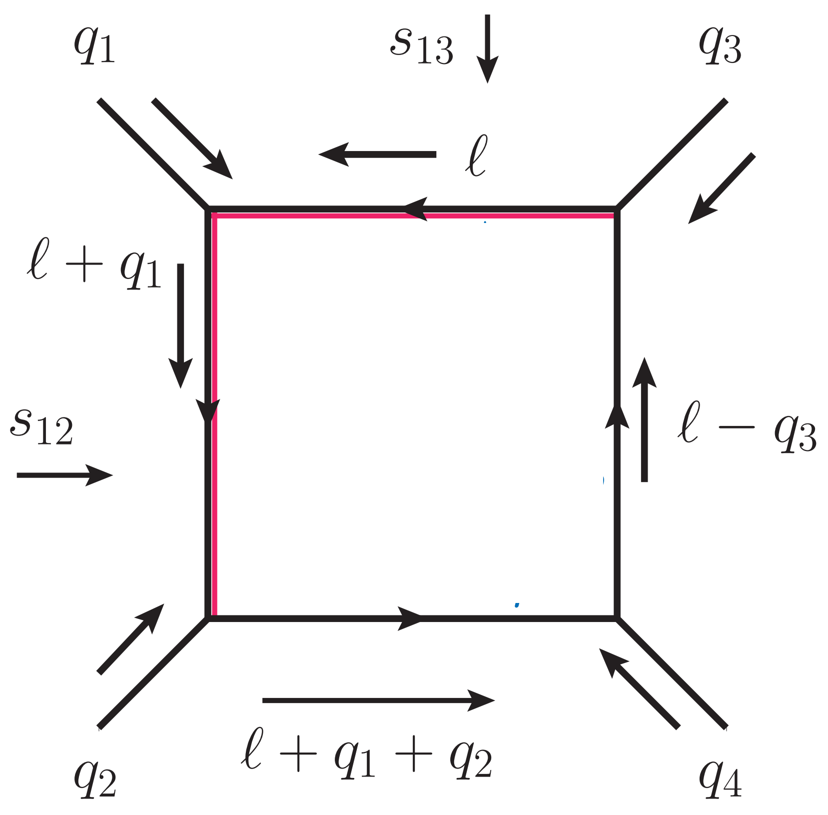
$\mathbf{u}^R$	order	interpretation	routing
(-2, -2, 0, 0)	$4 - 2(\epsilon + \alpha + \beta)$	c_1	ℓ
(0, -2, -2, 0)	$4 - 2(\epsilon + \beta + \gamma)$	c_2	$\ell - q_1$
(-2, 0, 0, -2)	$4 - 2(\epsilon + \alpha + \delta)$	c_3	$\ell + q_3$
(0, 0, -2, -2)	$4 - 2(\epsilon + \gamma + \delta)$	c_4	$\ell - q_1 - q_2$
(0, 0, 0, 0)	0	h	n/a

Using a set of possible loop momenta modes can systematically search for momentum routing to give a momentum space interpretation

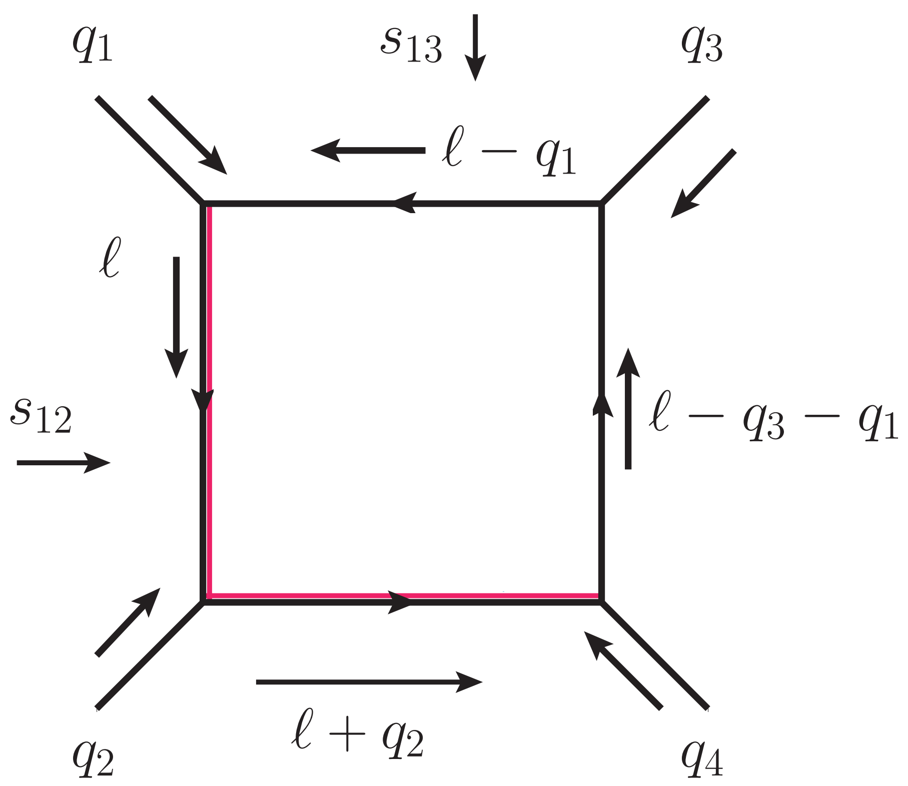
Implemented in pySecDec by Y. Ulrich (TBA)

High-Energy Expansion of $gg \to HH$ @ 1-loop

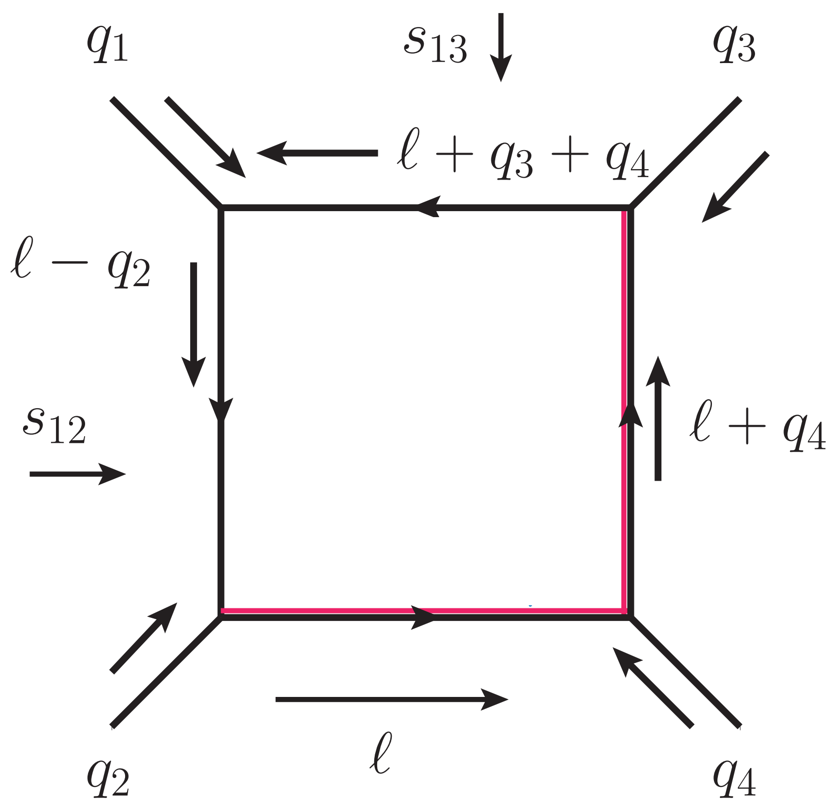
Collinear Regions



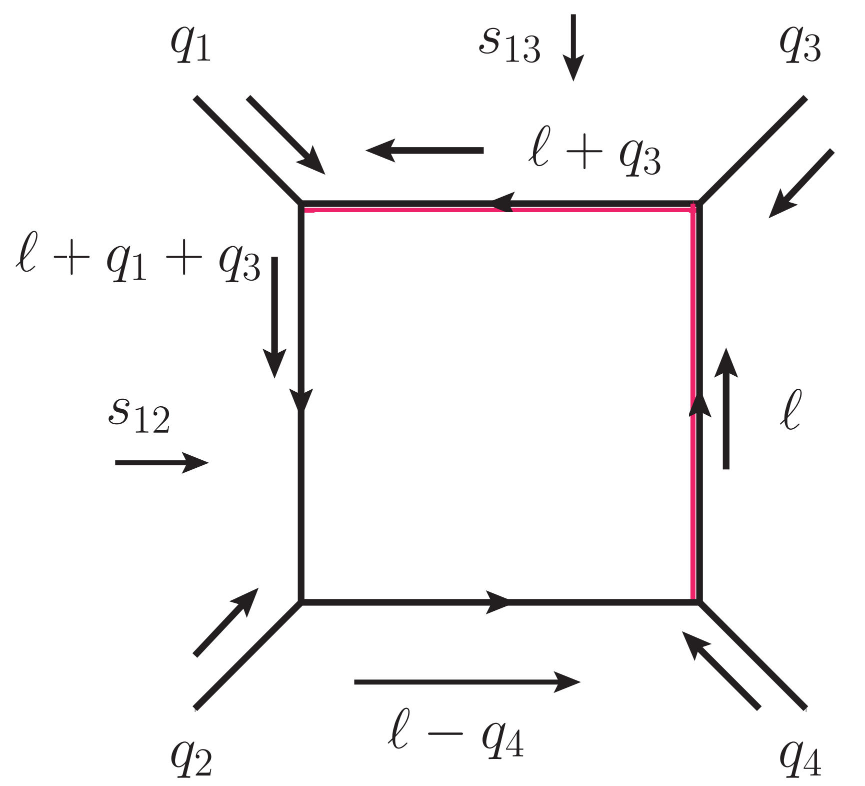
$\mathbf{u}^{(1)} = (-2, -2, 0, 0)$



$\mathbf{u}^{(2)} = (0, -2, -2, 0)$



$\mathbf{u}^{(4)} = (0, 0, -2, -2)$



$\mathbf{u}^{(3)} = (-2, 0, 0, -2)$

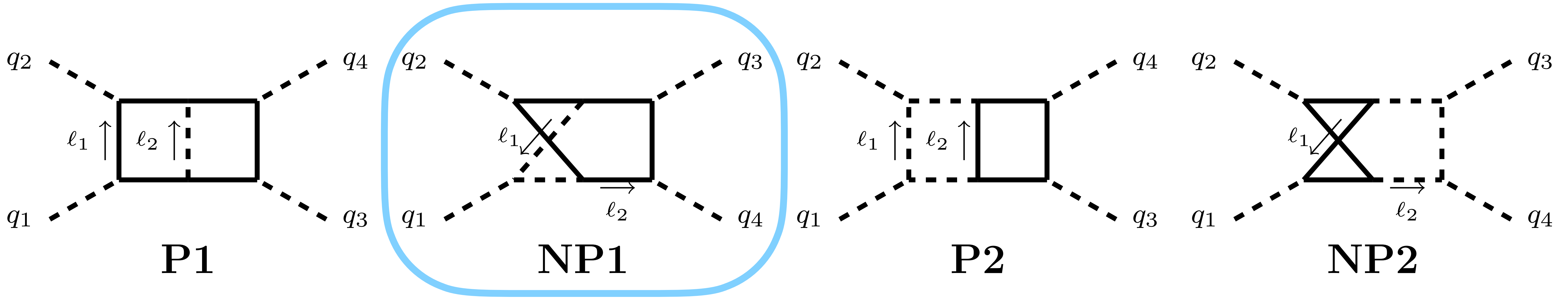
Collinear q_1 region $\quad \ell^\mu = Q(1, \lambda^2, \lambda)$

