Two-loop QCD helicity amplitudes for tt+jet production at leading colour

Towards NNLO QCD corrections

Colomba Brancaccio

UNIVERSITÀ DI TORINO

Based on: arXiv:2412.13876

With: Simon Badger, Matteo Becchetti, Heribertus Bayu Hartanto, Simone Zoia

(Thanks to Gaia Fontana for the wonderful drawings)

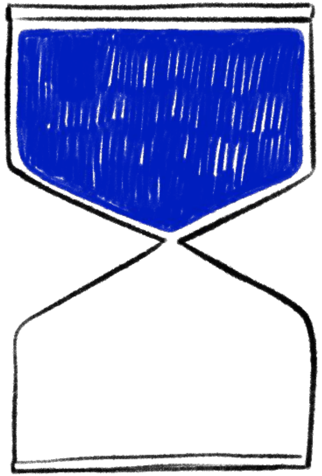
TUM/MPP Collider Seminar

20th May 2025

TABLE OF CONTENTS

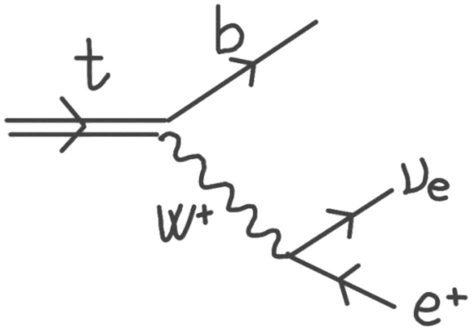
INTRODUCTION

From top quark physics
to 2→3 amplitude frontier



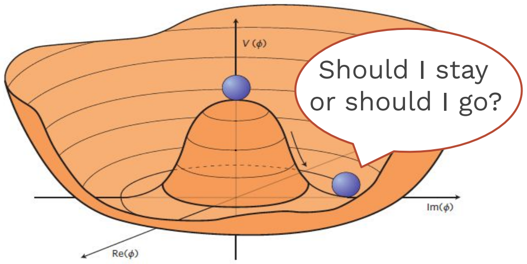
The top quark

- To-date **heaviest** fundamental particle

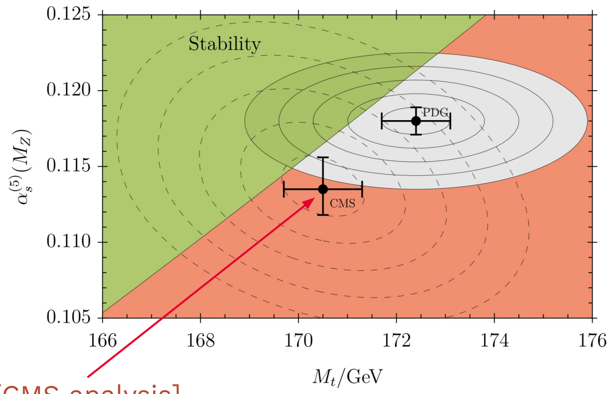


- **Decays** before hadronising
- **Information about its spin state can be inferred** from decay product distributions

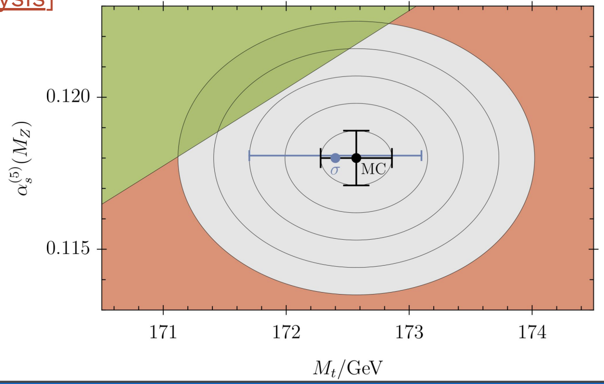
- Largest coupling to the Higgs boson
- Affects the **EW vacuum stability**



Vacuum stability



[CMS analysis]



Vacuum Stability in the Standard Model and Beyond

Gudrun Hiller,[1,2] Tim Höhne,[1] Daniel F. Litim,[2] and Tom Steudtner[1]

[1] *TU Dortmund University, Department of Physics, Otto-Hahn-Str.4, D-44221 Dortmund, Germany*
[2] *Department of Physics and Astronomy, University of Sussex, Brighton, BN1 9QH, U.K.*

We revisit the stability of the Standard Model vacuum, and investigate its quantum effective potential using the highest available orders in perturbation theory and the most accurate determination of input parameters to date. We observe that the stability of the electroweak vacuum centrally depends on the values of the top mass and the strong coupling constant. We estimate that reducing their uncertainties by a factor of two to three is sufficient to establish or refute SM vacuum stability at the 5σ level. We further investigate vacuum stability for a variety of singlet scalar field extensions with and without flavor using the Higgs portal mechanism. We identify the

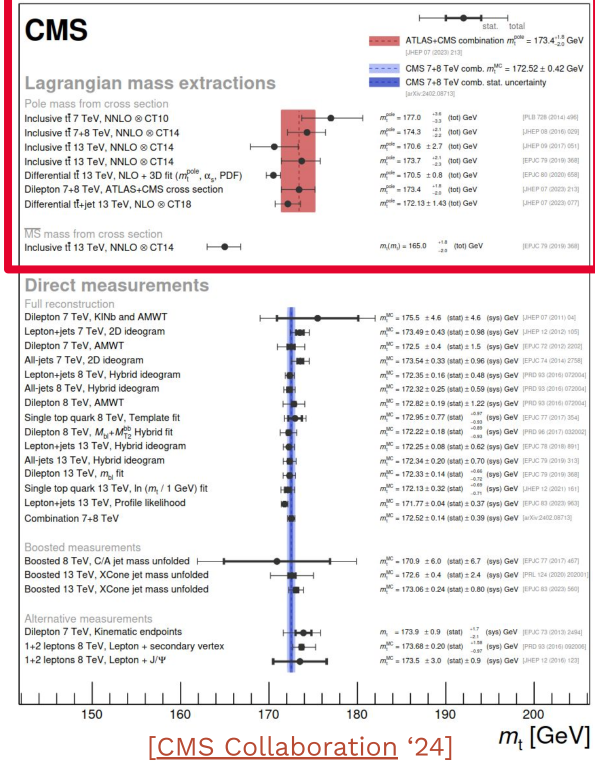
• • •

Higgs stability at the 5σ level or above. Assuming that central values do not change, we find that a 5σ signature necessitates the uncertainty in the top mass M_t to come down to the $200 - 300$ MeV range. Future e^+e^--colliders

• • •

[arXiv:2401.08811 '24]

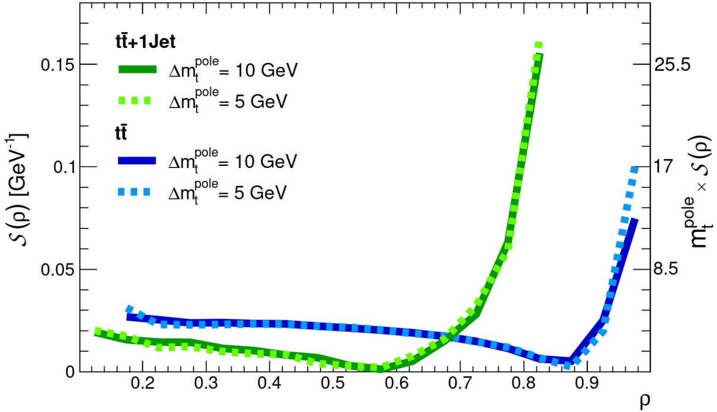
Indirect mass extraction



[CMS Collaboration '24]

Limitations from missing **higher perturbative orders**

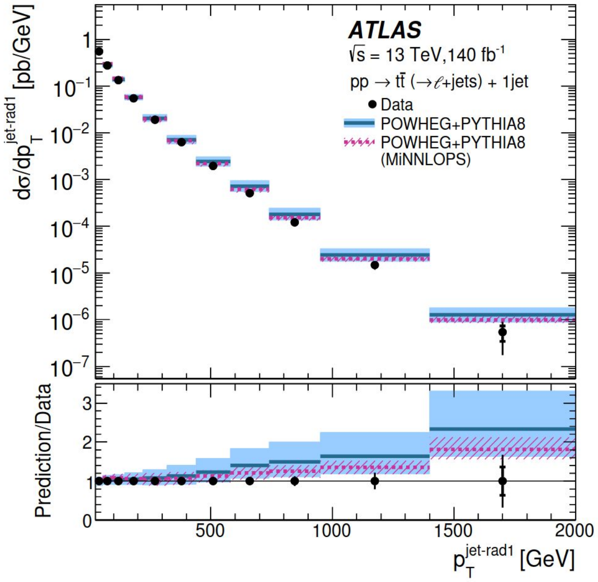
tt+jet production highly sensitive to top quark mass value