$$\int \underbrace{\frac{d^d\ell}{(2\pi)^d}}_{\lambda^4} \underbrace{\frac{1}{\ell^2 - m_t^2}}_{\frac{1}{\lambda^2}} \underbrace{\frac{1}{(\ell + q_1)^2 - m_t^2}}_{\frac{1}{\lambda^2}} \frac{1}{2(\ell + q_1)\cdot q_2} \frac{1}{-2\ell \cdot q_3}$$

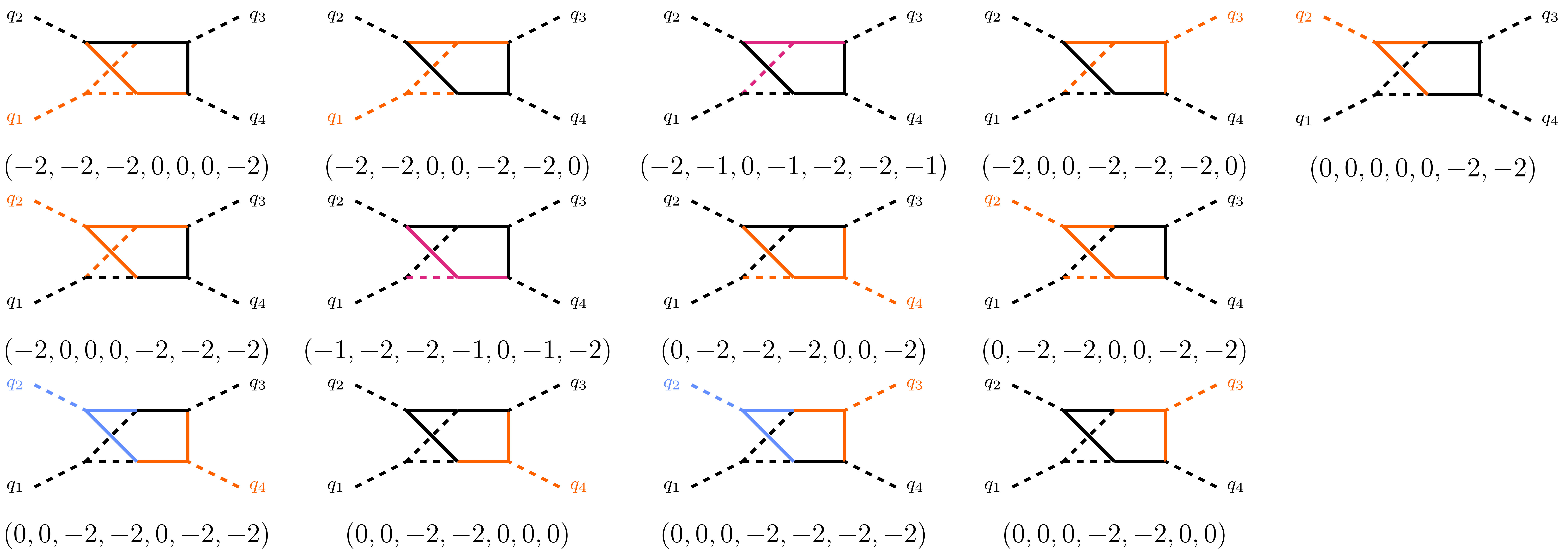
Collinear regions are also leading power at the level of scalar integrals!

High-Energy Expansion of $gg \to HH$ @ 2-loops

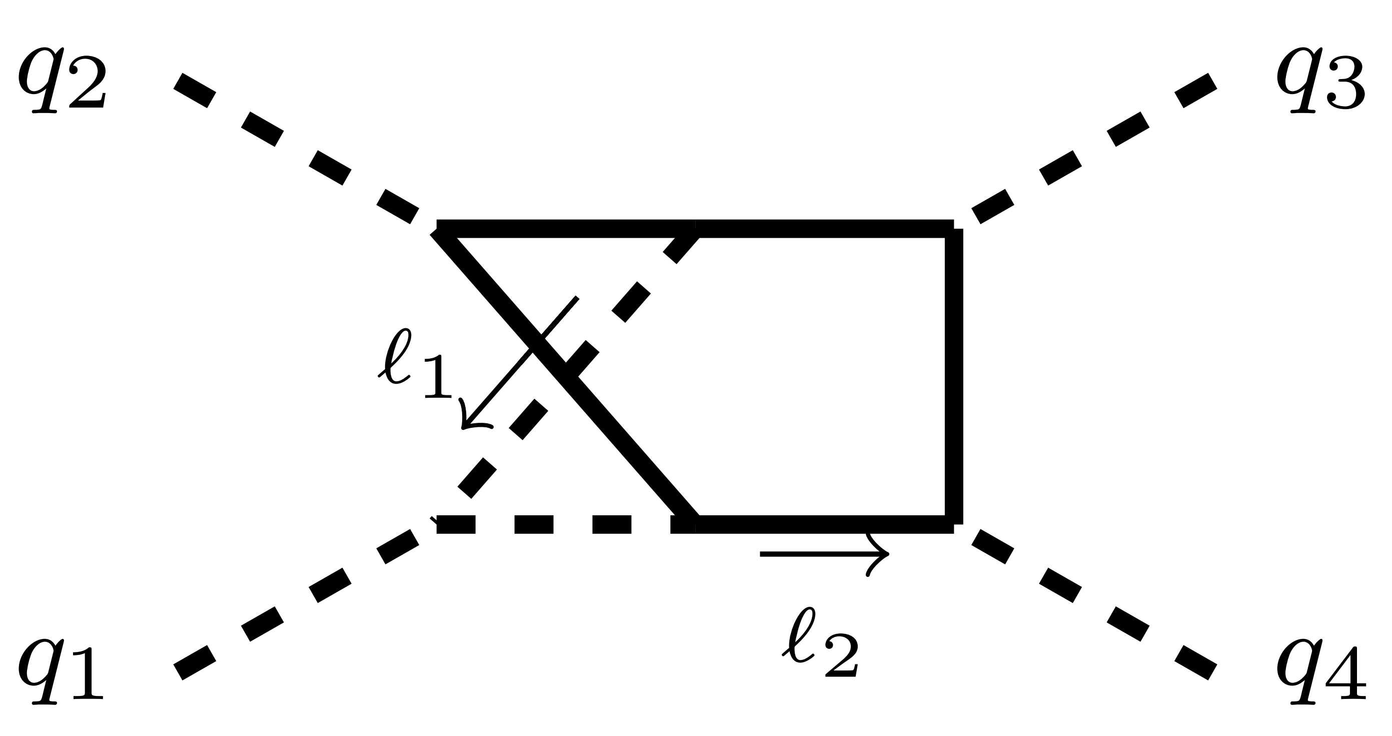
Topologies



Regions (Parameter Space)



High-Energy Expansion of $gg \to HH$ @ 2-loops



New features:

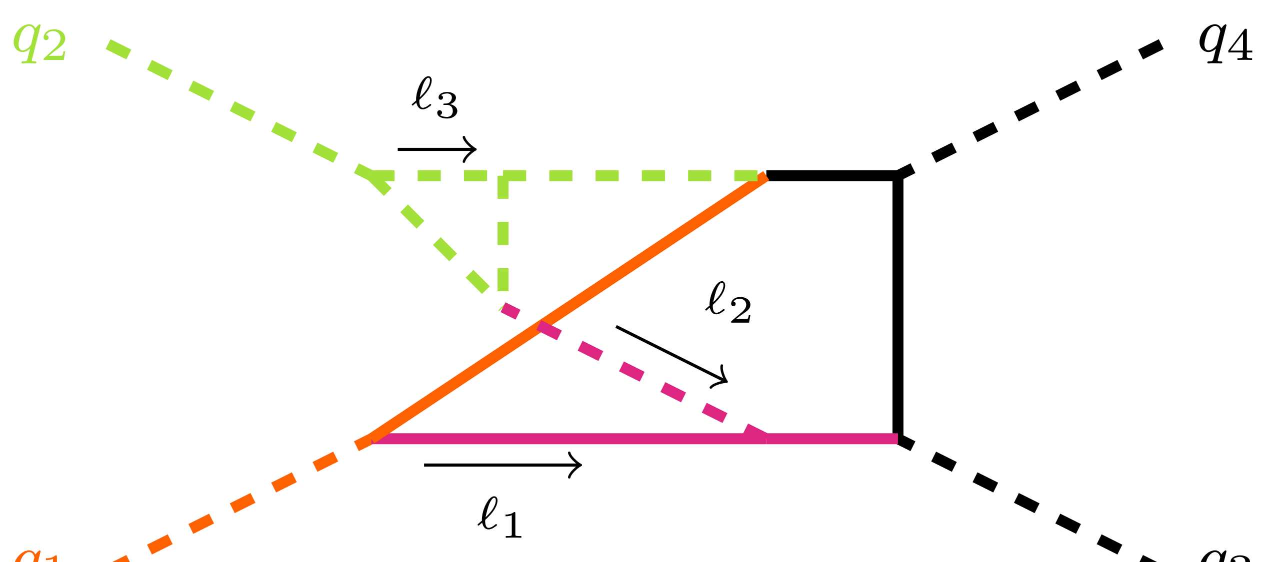
1. Soft modes appear $l_S^\mu = Q(\lambda, \lambda, \lambda)$
2. Soft regions are power enhanced at level of scalar integral

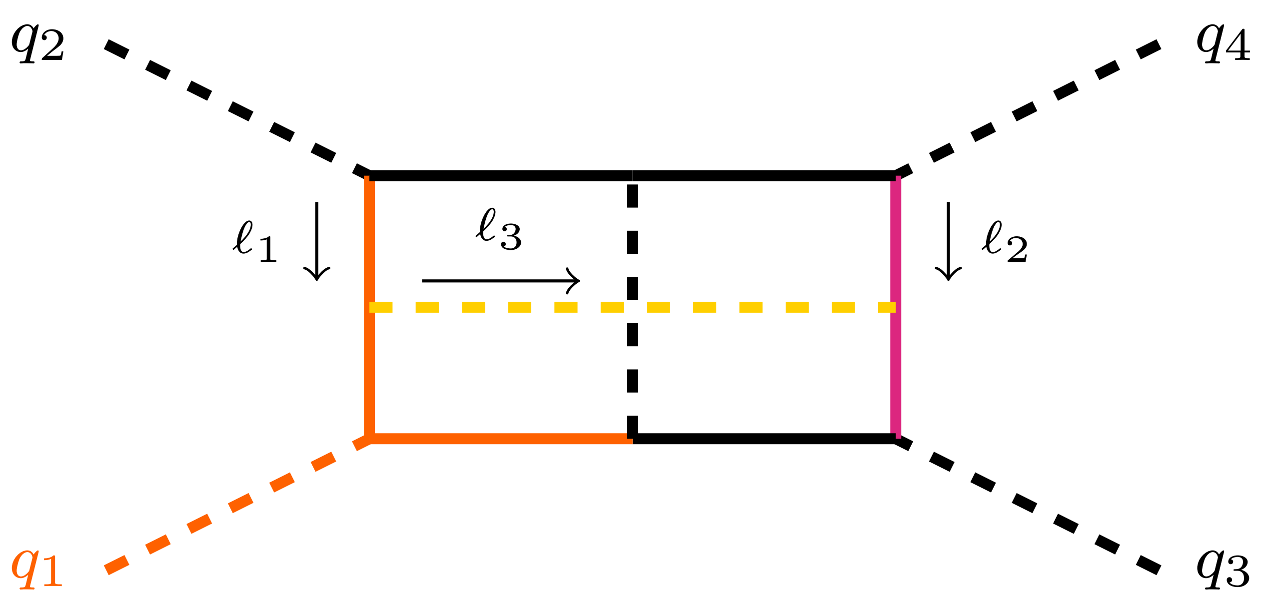
$\mathbf{u}^R$	order	interpretation	routing
(-2, -2, -2, 0, 0, 0, -2)	-4ϵ	c_1c_1	$\ell_1,\ \ell_2$
(-2, -2, 0, 0, -2, -2, 0)	-4ϵ	c_1c_1	$\ell_1,\ \ell_2 - q_3 - q_4$
(-2, -1, 0, -1, -2, -2, -1)	$-1-4\epsilon$	ss	$\ell_1,\ \ell_2 - q_3 - q_4$
(-2, 0, 0, -2, -2, -2, 0)	-4ϵ	c_3c_3	$\ell_1,\ \ell_2 - q_4$
(-2, 0, 0, 0, -2, -2, -2)	-4ϵ	c_2c_2	$\ell_1,\ \ell_2 - q_3 - q_4$
(-1, -2, -2, -1, 0, -1, -2)	$-1-4\epsilon$	ss	$\ell_1 - q_1,\ \ell_2$
(0, -2, -2, -2, 0, 0, -2)	-4ϵ	c_4c_4	$\ell_1 - q_1,\ \ell_2$
(0, -2, -2, 0, 0, -2, -2)	-4ϵ	c_2c_2	$\ell_1 - q_1,\ \ell_2$
(0, 0, -2, -2, 0, -2, -2)	-4ϵ	$c_4\bar{c}_2$	$\ell_1 - \ell_2 + q_3 + q_4,\ \ell_1$
(0, 0, -2, -2, 0, 0, 0)	-2ϵ	c_4h	$\ell_1 - \ell_2 + q_3 + q_4,\ \ell_1$
(0, 0, 0, -2, -2, -2, -2)	-4ϵ	$c_3\bar{c}_2$	$\ell_1 - \ell_2 + q_3,\ \ell_1 - q_4$
(0, 0, 0, -2, -2, 0, 0)	-2ϵ	c_3h	$\ell_1 - \ell_2 + q_3,\ \ell_1 - q_4$
(0, 0, 0, 0, 0, -2, -2)	-2ϵ	hc_2	$\ell_1,\ \ell_1 + \ell_2 - q_3 - q_4$
(0, 0, 0, 0, 0, 0, 0)	0	hh	n/a

Can again find momentum space interpretation

High-Energy Expansion of $gg \to HH$ @ 3-loops

Considering $gg \to HH$ diagrams at 3-loops we systematically checked for new loop momenta modes





Indeed find new modes entering

Hard-collinear $l^{\mu}_{HC_i} = Q(1, \lambda, \lambda^{\frac{1}{2}})$

Soft-collinear $l^{\mu}_{SC_i} = Q(\lambda, \lambda^2, \lambda^{\frac{3}{2}})$

Ultra-soft $l^{\mu}_{US} = Q(\lambda^2, \lambda^2, \lambda^2)$

Expect new modes entering at each loop order, consistent with results in the literature

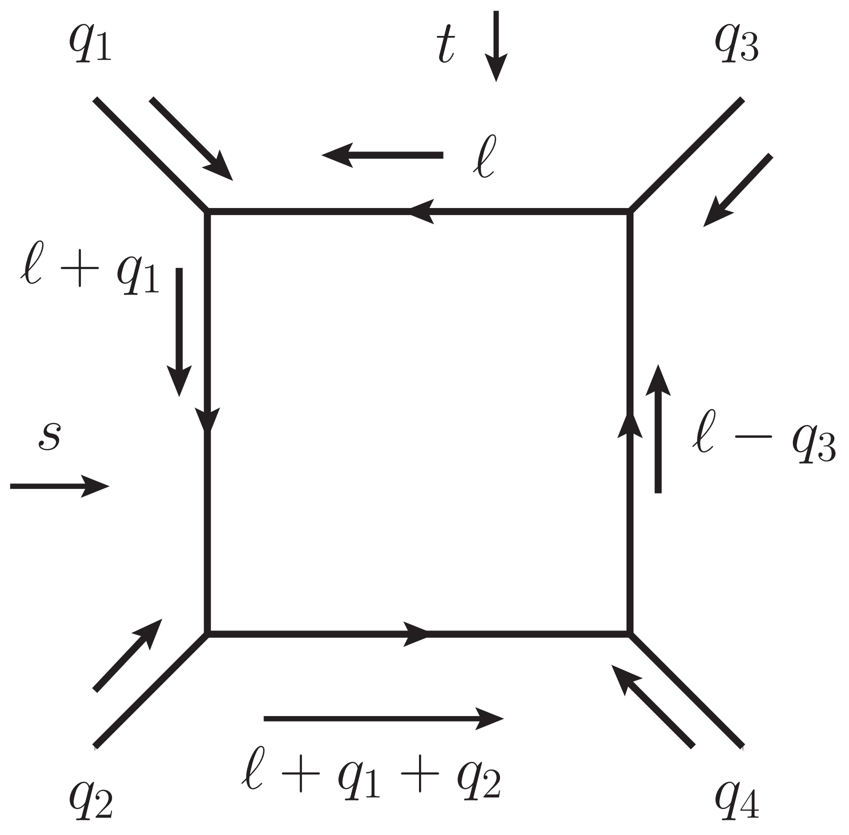
Ma 23

Application to $gg \to HH$:
Amplitude Level

Amplitude Level Results @ 1-loop

Can compute amplitude level results for each region, at the 1-loop level:

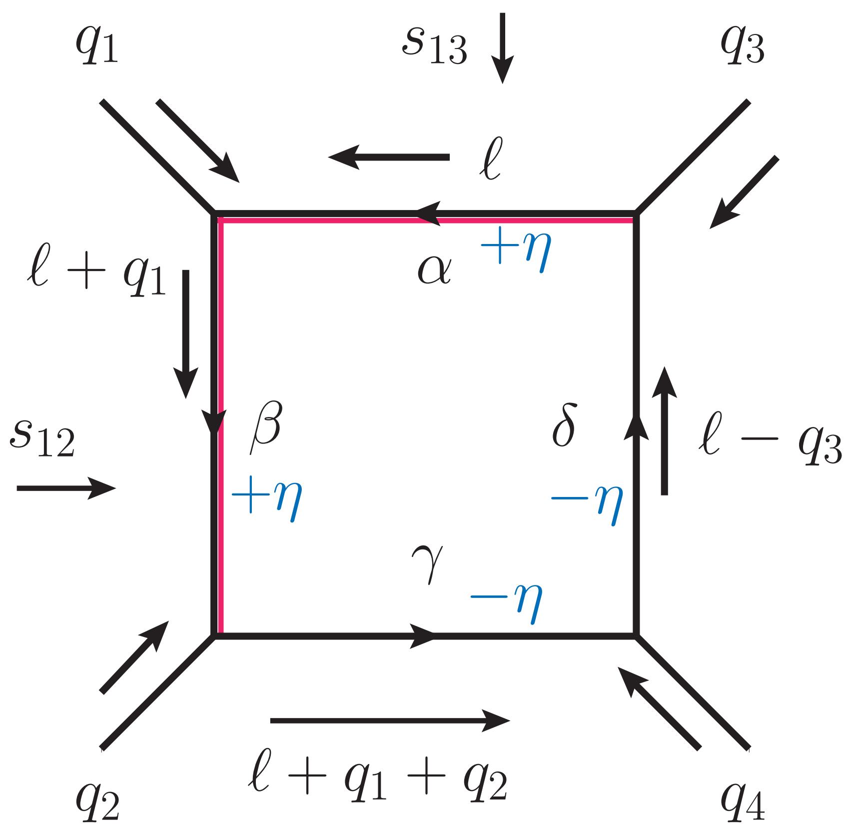
Hard region



Leading Power (LP)

$$A^{(h)}_{1,y_t^2} = \frac{4y_t^2}{s}\Bigg\{2 - 2m_t^2\Bigg[-\frac{2}{\epsilon^2 s} - \frac{1}{\epsilon}\left(\frac{s^2+2tu}{stu}\,l_s + \frac{l_t}{u} + \frac{l_u}{t}\right) + \frac{-l_s^2+2l_t^2+2l_u^2}{s} + \frac{l_s\,l_t}{t} + \frac{l_s\,l_u}{u} + \frac{(t-u)^2\,l_t\,l_u}{stu} - \left(\frac{2}{s}+\frac{1}{t}+\frac{1}{u}\right)l_s - \frac{t\,l_t}{su} - \frac{u\,l_u}{st} + \frac{60+13\pi^2}{6s}\Bigg] + \mathcal{O}(m_t^4)\Bigg\}$$