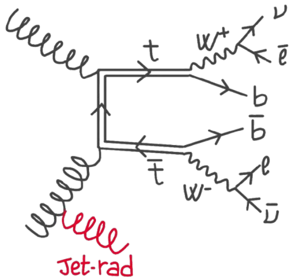
[Alioli, Fernandez, Fuster, Irles, Moch, Uwer, Vos '13]

tt+jet production at the LHC

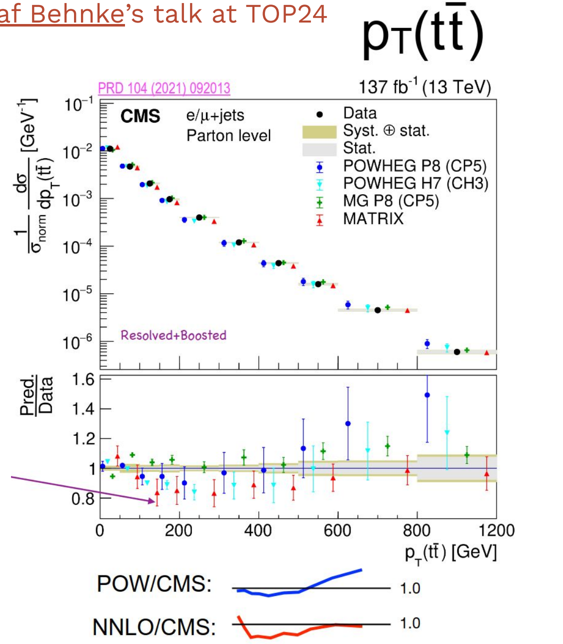
tt+jet is crucial also for **precision tests** of the Standard Model



[ATLAS Collaboration '24]



From Olaf Behnke's talk at TOP24



⇒ Both NNLO and POW exhibit some wiggles around the data
⇒ Need NNLO

Need N^{3}LO tt ⟶ Need NNLO tt+jet

Towards NNLO QCD corrections tt+jet

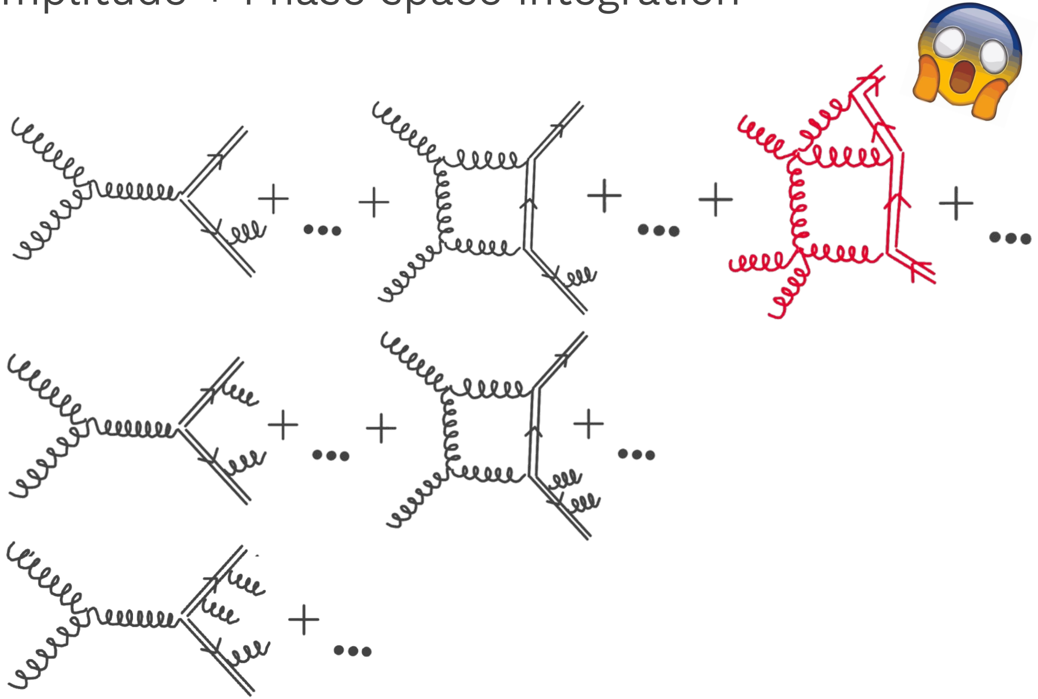
Cross section = Amplitude + Phase space integration

$$\sigma_n^{NNLO}$$

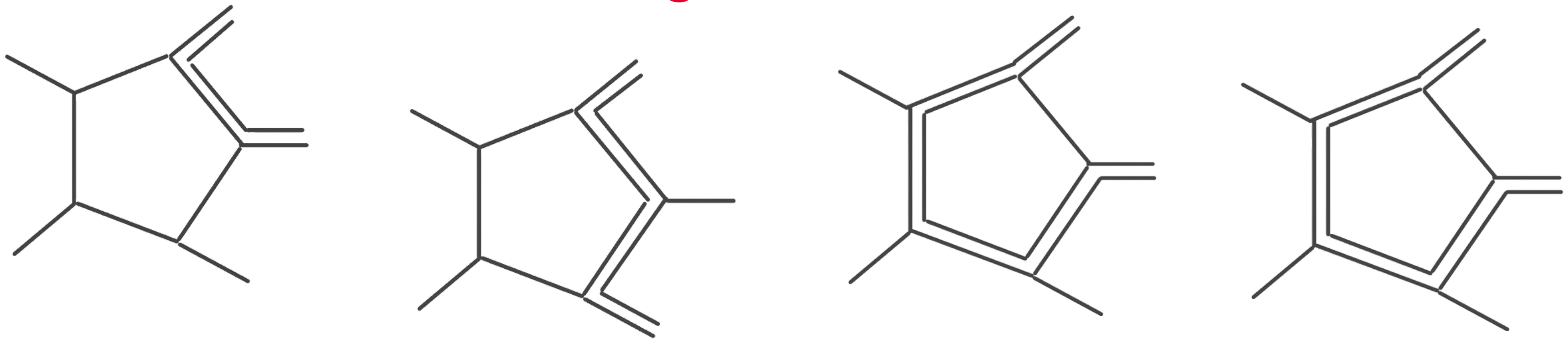
$$= \int_n \mathrm{d}\sigma^B + \mathrm{d}\sigma^V + \mathrm{d}\sigma^{VV}$$

$$+ \int_{n+1} \mathrm{d}\sigma^R + \mathrm{d}\sigma^{RV}$$

$$+ \int_{n+2} \mathrm{d}\sigma^{RR}$$



Current status of tt+jet corrections



- **NLO QCD** corrections
  [Dittmaier, Uwer, Weinzierl, '07]
- Mixed QCD and EW corrections
  [Gütschow, Lindert, Schönherr '18]
- Full off-shell decays and interfaces with parton shower
  [Melnikov and Schulze '10]
  [Alioli, Moch, Uwer '12]
  [Bevilacqua, Czakon, Hartanto, Kraus, Worek '15-'16]

Nowadays NLO is automated

OpenLoops2
[Buccioni, Lang, Lindert, Maierhöfer, Pozzorini, Zhang, Zoller '19]

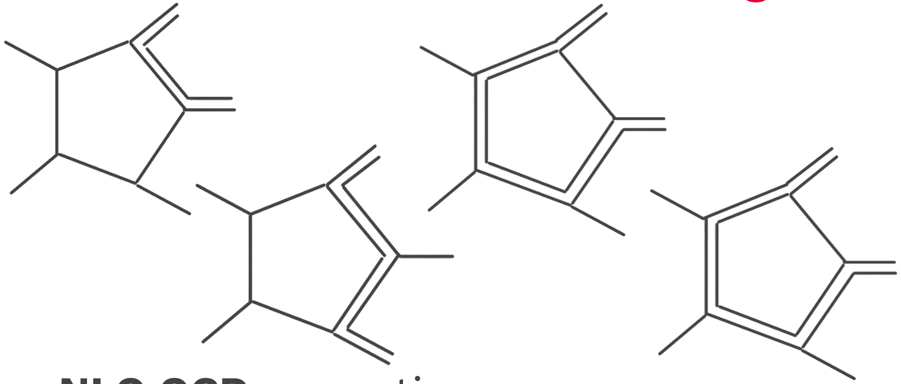
Helac-NLO
[Bevilacqua, Czakon, Garzelli, van Hameren, Kardos, Papadopoulos, Pittau, Worek '11]

MadGraph5_aMC@NLO
[Alwall, Frederix, Frixione, Hirschi, Maltoni, Mattelaer, Shao, Stelzer, Torrielli, Zaro '14]

Implemented within the framework

Powheg-Box
[Alioli, Nason, Oleari, Re '10]

Current status of tt+jet corrections

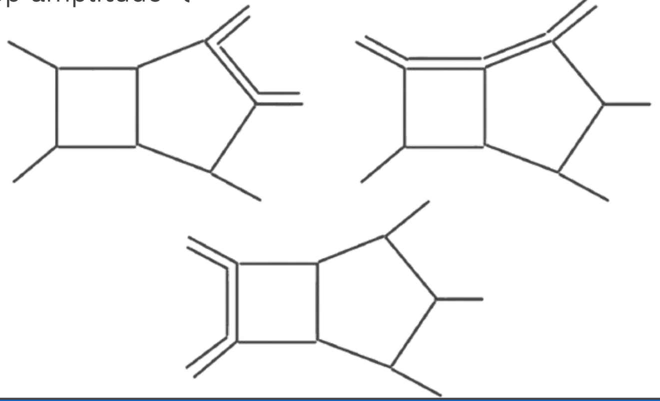


- **NLO QCD** corrections
  [Dittmaier, Uwer, Weinzierl, '07]
- Mixed QCD and EW corrections
  [Gütschow, Lindert, Schönherr '18]
- Full off-shell decays and interfaces with parton shower
  [Melnikov and Schulze '10]
  [Alioli, Moch, Uwer '12]
  [Bevilacqua, Czakon, Hartanto, Kraus, Worek '15-'16]