Collinear q_1 region



Next-to-Leading Power (NLP)

$$A^{(c_1)}_{1,y_t^2} = \frac{-4y_t^2 m_t^2\,\mu^{2\epsilon}e^{\gamma_E\epsilon}\Gamma(2\eta)^2\Gamma(\epsilon+2\eta)}{s(stu)(1-2\eta-\epsilon)\Gamma(4\eta)\Gamma(1+\eta)^2}\left(\frac{1}{m_t^2}\right)^{2\eta+\epsilon}$$
$$\times\Big\{(-t)^\eta(-u)^\eta\Big[s^2\left(1+2\epsilon^2-(2+\eta)\epsilon\right)-2\eta\,tu\Big]$$
$$+(-s)^\eta(-t)^\eta\Big[su\left(1+2\epsilon^2-(2+\eta)\epsilon\right)-tu(1-3\eta-(2-5\eta)\epsilon)\Big]$$
$$+(-s)^\eta(-u)^\eta\Big[st\left(1+2\epsilon^2-(2+\eta)\epsilon\right)-tu(1-3\eta-(2-5\eta)\epsilon)\Big]\Big\}$$

Generates $\log(m_t^2)$ at NLP

Amplitude Level Results @ 2-loop

Could compute each region at 2-loops (tedious), can instead examine numerator prior to reduction

Region $c_1 c_1$

$$\ell_1^\mu \sim \ell_2^\mu \sim Q(1, \lambda^2, \lambda)$$

$$l_1^2 \sim \lambda^2 Q^2, \quad l_2^2 \sim \lambda^2 Q^2, \quad l_1 \cdot l_2 \sim \lambda^2 Q^2,$$
$$l_1 \cdot q_1 \sim \lambda^2 Q^2, \quad l_2 \cdot q_1 \sim \lambda^2 Q^2,$$
$$l_1 \propto q_1, \quad l_2 \propto q_1 .$$

Region ss

$$\ell_1^\mu \sim \ell_2^\mu \sim Q(\lambda, \lambda, \lambda)$$

$$l_1^2 \sim \lambda^2 Q^2, \quad l_2^2 \sim \lambda^2 Q^2, \quad l_1 \cdot l_2 \sim \lambda^2 Q^2,$$
$$l_1 \cdot q_2 \sim \lambda^2 Q^2, \quad l_1 \cdot q_3 \sim \lambda^2 Q^2, \quad l_1 \cdot q_4 \sim \lambda^2 Q^2,$$
$$l_2 \cdot q_2 \sim \lambda^2 Q^2, \quad l_2 \cdot q_3 \sim \lambda^2 Q^2, \quad l_2 \cdot q_4 \sim \lambda^2 Q^2,$$

Inserting into amplitude, projecting form factors and computing traces
Numerator gives a λ^2 suppression for all soft/collinear regions

Consistent with the result of Steinhauser et al. for the $m_t \to 0$ limit

Davies, Mishima, Steinhauser, Wellmann 18;

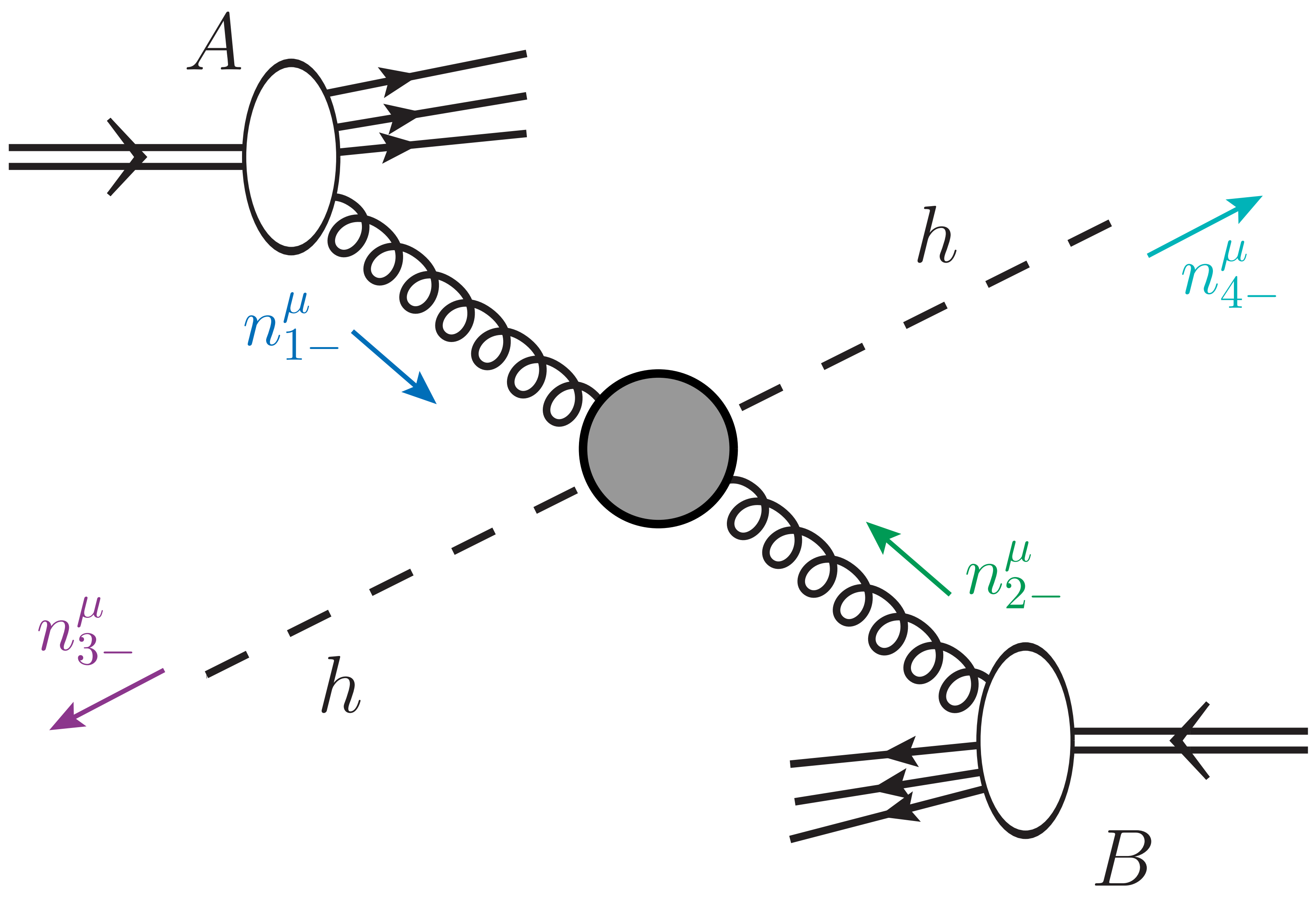
Suggests that regions other than the hard region are **helicity suppressed** by at least $\lambda \sim m_t$

Effective field theory Analysis

Study amplitude using Soft Collinear Effective Theory (SCET)

[C. Bauer, S. Fleming, D. Pirjol and I. Stewart, hep-ph/0011336] [C. Bauer, D. Pirjol, I. Stewart, hep-ph/0109045]
[M. Beneke, A. Chapovsky, M. Diehl, T. Feldmann, hep-ph/0206152] [M. Beneke, T. Feldmann, hep-ph//0211358]

$$\psi(x) \to \underbrace{\psi_1(x) + \ldots + \psi_N(x)}_{N\,\text{collinear fermion fields}} + q(x) \qquad \mathscr{L}_{\text{SCET}} = \sum_{i=1}^{N} \mathscr{L}_{c_i} + \mathscr{L}_{\text{soft}}$$



Lagrangians belong to a specific collinear direction
Can be expanded in powers of the small parameter

$$\mathscr{L}_{c_i} = \underbrace{\mathscr{L}^{(0)}_{c_i}}_{\text{LP}} + \underbrace{\mathscr{L}^{(1)}_{c_i}}_{\mathcal{O}(\lambda^1)} + \underbrace{\mathscr{L}^{(2)}_{c_i}}_{\mathcal{O}(\lambda^2)} + \ldots$$

Keep collinear, anti-collinear, and soft degrees of freedom
Hard modes are integrated out

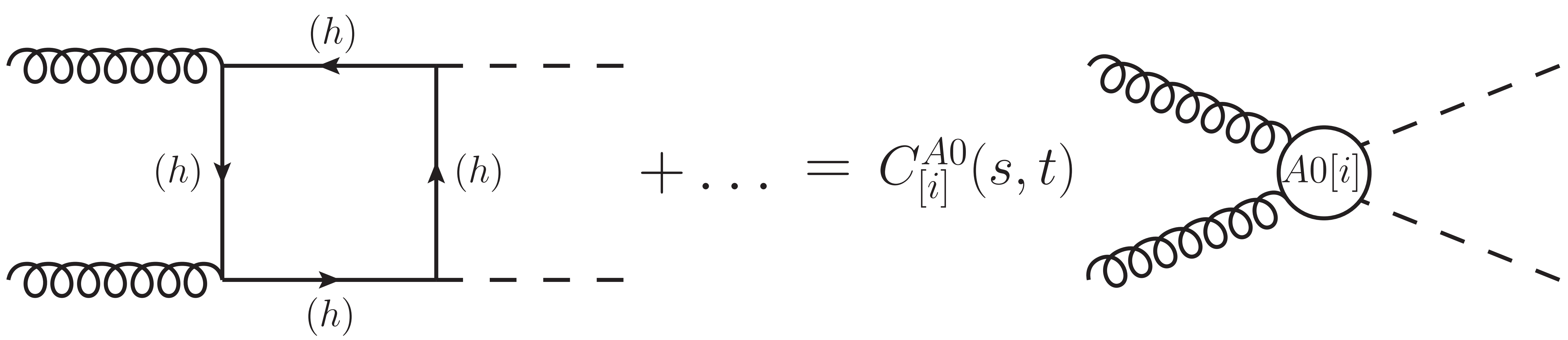
Generic N-jet operator has the form:

M. Beneke, M. Garny, R. Szafron, J. Wang, 17, 17, 18, 19

$$J = \int \left[\prod_{ik} dt_{i_k}\right] C(\{t_{i_k}\}) \prod_{i=1}^{N} J_{c_i}(t_{i_1}, t_{i_2} \ldots)$$

Leading Power Analysis

Leading power matching $J_{\mathrm{LP}}^{[i]}(t_1,t_2,t_3,t_4) = y_t^2\, P_i^{\mu\nu}\, \mathscr{A}_{c_1\perp_1\,\mu}(t_1 n_{1+})\, \mathscr{A}_{c_2\perp_2\,\nu}(t_2 n_{2+})\, h_{c_3}(t_3 n_{3+})\, h_{c_4}(t_4 n_{4+})$



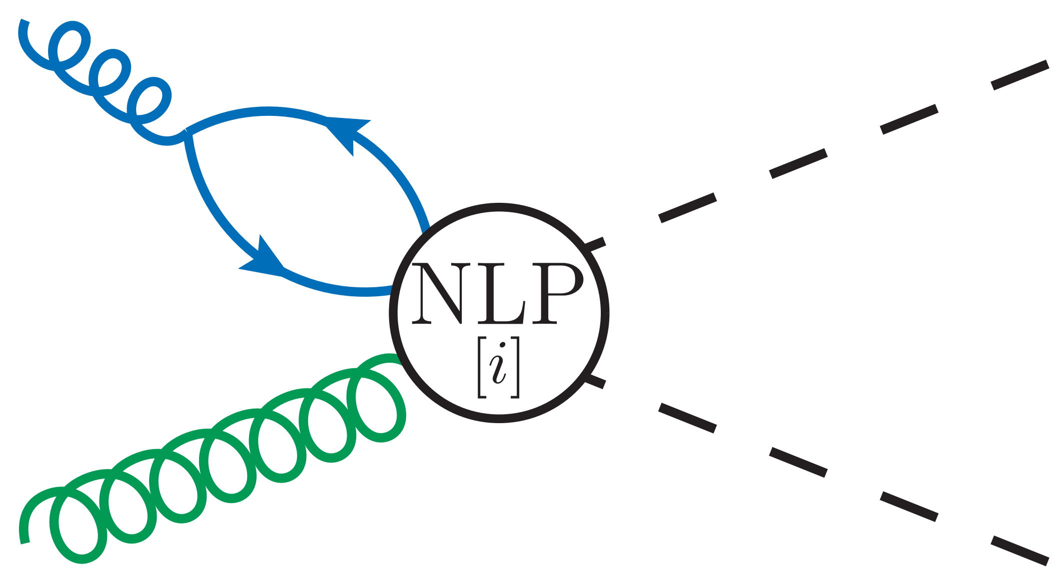
Collinear Regions c_1, c_2 $\quad \mathcal{M}_{\mathrm{LP}}^{\mathrm{QCD}} \propto \left(\bar{v}_{c_1}(\bar{r}q_1)\frac{\not{n}_{1+}}{2}u_{c_1}(rq_1)n_{3-\nu}\varepsilon^{\nu}_{\perp_1}(q_2) + \bar{v}_{c_1}(\bar{r}q_1)\frac{\not{n}_{1+}}{2}\gamma_5 u_{c_1}(rq_1)n^{\mu}_{3-}i\epsilon^{\perp_1}_{\mu\nu}\varepsilon^{\nu}_{\perp_1}(q_2) \right)$

Relevant operator structures

$$J_{S_i}(\{t_{i_1},t_{i_2}\}) = \bar{\chi}_{c_i}(t_{i_2}n_{i+})\frac{\not{n}_{i+}}{2}\,\chi_{c_i}(t_{i_1}n_{i+}),$$

$$J_{P_i}(\{t_{i_1},t_{i_2}\}) = \bar{\chi}_{c_i}(t_{i_2}n_{i+})\frac{\not{n}_{i+}}{2}\,\gamma_5\,\chi_{c_i}(t_{i_1}n_{i+}),$$

Mixing with the external gluon is forbidden at LP



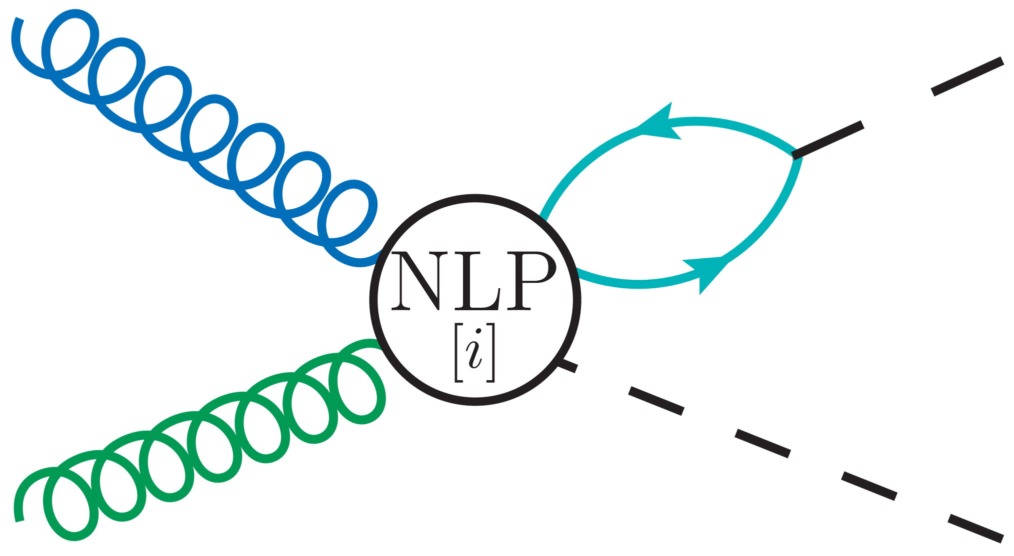
Leading Power Analysis

Collinear Regions c_3, c_4

$$\mathcal{M}_{\mathrm{LP}}^{\mathrm{QCD}} \sim i g_s^2 \mathbf{T}^B\,\mathbf{T}^A \left[\frac{g_W y_t}{2}\right] \bar{v}_{c_3}(q') \frac{1}{\bar{r}(n_{3+}q_3)} \left[\frac{2 n_{3-\mu}}{(n_{1+}q_1) n_{3-}\cdot n_{1-}} \frac{\not{n}_{3+}}{2} \gamma_{\nu\perp_3} u_{c_3}(q) + \frac{1}{r(n_{3+}q_3)} n_{3+\nu} \frac{\not{n}_{3+}}{2} \gamma_{\mu\perp_3} u_{c_3}(q) \right] \varepsilon^{\nu}_{\perp_2}(q_2)\ \varepsilon^{\mu}_{\perp_1}(q_1)$$

Situation reversed, structures appearing at LP are vector-like

Mixing with the external Higgs is forbidden at LP

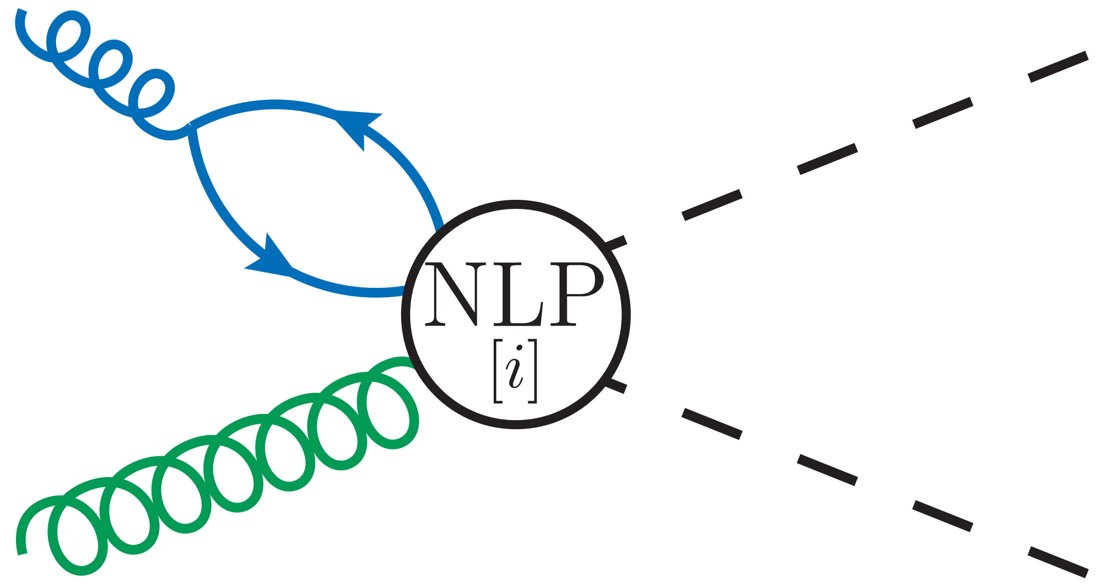


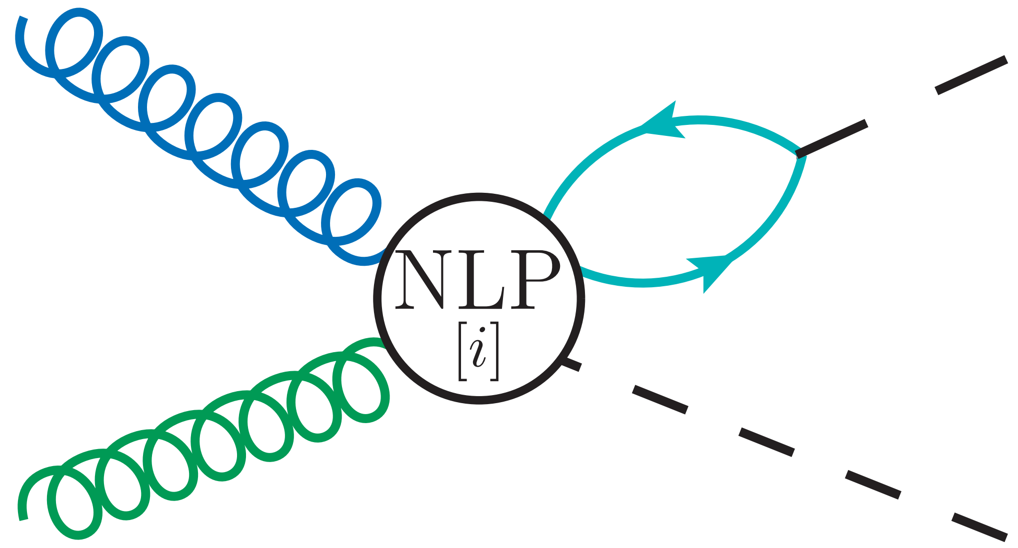
Result holds to all orders in α_s due to helicity conservation for $m_t \to 0$