- **NNLO QCD** corrections needed

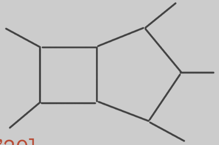
➡ initial steps towards this challenge

- One-loop at $O(\varepsilon^2)$: [Badger, Becchetti, Chaubey, Marzucca, Sarandrea '22] [Bera, **CB**, Canko, Hartanto, '25]
- DEs for two-loop planar topologies: [Badger, Becchetti, Chaubey, Marzucca '23] [Badger, Becchetti, Giraudo, Zoia '24] [Becchetti, Dlapa, Zoia '25]
- Numerical evaluation two-loop amplitude: [Badger, Becchetti, **CB**, Hartanto, Zoia '24]



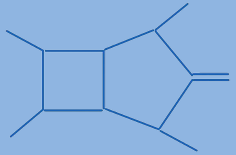
Current frontier of 2 → 3 scattering amplitudes

Massless external particles:



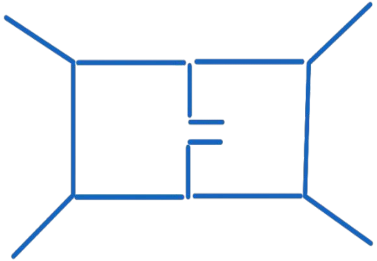
- pp → γγγ [Abreu, Page, Pascual, Sotnikov '20] [Chawdhry, Czakon, Mitov, Poncelet '21] [Abreu, De Laurentis, Ita, Klinkert, Page, Sotnikov '23]
- pp → γγj [Agarwal, Buccioni, von Manteuffel, Tancredi '21] [Chawdhry, Czakon, Mitov, Poncelet '21] [Badger, Brönnum-Hansen, Chicherin, Gehrmann, Hartanto, Henn, Marcoli, Moodie, Peraro, Zoia '21] [Buccioni, Chen, Feng, Gehrmann, Huss, Marcoli, '25]
- pp → γjj [Badger, Czakon, Hartanto, Moodie, Peraro, Poncelet, Zoia '23]
- pp → jjj [Abreu, Febres Cordero, Ita, Page, Sotnikov '21] [De Laurentis, Ita, Klinkert, Sotnikov '23] [Agarwal, Buccioni, Devoto, Gambuti, von Manteuffel, Tancredi '23] [De Laurentis, Ita, Sotnikov '23]

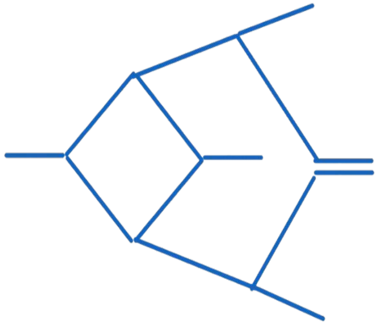
One massive external particle leading colour:



- pp → Wbb [Badger, Hartanto, Zoia '21] [Hartanto, Poncelet, Popescu, Zoia '22]
- pp → Wjj [Abreu, Febres Cordero, Ita, Klinkert, Page, Sotnikov '22]
- pp → Hbb [Badger, Hartanto, Krys, Zoia '21]
- pp → Wγj [Badger, Hartanto, Krys, Zoia '22]
- pp → W/Zbb [Buonocore, Devoto, Kallweit, Mazzitelli, Rottoli, Savoini '22] [Mazzitelli, Sotnikov, Wiesemann '24]

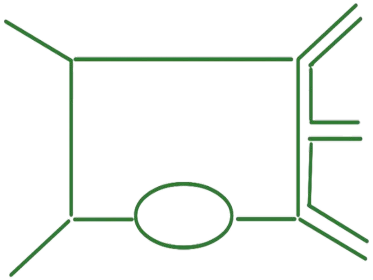
Current frontier of 2 → 3 scattering amplitudes

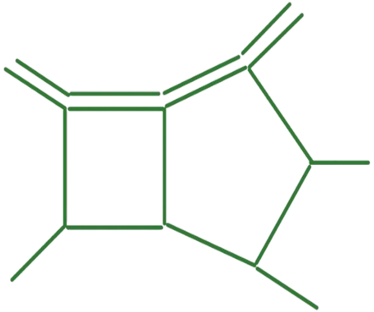




One massive external particle full colour:

- pp → Wγγ [Badger, Hartanto, Wu, Zhang, Zoia '24]
- pp → Hbb [Badger, Hartanto, Poncelet, Wu, Zhang, Zoia '24]





Numerical evaluations with more internal masses:

- qq → ttH (nf part) [Agarwal, Heinrich, Jones, Kerner, Klein, Lang, Magerya, Olsson '24]
- gg → ttg (leading colour) [Badger, Becchetti, **CB**, Hartanto, Zoia '24]

Growing challenges for phenomenology

Massless amplitudes

Massless internal propagators

Logs, Classical Polylogs, MLPs ...

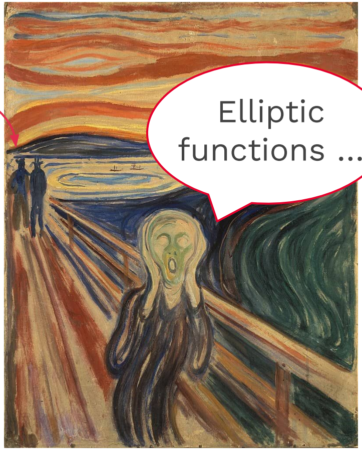
- Analytic cancellation of the poles
- Compact expressions
- Fast and stable numerics

↓

Pentagon functions approach

Internal masses

Massive internal propagators



Elliptic functions ...

- Functions under investigation

↓

How to obtain the same advantages of the massless case?

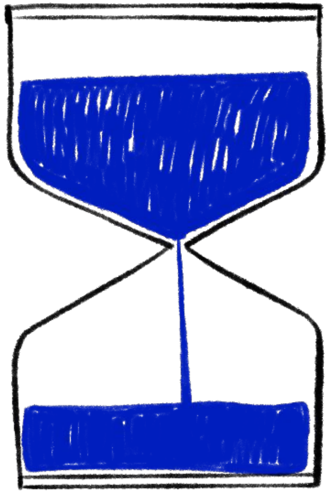
Plan for the next ~half an hour

- **Leading colour** approximation
- **Helicity** amplitudes
- Integral families
- **Finite field** methods
- Review pentagon functions approach
- **Extend the pentagon function strategy** to non-canonical DEs
  $\longrightarrow$ minimise the impact of non-polylogarithmic functions
- Numerical evaluation of special functions
- **Numerical evaluation and analytic reconstruction** of the amplitude

AMPLITUDE COMPUTATION

Finite field methods &

Special function basis



Colour decomposition

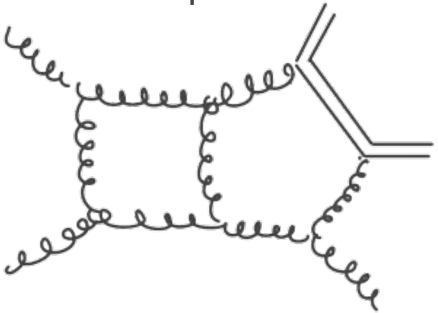
- Consider all diagrams contributing to the process (gg→ttg @ 2 loops)

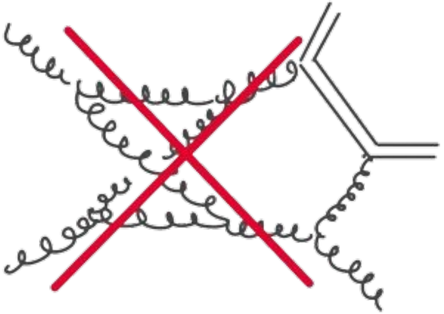
[QGRAF]

$$A^{(L)}(\vec{x},\epsilon) = \sum \Big(\quad \Big)$$

- **Colour expansion** ➡ take the leading colour limit
  ➡ reduce the complexity of the loop integrals

At two loops:





diagrams	tot	LC
tree-level	16	6
1-loop	384	77
2-loop	11370	1357

Leading colour ∝ Nc² ➡ only planar diagrams

Helicity amplitudes

$$A^{(L)} = \sum_{i=1}^{N} \mathcal{T}_i \, \mathcal{F}_i^{(L)}$$

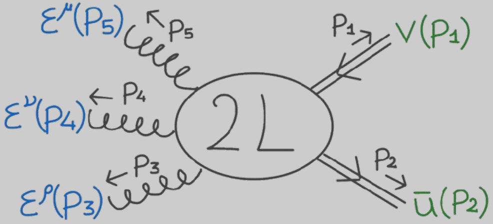
Tensor structure

Ex. gg -> ttg

$$\mathcal{T}_1 = m_t^2 \; \bar{u}(p_2) v(p_1) \varepsilon(p_3, q_3) \cdot p_1 \; \varepsilon(p_4, q_4) \cdot p_1 \; \varepsilon(p_5, q_5) \cdot p_1$$

$$\vdots$$

$$\mathcal{T}_{32} = \ldots$$



Form factors

$$\mathcal{F}_i^{(L)} = \sum_{j=1}^{N} \left(\Theta^{-1}\right)_{ij} \underbrace{\sum_{\text{pol.}} \mathcal{T}_j^{\dagger} A^{(L)}}_{A^{(L),\text{proj}}}$$

with $\Theta_{ij} = \mathcal{T}_i^{\dagger} \mathcal{T}_j$

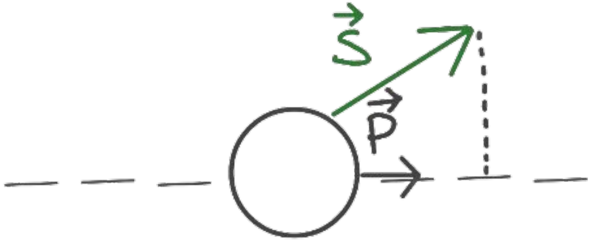
Specify helicities: $$A^{(L),h_1 h_2 h_3 h_4 h_5} = \sum_{i=1}^{N} \mathcal{T}_i^{h_1 h_2 h_3 h_4 h_5} \; \mathcal{F}_i^{(L)}$$