Next-to-Leading Power

Structure of the amplitude allows mixing with external gluon/Higgs

Expect contributions from collinear/soft regions





Overview of Structure

The $gg \to HH$ amplitude in the $\overline{\text{MS}}$ scheme has the following leading power structure

$$l_\mu = \log(\mu_t^2/s)$$

$$l_m = \log(\mu_t^2/s), \log(m_t^2/s)$$

$$\text{LO}: \alpha_s y_t^2 (c_0 + m_t n_0),$$

$$\text{NLO}: \alpha_s^2 y_t^2 (a_1 l_\mu + c_1 + m_t n_1),$$

$$\text{NNLO}: \alpha_s^3 y_t^2 (a_2 l_\mu^2 + b_2 l_m + c_2 + m_t n_2),$$

$$\text{N}^3\text{LO}: \alpha_s^4 y_t^2 (a_3 l_\mu^3 + b_3 l_m^2 + d_3 l_m + c_3 + m_t n_3),$$

$$\text{N}^i\text{LO}: \alpha_s^{i-1} y_t^2 (a_i l_\mu^i + b_4 l_m^{i-1} + d_i l_m^{i-2} + \ldots + c_i + m_t n_i) .$$

Master integrals known

Caola, von Manteuffel, Tancredi 20;

Bargiela, Caola, von Manteuffel, Tancredi 22;

Leading log structure generated to all orders by RG running

$$m^{\text{LL}}(\mu) = M \exp\left[a_{\gamma_m}^{\text{LL}}(\mu)\right] z_m(M)$$

$$a_{\gamma_m}^{\text{LL}}(\mu) = \frac{3C_F}{2\beta_0} \ln\left(1 - \frac{\alpha_s(\mu)}{2\pi}\beta_0 \ln\left(\frac{\mu^2}{M^2}\right)\right)$$

LP LL

Known from RG running of top-quark mass

LP NLL

RG running + massification

Penin 06; Moch, Mitov 07; Becher, Melnikov 07; Engel et al 19; Wang, Xia, Yang, Ye 23;

LP Constant

Hard region ($m_t = 0$) contribution only, known to NLO

Leading Power Expansion

We study the LO and NLO finite virtual corrections

$$A^{\text{fin}}_{i,j} = \frac{\alpha_s}{2\pi} A^{(0)}_{i,j} + \left(\frac{\alpha_s}{2\pi}\right)^2 A^{(1)}_{i,j} + \mathcal{O}(\alpha_s^3)$$

Use ``SCET'' IR scheme for virtuals Becher, Neubert 09, 13;

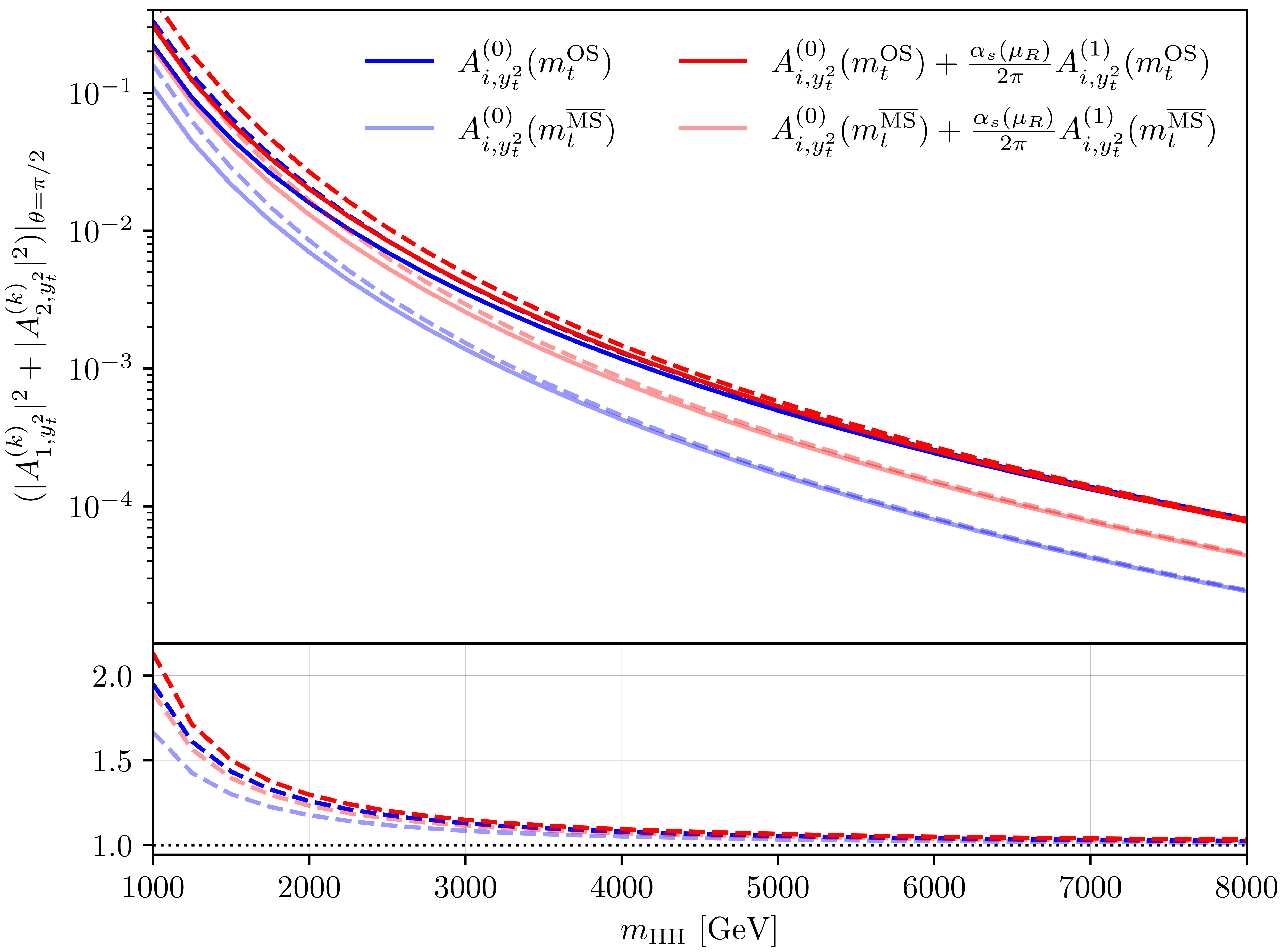
Neglecting real contributions

Solid Lines: Full TH result

Davies, Mishima, Steinhauser, Wellmann 18;

Dashed Lines: Leading power expansion

Leading power is a good approximation for $\sqrt{s} \gtrsim 1$ TeV, focus on the very high energy behaviour of the amplitude,



Mass Scheme Uncertainty

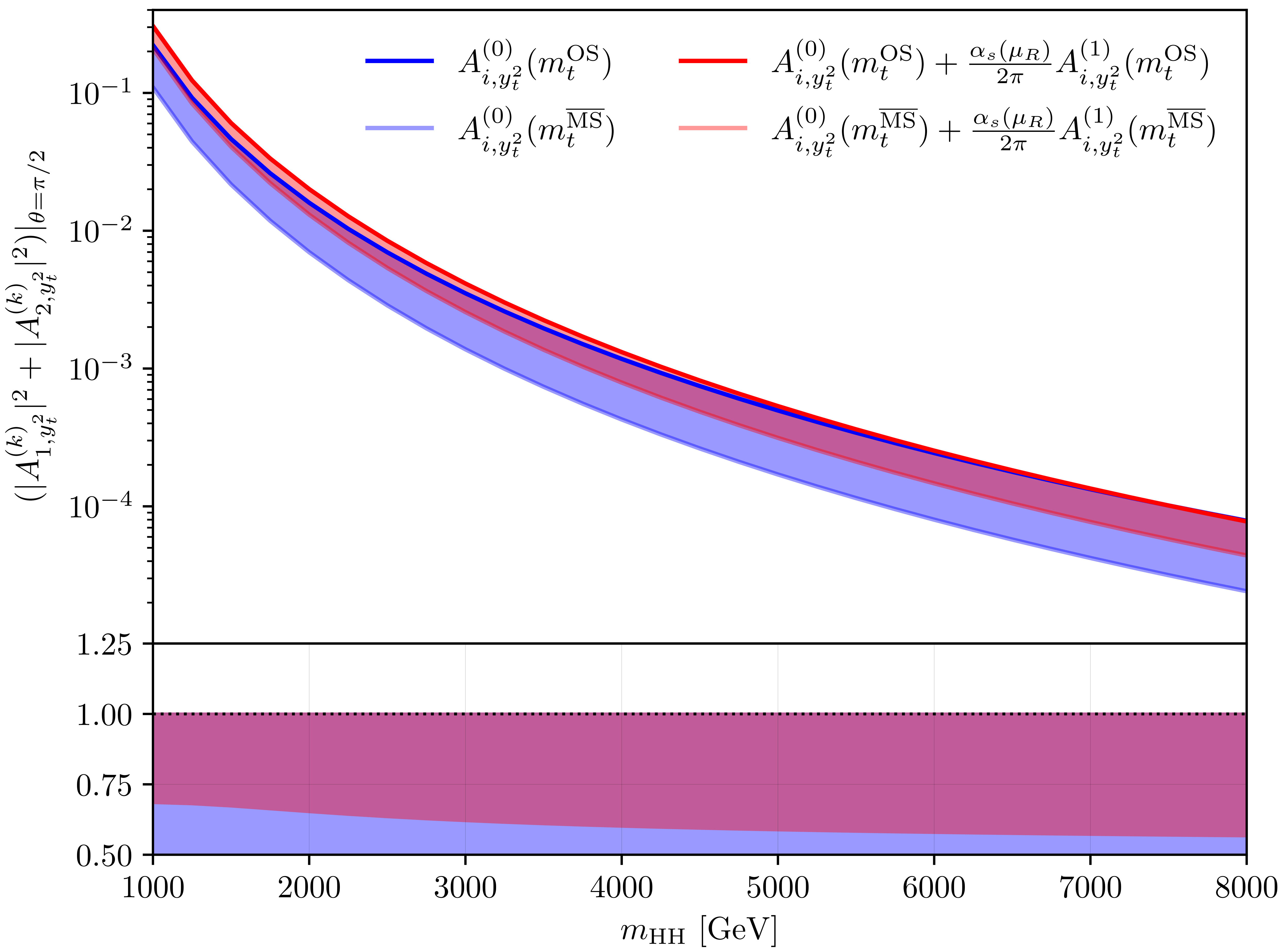
Estimate as the difference between the squared/interfered amplitude evaluated with top-quark mass in different schemes