Massive spinors

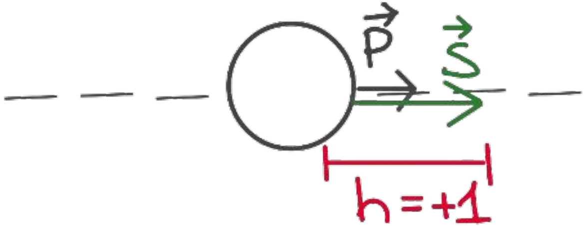
Helicity amplitudes encode spin correlation information
→ inclusion of top-quark decay in narrow-width approximation

Helicity: projection of the spin along the direction of momentum

Massless case:



Example

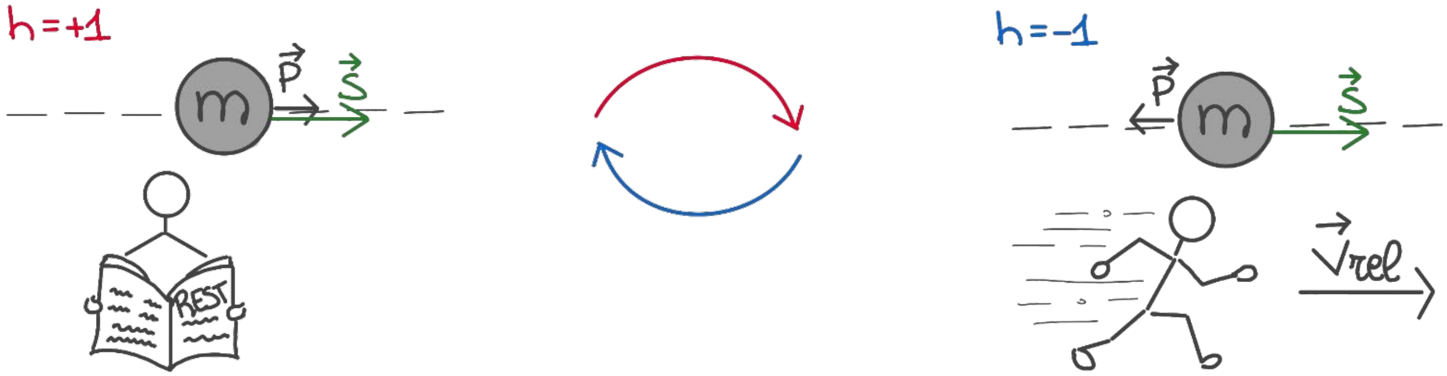


Massive spinors

Helicity amplitudes encode spin correlation information
→ inclusion of top-quark decay in narrow-width approximation

Helicity: projection of the spin along the direction of momentum

Massive case:



Massive spinors

Helicity amplitudes encode spin correlation information
→ inclusion of top-quark decay in narrow-width approximation

Helicity: projection of the spin along the direction of momentum

Massive case:

$$u_-(p,m) = \frac{\langle p^\flat n\rangle}{m}\left(u_+(p,m)|_{p^\flat \leftrightarrow n}\right)$$

$$u_+(p,m) = \frac{(\not{p}+m)|n\rangle}{\langle p^\flat\, n\rangle} \qquad u_-(p,m) = \frac{(\not{p}+m)|n]}{[p^\flat\, n]}$$

where n an arbitrary light-like momentum and $p^{\flat,\mu} = p^\mu - \frac{m^2}{2p\cdot n} n^\mu$

see [Kleiss, Stirling '85] [Arkani-Hamed, Huang, Huang '17]

Reduction to Master Integrals (MIs)

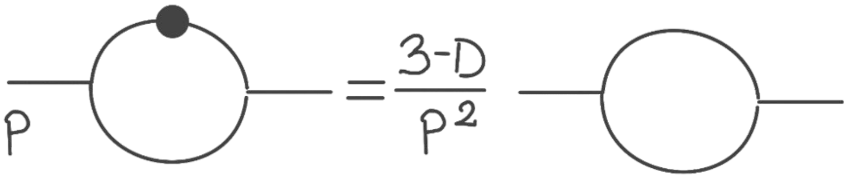
- The amplitude is written in terms of a large set of Feynman integrals:

$$\boxed{A^{(L),\mathrm{proj}}(\vec{x},\epsilon) = \sum_i c_i(\vec{x},\epsilon) I_i(\vec{x},\epsilon)}$$

i.e. $I(\vec{x},\epsilon) = \int \frac{d^{\mathrm{D}}k_1 d^{\mathrm{D}}k_2}{k_1^2(k_1+p_1)^2(k_1+p_1+p_2)^2 \dots}$ $\quad \mathrm{D} = 4-2\epsilon$

- Feynman integrals ⟶ linear combination of an integral basis (alias MIs)
  ⟶ obtained solving **Integration by Parts Identities (IBPs)**

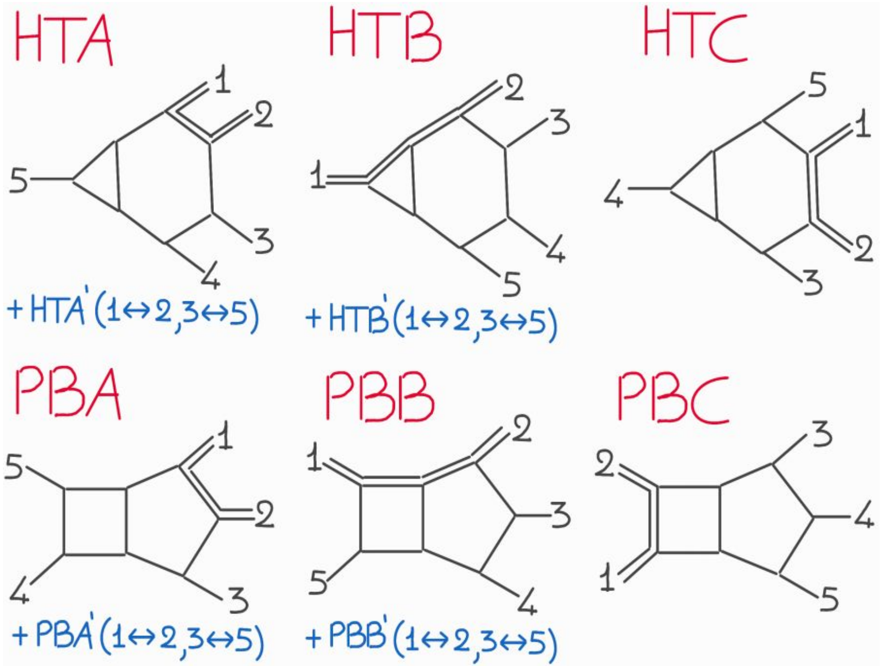
[Tkachov, '81][Chetyrkin, Tkachov, '81]
[Laporta, '00]



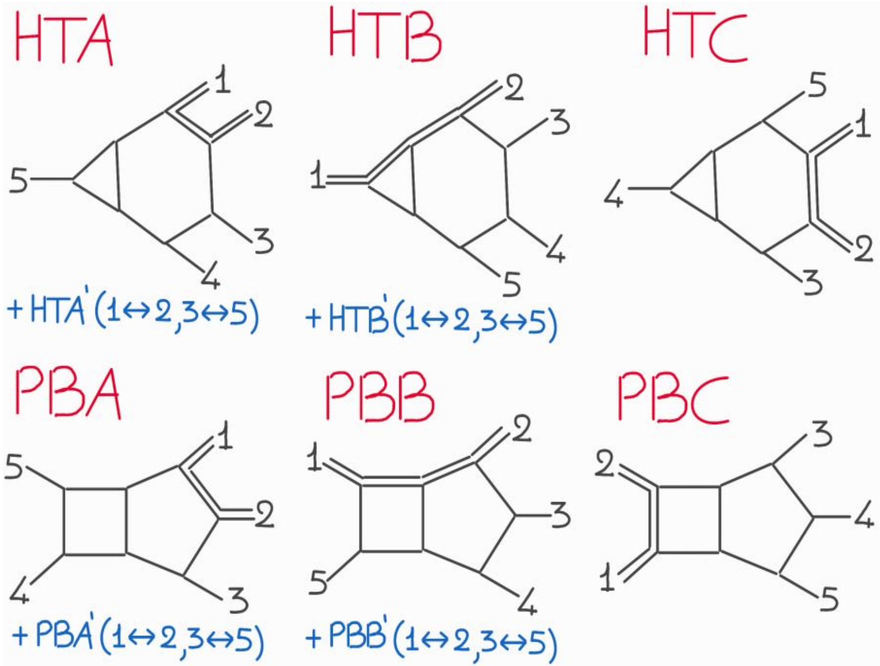
- **IBP solution dominates the total evaluation time**
  ⟶ IBPs generated in an optimised way thanks to *NeatIBP* + *FiniteFlow*