On-Shell: $m_t = m_t^{\mathrm{OS}} = 173.21\ \mathrm{GeV}$
$\overline{\mathrm{MS}}$: $m_t = m_t^{\overline{\mathrm{MS}}}(\mu_t = \sqrt{s})$

Very large uncertainty obtained

LO: ~60-70% (blue band)

NLO: ~30-40% (red band)



Mass Scheme Uncertainty

Including **known** LP LL to all orders via

$$A^{(j,\mathrm{LL})}_{i,y_t^2}(m_t^{\mathrm{OS}}) = \left(\frac{m^{\mathrm{LL}}(\mu_t)}{m_t^{\mathrm{OS}}}\right)^2 A^{(j)}_{i,y_t^2}(m_t^{\mathrm{OS}})$$

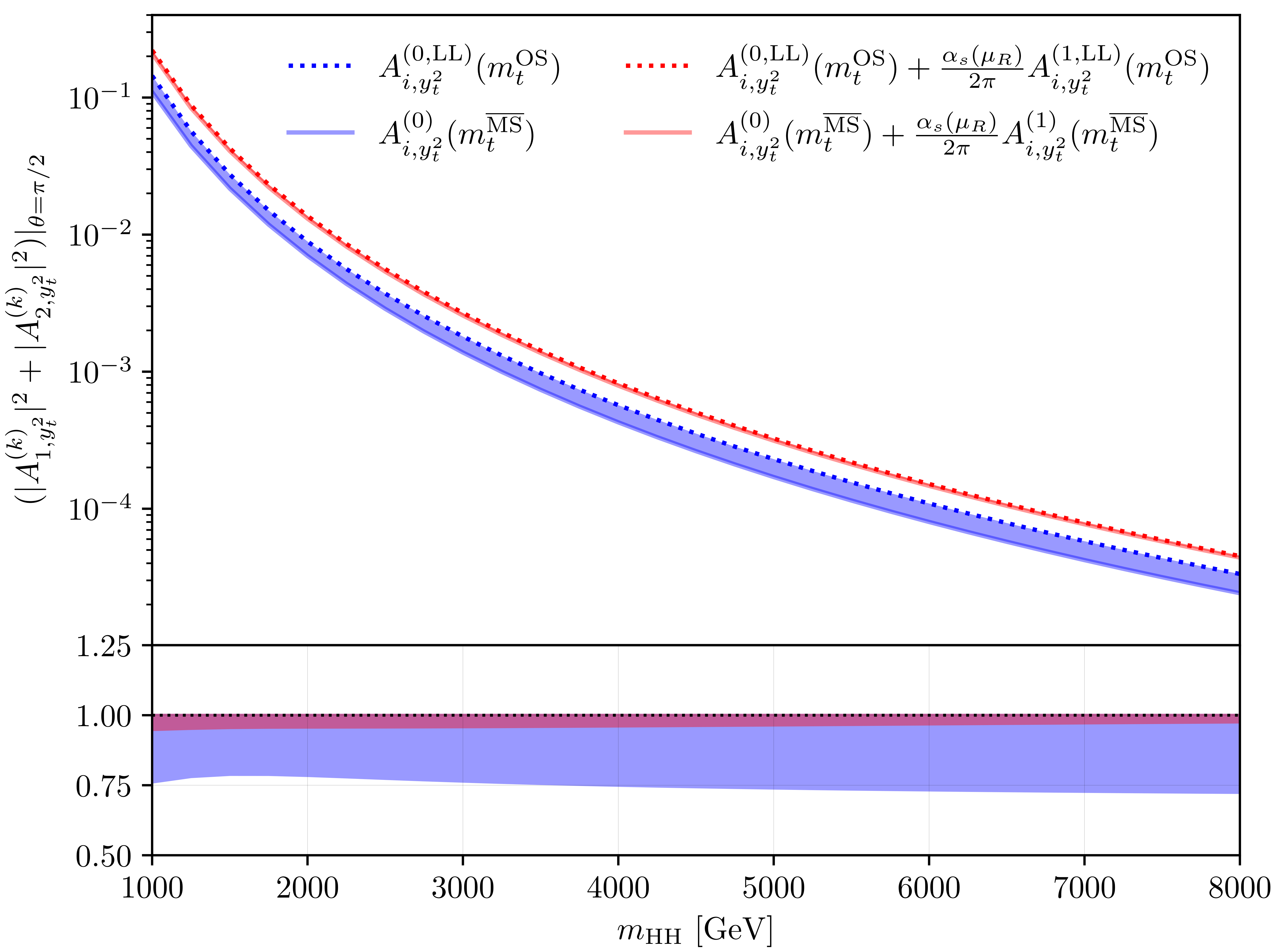
which includes the **green** tower of logarithms to all orders

Uncertainty significantly reduced

LO: ~25-30% (blue band)

NLO: ~3-4% (red band)

Do not advocate a particular scheme, but argue that the known LP LL should be included and not assigned as part of the uncertainty



Mass Scheme Uncertainty

Alternatively, can assess the mass scheme uncertainty by sticking to $\overline{\mathrm{MS}}$ and varying $\mu_t^2 \in (s/4,\ 4s)$

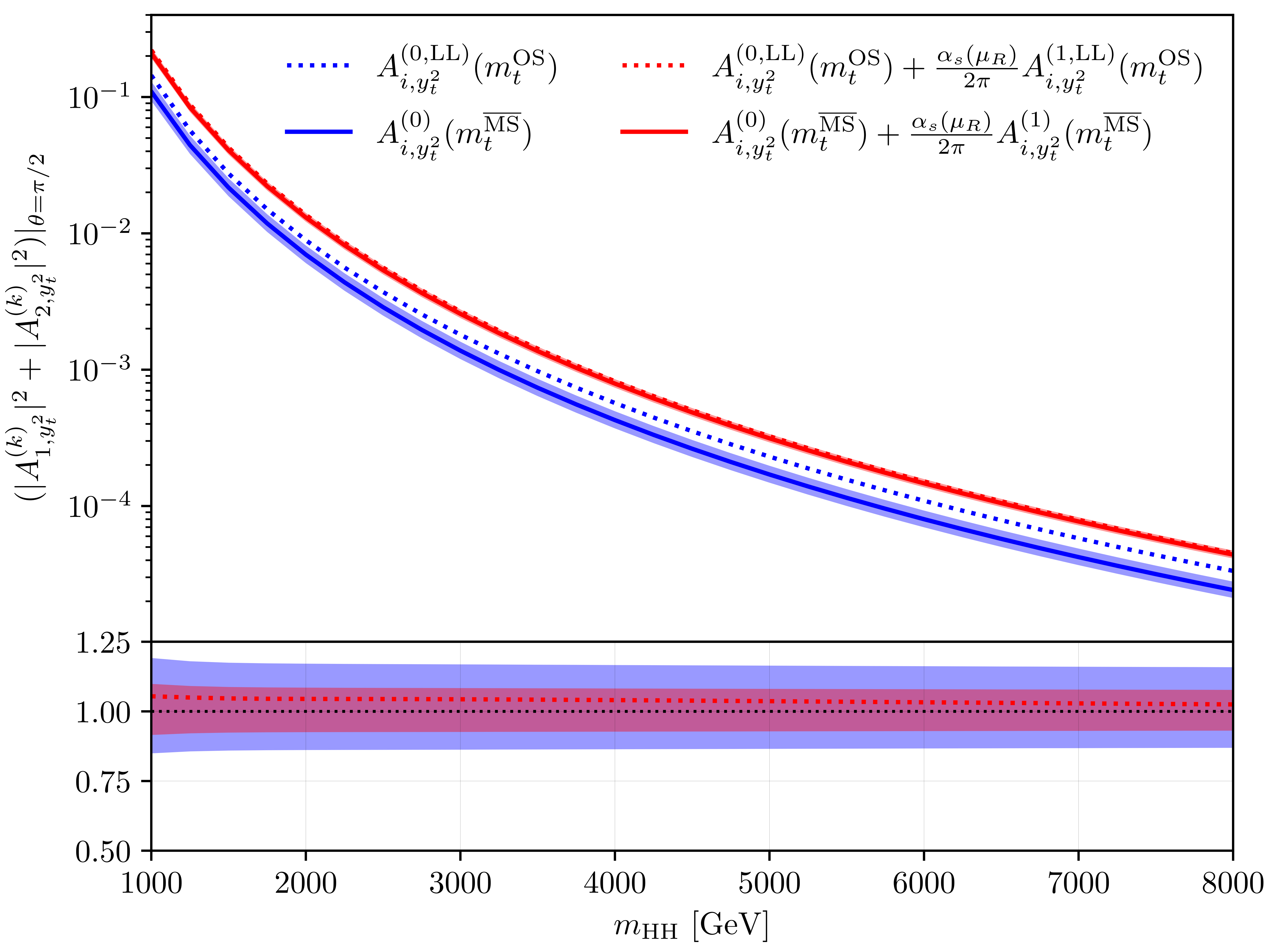
Uncertainty

LO: ~13-16% (blue band)

NLO: ~7-8% (red band)

This uncertainty can be reduced by computing the LP amplitude at NNLO (requires only $m_T = 0$ result)

Conservative: take envelope of $\mathrm{OS}^{\mathrm{LL}}$ and $\overline{\mathrm{MS}}$ as uncertainty



Conclusion

Studied $gg \to HH$ in the very high-energy limit

Method of Regions showed that only hard region contribution at leading power (LO, NLO)

Argued using tools of SCET that this generalises to all orders in α_s

Leading small-m_t logarithms known to all orders from RG running

Results

Studied impact of including LP LL on finite virtual amplitude at $\sqrt{s} > 1\ \mathrm{TeV}$

Mass scheme uncertainty at NLO reduced from 30-40% → 7-8% if we consistently include known LLs

Outlook

Analysis can be improved with NNLO finite coefficient, including NLL (massification)

NLP formalism significantly more involved, framework being developed

Interesting to look at $t\bar{t}$ threshold region & full NNLO calculation still desperately needed

Thank you for listening!

Backup

Sector Decomposition in a Nutshell

Can exchange loop integrals for integrals over Feynman parameters

$$I \sim \int \mathrm{d}^d k_1 \ldots \mathrm{d}^d k_l \frac{1}{\prod_{i=1}^{N} (q_i - m_i)^{\nu_i}} \leftrightarrow I \sim \int_{\mathbb{R}^{N+1}_{>0}} [\mathrm{d}\boldsymbol{x}]\, \boldsymbol{x}^{\boldsymbol{\nu}} \frac{[\mathcal{U}(\boldsymbol{x})]^{N-(L+1)D/2}}{[\mathcal{F}(\boldsymbol{x}, \mathbf{s}) - i\delta]^{N-LD/2}} \delta(1 - H(\boldsymbol{x}))$$

$\mathcal{U}, \mathcal{F}$ are *polynomials* in FP x

Singularities

1. UV/IR singularities when some $\{x\} \to 0$ simultaneously $\Longrightarrow$ Sector Decomposition
2. Thresholds when $\mathcal{F}$ vanishes inside integration region $\Longrightarrow$ $i\delta$

Sector decomposition

Find a local change of coordinates for each singularity that factorises it (blow-up)

Hepp 66; Roth, Denner 96; Binoth, Heinrich 00; Heinrich 08

Sector Decomposition in a Nutshell

$$I \sim \int_{\mathbb{R}^N_{>0}} [\mathrm{d}\boldsymbol{x}]\, \boldsymbol{x}^{\boldsymbol{\nu}} \left(c_i\, \boldsymbol{x}^{\mathbf{r}_i}\right)^t$$

$$\mathcal{N}(I) = \mathrm{convHull}(\mathbf{r}_1, \mathbf{r}_2, \ldots) = \bigcap_{f \in F} \left\{ \mathbf{m} \in \mathbb{R}^N \mid \langle \mathbf{m}, \mathbf{n}_f \rangle + a_f \geq 0 \right\}$$

Normal vectors incident to each extremal vertex define a local change of variables*

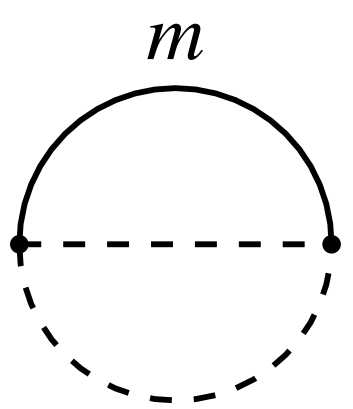
Kaneko, Ueda 10

$$x_i = \prod_{f \in S_j} y_f^{\langle \mathbf{n}_f, \mathbf{e}_i \rangle}$$

$$I \sim \sum_{\sigma \in \Delta^T_{\mathcal{N}}} |\sigma| \int_0^1 \left[\mathrm{d}\mathbf{y}_f\right] \underbrace{\prod_{f \in \sigma} y_f^{\langle \mathbf{n}_f, \boldsymbol{\nu} \rangle - t a_f}}_{\textbf{Singularities}} \Bigg(\underbrace{c_i \prod_{f \in \sigma} y_f^{\langle \mathbf{n}_f, \mathbf{r}_i \rangle + a_f}}_{\textbf{Finite}} \Bigg)^t$$

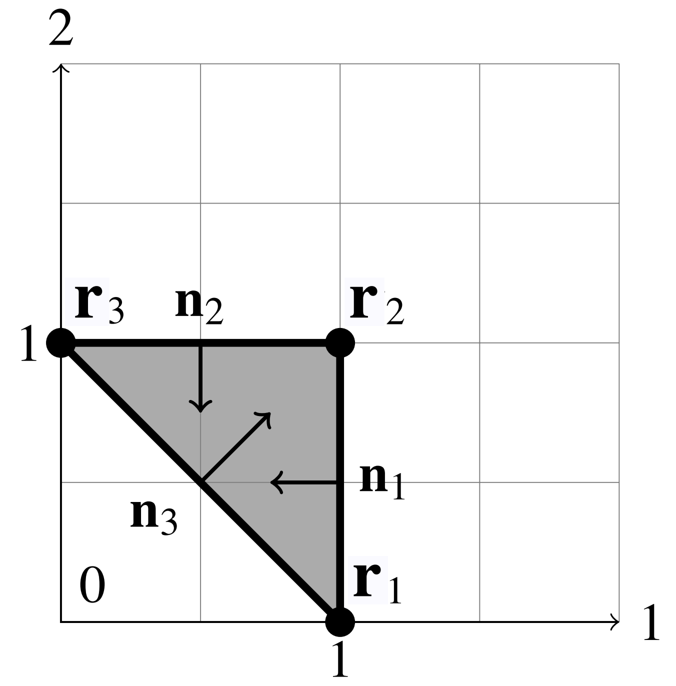
*If $|S_j| > N$, need triangulation to define variables (simplicial normal cones $\sigma \in \Delta^T_{\mathcal{N}}$)

Sector Decomposition in a Nutshell

$I =$ 

$= -\Gamma(-1+2\varepsilon)\left(m^2\right)^{1-2\varepsilon}\int_0^\infty \frac{\mathrm{d}x_1\mathrm{d}x_2}{\left(x_1^1x_2^0 + x_1^1x_2^1 + x_1^0x_2^1\right)^{2-\varepsilon}}.$

$$\mathbf{r}_1 = \begin{pmatrix}1\\0\end{pmatrix}, \mathbf{r}_2 = \begin{pmatrix}1\\1\end{pmatrix}, \mathbf{r}_3 = \begin{pmatrix}0\\1\end{pmatrix}$$

$\mathcal{N}(I) =$ 

$$= \quad \begin{matrix} \mathbf{n}_1 = \begin{pmatrix}-1\\0\end{pmatrix} & \mathbf{n}_2 = \begin{pmatrix}0\\-1\end{pmatrix} & \mathbf{n}_3 = \begin{pmatrix}1\\1\end{pmatrix} \\ a_1 = \quad 1 & a_2 = \quad 1 & a_3 = \quad -1 \end{matrix}$$

For each vertex make the local change of variables

e.g. $\mathbf{r}_1$: $x_1 = y_1^{-1}y_3^1,\ x_2 = y_1^0y_3^1$, $\mathbf{r}_2 : x_1 = y_1^{-1}y_2^0,\ x_2 = y_1^0y_2^{-1}$, $\mathbf{r}_3 : x_1 = y_2^0y_3^1, x_2 = y_2^{-1}y_3^1$

$$I = -\Gamma(-1+2\varepsilon)\left(m^2\right)^{1-2\varepsilon}\int_0^1 \mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3\,\frac{y_1^{-\varepsilon}\,y_2^{-\varepsilon}\,y_3^{-1+\varepsilon}}{(y_1+y_2+y_3)^{2-\varepsilon}}\left[\delta(1-y_2)+\delta(1-y_3)+\delta(1-y_1)\right]$$

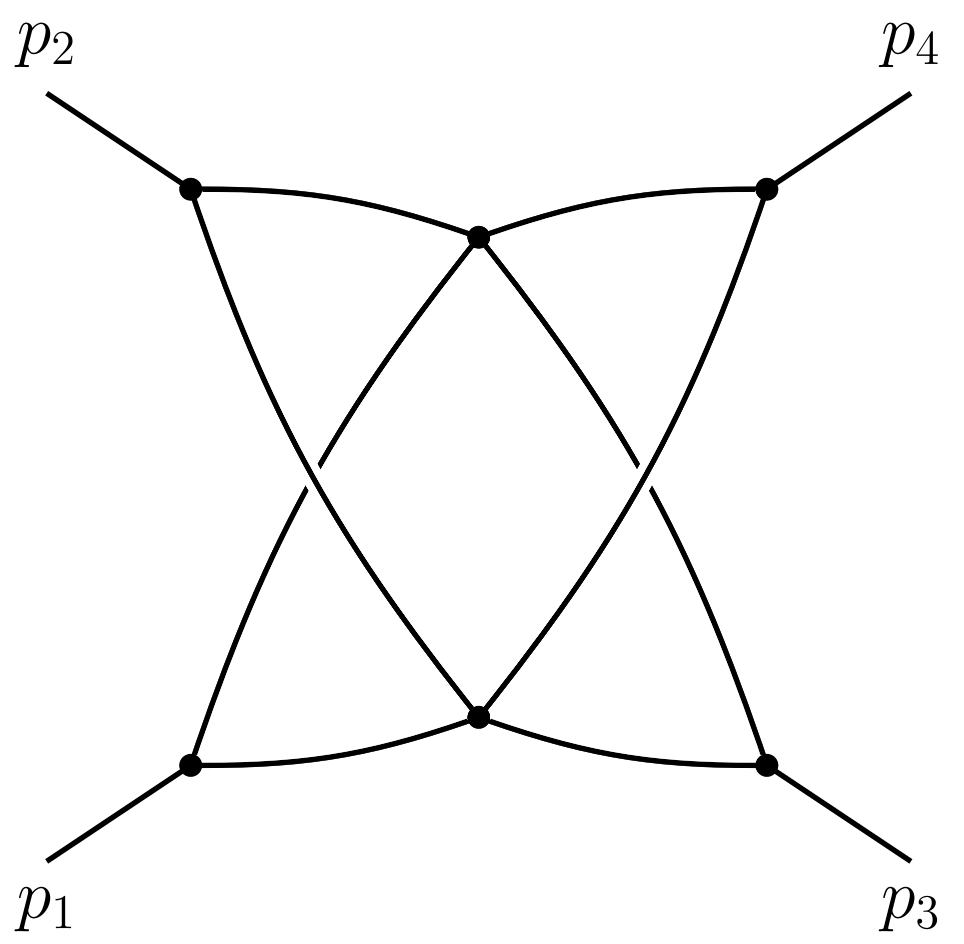
Schlenk 2016

Some Future Developments

Computing integrals with leading Landau singularities inside the integration domain

w/ Gardi, Herzog, Ma (WIP)

Impossible → Possible to compute



$$I = \epsilon^{-4}\left[8.3400403920\mathbf{28} - 52.3598775598\mathbf{347}i\right] + \mathcal{O}\left(\epsilon^{-3}\right)$$
$$I^{\text{ana.}} = \epsilon^{-4}\left[8.34004039223768 - 52.35987755984493i\right] + \mathcal{O}\left(\epsilon^{-3}\right)$$

Avoiding contour deformation even in the Minkowski/physical regime

w/ Olsson, Stone (WIP)

Speedup ~100-1000x for some cases