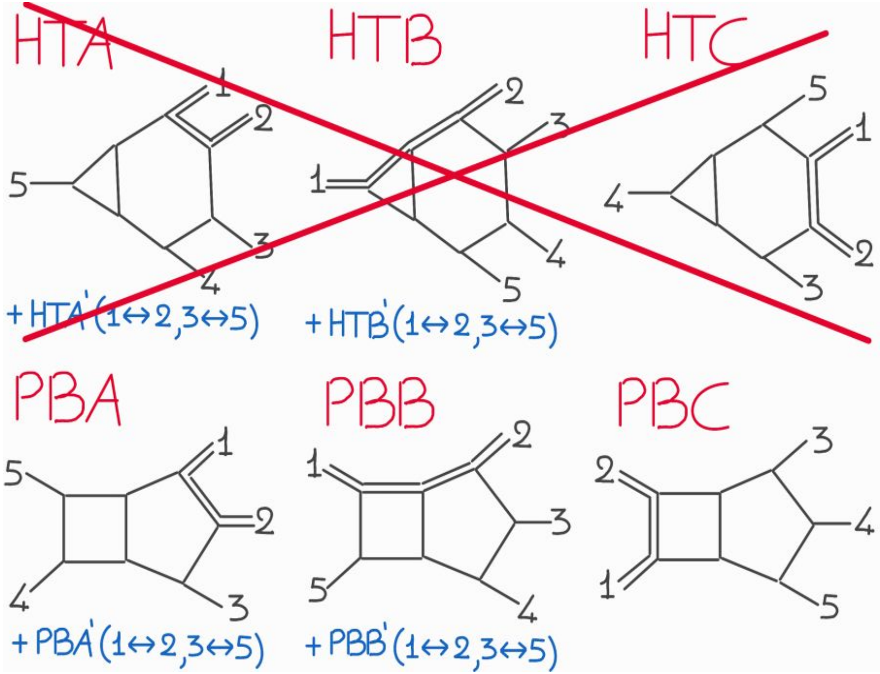
[Wu, Boehm, Ma, Xu, Zhang '23] [Peraro, '19]

Integral families



Integral families





Algebraic Vs analytic complexity

Amplitude in terms of MIs

$$A^{(2),\mathrm{proj}}(\vec{x},\epsilon) = \sum_i \boxed{d_i(\vec{x},\epsilon)}\,\boxed{f_i(\vec{x},\epsilon)}$$

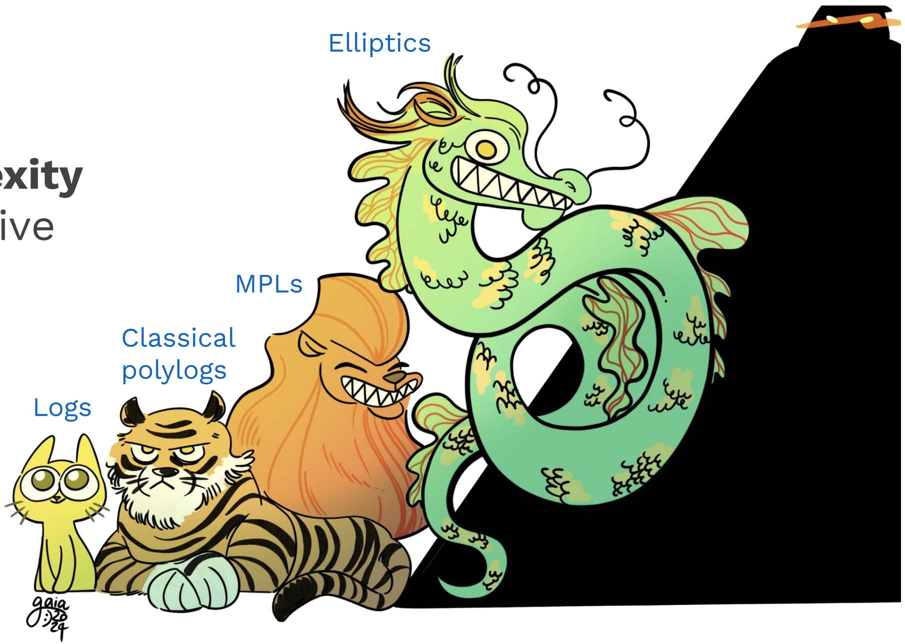
Algebraic complexity
→ 6 variables

Analytic complexity
→ internal massive propagators

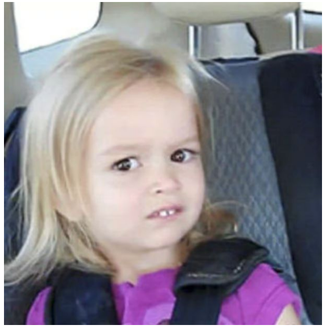
loops

Difficulty grows with:
- Loop order
- Number of particles

particles



Algebraic complexity



Intermediate steps in scattering amplitude computations can produce very **large expressions**

Replace symbolic operations with numerical evaluations in a **finite field**
(integers mod prime P)

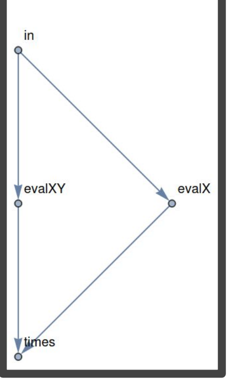
[von Manteuffel, Schabinger '14] [Peraro '16]

Numerical framework: *FiniteFlow*

[Peraro '19]

```
<< FiniteFlow`

FUNC1 = x^4 + y^8;
FUNC2 = x^2;

FFNewGraph[graphXY, in, {x, y}];
FFAlgRatFunEval[graphXY, evalXY, {in}, {x, y}, {FUNC1}];
FFGraphOutput[graphXY, evalXY];

FFGraphEvaluate[graphXY, RandomInteger[{10^6, 10^7}, 2]]

{5 374 497 635 291 780 286}

FFAlgRatFunEval[graphXY, evalX, {in},  {x, y}, {FUNC2}];
FFAlgMul[graphXY, times, {evalX, evalXY}];
FFGraphOutput[graphXY, times];

FFReconstructFunction[graphXY, {x, y}]

{x^6 + x^2 y^8}
```

```
FFGraphDraw[graphXY]
```

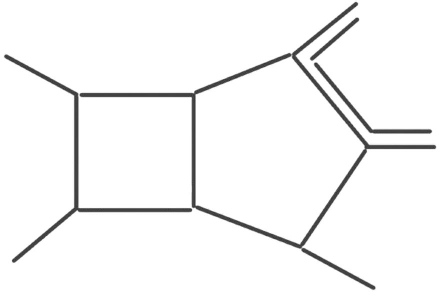


Differential equations for MIs

MIs satisfy the following **differential equation**:

$$\mathrm{d}\vec{f}(\vec{x},\epsilon) = \mathrm{d}A(\vec{x},\epsilon)\ \vec{f}(\vec{x},\epsilon)$$

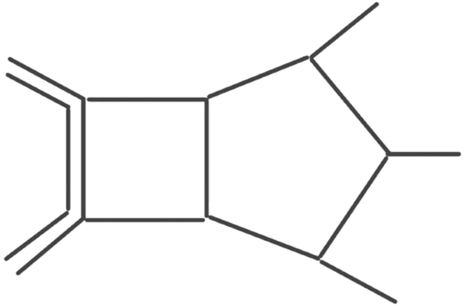
[Kotikov '91-'93]
[Bern, Dixon, Kosower '94]
[Remiddi '97]
[Gehrmann, '00]
[Henn, '13]



For **PBA** and **PBC**:

[Badger, Becchetti, Chaubey, Marzucca '23]
[Badger, Becchetti, Giraudo, Zoia '24]

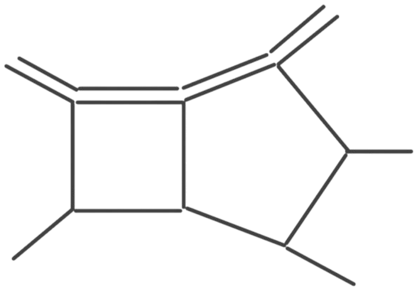
$$\mathrm{d}A_{\tau,\sigma}(\vec{x}) = \underbrace{\epsilon}_{\epsilon\text{ factorised}} \sum_i c_i \underbrace{\mathrm{dlog}\, W_i(\vec{x})}_{\text{dlog-form}}$$



Differential equations for MIs

MIs satisfy the following **differential equation**:

$$\boxed{\mathrm{d}\vec{f}(\vec{x},\epsilon) = \mathrm{d}A(\vec{x},\epsilon)\ \vec{f}(\vec{x},\epsilon)}$$



For **PBB**:

[Badger, Becchetti, Giraudo, Zoia '24]

$$\mathrm{d}A(\vec{x},\epsilon) = \sum_{k=0}^{2} \underbrace{\epsilon^k}_{\text{DEs quadratic in } \epsilon} \sum_i c_{ki}\, \underbrace{\omega_i(\vec{x})}_{\text{one-form}}$$

solution in terms of elliptic functions

DANGER

Canonical form in the new paper by Becchetti, Dlapa, Zoia

~~Analytic Solution~~

Numerical Integration

Laurent expansion

- Expand the amplitude around $\epsilon = 0$ to compare the **pole structure** against UV renormalization and IR subtraction

- The Laurent expansion of the MIs is

$$\vec{f}(\vec{x}, \epsilon) = \sum_{k=0}^{4} \epsilon^k \vec{f}^{(k)}(\vec{x})$$

- Construction of a **basis of algebraically independent MIs components** in terms of which we can express the amplitude

$$A^{(2L),\text{proj}}(\epsilon, \vec{x}) = \sum_i \sum_{k=-4}^{0} \epsilon^k r_{ki}(\vec{x}) \mathrm{F}_i(\vec{x})$$

Pentagon functions approach

We would like to achieve:

1. **Analytic UV/IR pole subtraction**
2. **Simplification of finite remainders**
3. **Improved numerical evaluation**

“**Pentagon functions**” approach particularly successful for 2-loop 5-point with no internal masses

see [Gehrmann, Henn, Jakubčík , Lim, Mella, Syrrakos, Tancredi, Bobadilla ‘24] [Gehrmann, Henn, Lo Presti ’18] [Chicherin, Sotnikov ’20] [Chicherin, Sotnikov, Zoia ’22] [Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia ’24]

Pentagon functions approach

How does the pentagon functions approach work?

Pentagon functions derivation: Step 0

The inputs of the algorithm are:

- **canonical DEs** for all topologies and permutations

$$\mathrm{d}\vec{f}_{\underbrace{\tau}_{\text{topology}},\underbrace{\sigma}_{\text{permutation}}}(\vec{x}) = \epsilon\,\mathrm{d}A_{\tau,\sigma}(\vec{x})\vec{f}_{\tau,\sigma}(\vec{x})$$

$$\mathrm{d}A_{\tau,\sigma}(\vec{x}) = \sum_{i=1}^{n} c_{\tau,\sigma}^{(i)}\,\mathrm{dlog}\,W_i(\vec{x})$$

- **Boundary values** for the MIs components (will be needed at Step 2 😉)

Numerical boundary values obtained with *AMFlow*

[Liu, Ma, Wang '18] [Liu, Ma '22]

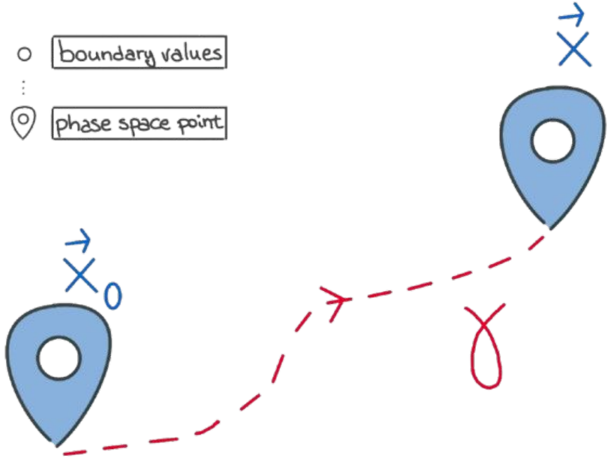
Pentagon functions derivation: Step 1

- Solution of the canonical DEs in terms of **Chen iterated integrals**

$$\vec{f}(\vec{x},\epsilon) = \sum_{k=0}^{4} \epsilon^k \vec{f}^{(k)}(\vec{x})$$

Laurent expansion up to the required order

$$\vec{f}^{(k)}(\vec{x}) = \underbrace{\int_\gamma \mathrm{d}A(\vec{x}')\vec{f}^{(k-1)}(\vec{x}')}_{\int_\gamma \mathrm{dlog}\, W_{i_k}(\vec{x}')[W_{i_1},\ldots,W_{i_{k-1}}]_{\vec{x}_0}(\vec{x}')} + \vec{f}^{(k)}(\vec{x}_0)$$



Chen iterated integrals $\equiv [W_{i_1},\ldots,W_{i_k}]_{\vec{x}_0}(\vec{x}')$

Pentagon functions derivation: Step 1

- Solution of the canonical DEs in terms of **Chen iterated integrals**
- Write the MIs components in terms of **symbols**

Symbol = iterated integral without the boundary information

[Goncharov, Spradlin, Vergu, Volovich '10]
[Duhr, Gangl, Rhodes '12]

Example

$$\mathrm{Li}_2(x) = -\int_0^1 \mathrm{dlog}(x) \circ \mathrm{dlog}(1-x)$$

$$\mathcal{S}\left(\mathrm{Li}_2(x)\right) = -\left[1-x, x\right]$$

Look for relations among MIs components exploiting that **symbols are independent**

Pentagon functions derivation: Step 1

- Solution of the canonical DEs in terms of **Chen iterated integrals**
- Write the MIs components in terms of **symbols**
- Select the MI components for the **basis at symbol level**

Include products of lower-weight functions

$$\left\{f_i^{(0)}\right\} \longrightarrow \{1\}$$

$$\left\{f_i^{(1)}\right\} \longrightarrow \left\{F_k^{(1)}\right\}$$

$$\left\{f_i^{(2)}\right\} \cup \left\{F_i^{(1)} \times F_j^{(1)}\right\} \longrightarrow \left\{F_k^{(2)}\right\}$$

ATTENTION!
Pentagon functions in progress

Tune ordering to streamline the amplitude

Preferred ordering:

- lower-weight functions over higher-weight functions
- 1-loop functions over 2-loop functions

Pentagon functions derivation: Step 1

- Solution of the canonical DEs in terms of **Chen iterated integrals**
- Write the MIs components in terms of **symbols**
- Select the MI components for the **basis at symbol level**
- At the end of this step we have

★ The **pentagon function basis** $\left\{F_i^{(k)}\right\}$
★ The **MIs components** as a polynomial in the $\left\{F_i^{(k)}\right\}$ basis **at symbol level**

Ex. $$f_{29}^{(2)} = 2\,F_{11}^{(1)}\,F_{13}^{(1)} + \left(F_{13}^{(1)}\right)^2 + F_{11}^{(2)}$$

Pentagon functions derivation: Step 2

Write the **MIs components as iterated integrals**

We make the following **ansatz**:

[Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia '24]

> All MIs components are polynomials in $\left\{ F_i^{(2)} \right\}$ + $\vec{c}\,\zeta_2$

$$f_i^{(2)} = \underbrace{\sum_j a_j F_j^{(2)} + \sum_{j \leq k} b_{jk} F_j^{(1)} F_k^{(1)}}_{\text{Fixed from symbol level analysis}} + c\,\zeta_2$$

Match with boundaries

$$\vec{f}^{\,(2)} = \sum_{i_1 i_2} M_{i_1 i_2} \left[W_{i_1} W_{i_2} \right] \vec{b}^{\,(0)} + \sum_i M_i \left[W_i \right] \vec{b}^{\,(1)} + \vec{b}^{\,(2)}$$

Boundaries fixed at Step 0

Example
At weight 2

Pentagon functions derivation: Step 2

Write the **MIs components as iterated integrals**

We make the following **ansatz**:

All MIs components are polynomials in $\left\{F_i^{(k)}\right\}$ + terms including $\zeta_2,\ \zeta_3$

At the end of the algorithm we have:

★ The **pentagon function basis** $\left\{F_i^{(k)}\right\}$

★ The **MIs components** as polynomials in $\left\{F_i^{(k)}\right\}$ and $\zeta_2,\ \zeta_3$ with coefficients in $\mathbb{Q}$

Ex.
$$f_{29}^{(2)} = 2\,F_{11}^{(1)}\,F_{13}^{(1)} + \left(F_{13}^{(1)}\right)^2 + F_{11}^{(2)} + \zeta_2$$

Pentagon functions derivation: non-canonical case

I thought I saw
a puddy ~~cat~~
non-canonical equation

Pentagon functions derivation: non-canonical case

- Choose the MIs so that the **non-polylogarithmic** functions are **pushed in the finite part**
  $\longrightarrow$ Numerical check of which $f_i^{(k)}(\vec{x}) = 0$

- **Possible because**
  - the pole structure of the 2 loop amplitude is set by 1 loop amplitude

- **Not obvious because**
  - arbitrary choice of the MIs (which have poles)
  - DEs couple "non-problematic" MIs to the problematic ones

DEs for tt+jet production

The pentagon functions approach applies straightforwardly to PBA and PBC

For PBB (and its permutation), we have:

$$\mathrm{d}\vec{f}_B(\vec{x},\epsilon) = \Omega_B(\vec{x},\epsilon)\vec{f}_B(\vec{x},\epsilon)$$

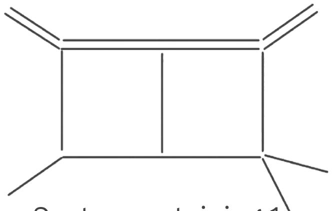
$$\Omega_B(\vec{x},\epsilon) = \sum_{k=0}^{2} \epsilon^k \Omega_B^{(k)}(\vec{x})$$

to indicate the topology PBB

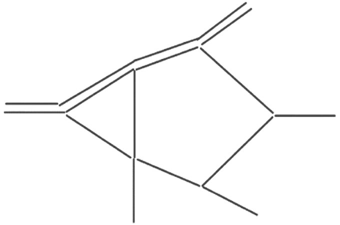
$$\Omega_B^{(k)}(\vec{x}) = \underbrace{\sum_i A_{k,i}^B \mathrm{dlog}\, W_i(\vec{x})}_{\text{Effort to shape the DEs in dlog-form as much as possible}} + \sum_j B_{k,j}^B \omega_j(\vec{x})$$

$\mathcal{Q}$ linearly independent non-logarithmic one-forms

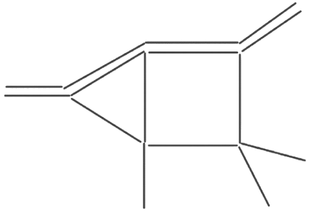
e.g. $\omega(x,y) = \omega_x(x,y)\mathrm{d}x + \omega_y(x,y)\mathrm{d}y$



Sector containing 1 “problematic” MI



Sector containing 2 “problematic” MIs



Sector containing 3 “problematic” MIs

Conspiracy among the boundary values

The DEs for the component $\vec{f}^{(k)}(\vec{x})$ are

$$\mathrm{d}\vec{f}^{(k)}(\vec{x}) = \Omega^{(0)}(\vec{x})\,\vec{f}^{(k)}(\vec{x}) + \Omega^{(1)}(X)\,\vec{f}^{(k-1)}(\vec{x}) + \Omega^{(2)}(X)\,\vec{f}^{(k-2)}(\vec{x})$$

with $k \geq 0$

$\vec{f}^{(-2)}(\vec{x}) = \vec{f}^{(-1)}(\vec{x}) = 0$

Conspiracy among the boundary values

The DEs for the coefficients $\vec{f}^{(k)}(\vec{x})$ are

with $k \geq 0$

$$\mathrm{d}\vec{f}^{(k)}(\vec{x}) = \cancel{\Omega^{(0)}(\vec{x})\,\vec{f}^{(k)}(\vec{x})} + \cancel{\Omega^{(1)}(X)\,\vec{f}^{(k-1)}(\vec{x})} + \cancel{\Omega^{(2)}(X)\,\vec{f}^{(k-2)}(\vec{x})} \qquad \vec{f}^{(-2)}(\vec{x}) = \vec{f}^{(-1)}(\vec{x}) = 0$$

$\boxed{k=0}$ $\quad \mathrm{d}\vec{f}^{(0)}(\vec{x}) = 0 \quad \longrightarrow$ From the knowledge of which coefficients vanish, based on sufficient numerical evaluation using AMFlow

$\longrightarrow \quad \vec{f}^{(0)}(\vec{x})$ Constant

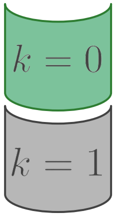
Conspiracy among the boundary values

The DEs for the coefficients $\vec{f}^{\,(k)}(\vec{x})$ are

with $k \geq 0$

$$\mathrm{d}\vec{f}^{\,(k)}(\vec{x}) = \cancel{\Omega^{(0)}(\vec{x})\,\vec{f}^{\,(k)}(\vec{x})} + \Omega^{(1)}(X)\,\vec{f}^{\,(k-1)}(\vec{x}) + \cancel{\Omega^{(2)}(X)\,\vec{f}^{\,(k-2)}(\vec{x})}$$

$$\vec{f}^{\,(-2)}(\vec{x}) = \vec{f}^{\,(-1)}(\vec{x}) = 0$$



$$\mathrm{d}\vec{f}^{\,(0)}(\vec{x}) = 0 \longrightarrow \vec{f}^{\,(0)}(\vec{x}) \text{ Constant}$$

$$\mathrm{d}\vec{f}^{\,(1)}(\vec{x}) = \sum_i A_{1,i}\,\mathrm{dlog}\,W_i(\vec{x})\,\vec{f}^{\,(0)} \longrightarrow \text{Canonical}$$

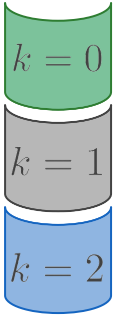
Conspiracy among the boundary values

The DEs for the coefficients $\vec{f}^{(k)}(\vec{x})$ are

$$\mathrm{d}\vec{f}^{(k)}(\vec{x}) = \cancel{\Omega^{(0)}(\vec{x})\,\vec{f}^{(k)}(\vec{x})} + \Omega^{(1)}(X)\,\vec{f}^{(k-1)}(\vec{x}) + \cancel{\Omega^{(2)}(X)\,\vec{f}^{(k-2)}(\vec{x})}$$

with $k \geq 0$

$$\vec{f}^{(-2)}(\vec{x}) = \vec{f}^{(-1)}(\vec{x}) = 0$$



$$\mathrm{d}\vec{f}^{(0)}(\vec{x}) = 0 \longrightarrow \vec{f}^{(0)}(\vec{x})$$ Constant

$$\mathrm{d}\vec{f}^{(1)}(\vec{x}) = \sum_i A_{1,i}\,\mathrm{dlog}\,W_i(\vec{x})\,\vec{f}^{(0)} \longrightarrow$$ Canonical

$$\mathrm{d}f^{(2)}_{15}(\vec{x}) = \frac{1}{24}\underbrace{\left[12\,f^{(1)}_{103}(\vec{x}_0)+8\,f^{(1)}_{110}(\vec{x}_0)+4\,f^{(1)}_{111}(\vec{x}_0)+3\,f^{(1)}_{118}(\vec{x}_0)-48\,f^{(1)}_{63}(\vec{x}_0)\right]}_{\text{vanishing for a conspiracy of the boundary values}}\omega_2(\vec{x})+\dots\,.$$

$$\downarrow$$

$$\mathrm{d}\vec{f}^{(2)}(\vec{x}) = \sum_i A_{2,i}\,\mathrm{dlog}\,W_i(\vec{x})\,\vec{f}^{(1)} \longrightarrow$$ Canonical

Conspiracy among the boundary values

The DEs for the coefficients $\vec{f}^{(k)}(\vec{x})$ are

with $k \geq 0$

$$\mathrm{d}\vec{f}^{(k)}(\vec{x}) = \cancel{\Omega^{(0)}(\vec{x})\,\vec{f}^{(k)}(\vec{x})} + \Omega^{(1)}(X)\,\vec{f}^{(k-1)}(\vec{x}) + \cancel{\Omega^{(2)}(X)\,\vec{f}^{(k-2)}(\vec{x})}$$

$$\vec{f}^{(-2)}(\vec{x}) = \vec{f}^{(-1)}(\vec{x}) = 0$$

$k=0$: $\mathrm{d}\vec{f}^{(0)}(\vec{x}) = 0 \longrightarrow \vec{f}^{(0)}(\vec{x})$ Constant

$k=1$: $\mathrm{d}\vec{f}^{(1)}(\vec{x}) = \sum_i A_{1,i}\,\mathrm{dlog}\,W_i(\vec{x})\,\vec{f}^{(0)} \longrightarrow$ Canonical

$k=2$: $\mathrm{d}\vec{f}^{(2)}(\vec{x}) = \sum_i A_{2,i}\,\mathrm{dlog}\,W_i(\vec{x})\,\vec{f}^{(1)} \longrightarrow$ Canonical

$k=3$: Same as for $k=2$

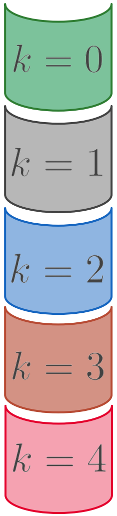
Conspiracy among the boundary values

The DEs for the coefficients $\vec{f}^{(k)}(\vec{x})$ are

$$\mathrm{d}\vec{f}^{(k)}(\vec{x}) = \Omega^{(0)}(\vec{x})\,\vec{f}^{(k)}(\vec{x}) + \Omega^{(1)}(X)\,\vec{f}^{(k-1)}(\vec{x}) + \Omega^{(2)}(X)\,\vec{f}^{(k-2)}(\vec{x})$$

with $k \geq 0$

$$\vec{f}^{(-2)}(\vec{x}) = \vec{f}^{(-1)}(\vec{x}) = 0$$



$\mathrm{d}\vec{f}^{(0)}(\vec{x}) = 0 \longrightarrow \vec{f}^{(0)}(\vec{x})$ Constant

$\mathrm{d}\vec{f}^{(1)}(\vec{x}) = \sum_i A_{1,i}\,\mathrm{dlog}\,W_i(\vec{x})\,\vec{f}^{(0)} \longrightarrow$ Canonical

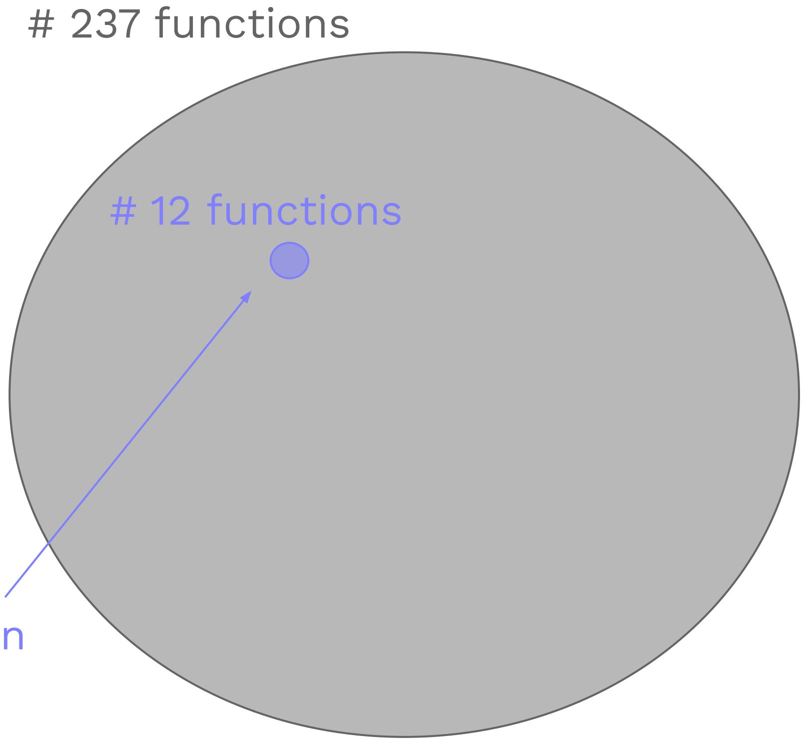
$\mathrm{d}\vec{f}^{(2)}(\vec{x}) = \sum_i A_{2,i}\,\mathrm{dlog}\,W_i(\vec{x})\,\vec{f}^{(1)} \longrightarrow$ Canonical

Same as for $k=2$

$\vec{f}_i^{(4)}(\vec{x}) = F_i^{(4^*)}$ $\longrightarrow$ Non-polylogarithmic special functions $\longrightarrow$ MIs of the elliptic sectors

Polylogarithmic Vs Non-polylogarithmic

- All special functions
- Non-polylogarithmic functions



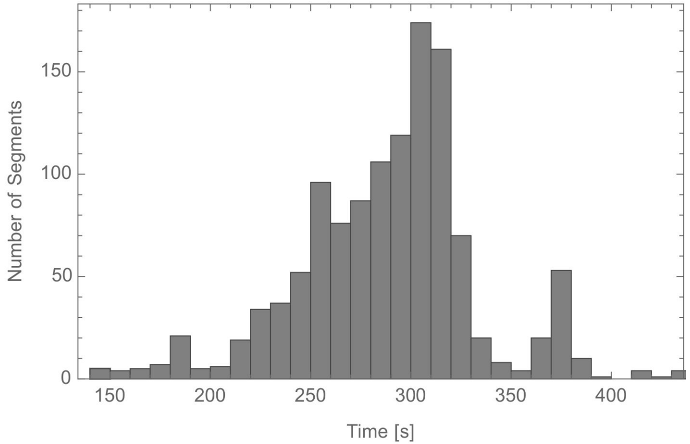
we know how to achieve
fast+stable numerics

need further investigation

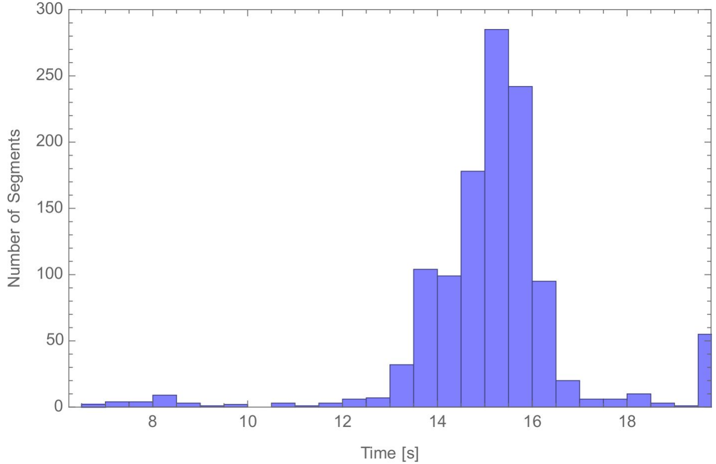
Numerical evaluation of special functions

We solve the DEs for the special functions using **generalised power series** [Moriello '20] method as implemented in *DiffExp* [Hidding '21]

Evaluation time per segment - Special functions



Evaluation time per segment - Non-polylog. functions



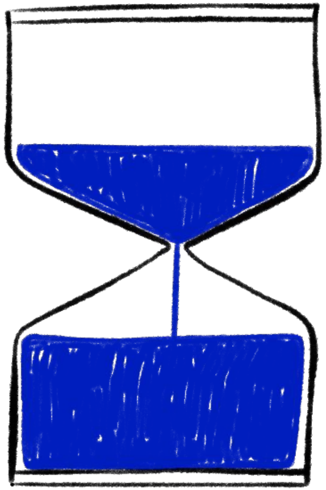
	PB_A	PB_B	PB_C	all MIs	special func. (all)	special func. (non-polylog.)
$\langle T \rangle$	$43\,s$	$77\,s$	$66\,s$	$309\,s$	$297\,s$	$16\,s$
σ	$7\,s$	$17\,s$	$14\,s$	$27\,s$	$65\,s$	$3\,s$

RESULTS

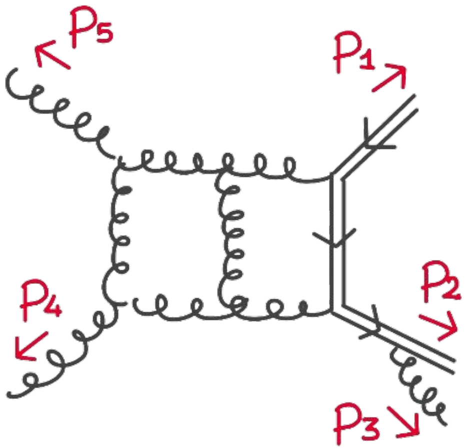
Two-loop gg→ttg helicity amplitudes:

Numerical evaluation

and analytic results



Notation and kinematics



Process:

$$g(-p_4) + g(-p_5) \to \bar{t}(p_1) + t(p_2) + g(p_3)$$

Kinematics:

$$\vec{x} = (d_{12}, d_{23}, d_{34}, d_{45}, d_{15}, m_t^2)$$

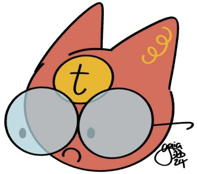
Spin structure basis for helicity states:

$$A^{(L)}(1_t^+, 2_{\bar{t}}^+, 3^{h_3}, 4^{h_4}, 5^{h_5}; n_t, n_{\bar{t}}) = m_t \Phi^{h_3 h_4 h_5} \times \sum_{i=1}^{4} \Theta_i(n_t, n_{\bar{t}})\, A^{(L),[i]}(1_t^+, 2_{\bar{t}}^+, 3^{h_3}, 4^{h_4}, 5^{h_5})$$

Finite remainder reconstruction

- Mass-renormalised amplitudes are gauge invariant
  ➡ **Gauge invariance check**
- UV/IR poles identified analytically and finite remainder computed directly
  ➡ **Pole check**
- Results cross-checked against independent computation of helicity amplitudes in terms of momentum-twistor variables
  ➡ **Simplification of the amplitude**

Sub-amplitude	max degrees MIs recon.	max degrees SF recon.
$A^{(2),[1]}(1_t^+, 2_{\bar{t}}^+, 3^{h_3}, 4^{h_4}, 5^{h_5})$	$404/393$	$314/303$
$A^{(2),[2]}(1_t^+, 2_{\bar{t}}^+, 3^{h_3}, 4^{h_4}, 5^{h_5})$	$398/389$	$305/296$
$A^{(2),[3]}(1_t^+, 2_{\bar{t}}^+, 3^{h_3}, 4^{h_4}, 5^{h_5})$	$411/402$	$321/312$
$A^{(2),[4]}(1_t^+, 2_{\bar{t}}^+, 3^{h_3}, 4^{h_4}, 5^{h_5})$	$420/411$	$326/317$



RESULTS

Numerical evaluation of the finite remainder

Numerical evaluation of finite remainders up to two loops in the leading color limit for gg → ttg production

Helicity	$R^{(0),[1]}$	$R^{(0),[2]}$	$R^{(0),[3]}$	$R^{(0),[4]}$
$+++$	$0.26326 - 0.0097514\,i$	0	0	0
$+-+$	$5.9619 - 0.16047\,i$	0	0	$-0.31659 - 0.097935\,i$
$++-$	$-5.9575 + 0.0089231\,i$	$-12.606 - 0.067440\,i$	$4.6564 + 0.024911\,i$	$-1.9692 - 0.010535\,i$

Helicity	$R^{(1),[1]}/R^{(0),[1]}$	$R^{(1),[2]}/R^{(0),[1]}$	$R^{(1),[3]}/R^{(0),[1]}$	$R^{(1),[4]}/R^{(0),[1]}$
$+++$	$38.396 - 5.8002\,i$	$71.982 - 4.0653\,i$	$-14.289 + 0.70866\,i$	$17.909 - 0.39528\,i$
$+-+$	$19.221 - 8.4151\,i$	$-4.8506 + 4.8015\,i$	$0.67096 - 0.09959\,i$	$-1.2201 + 2.1594\,i$
$++-$	$20.369 - 19.991\,i$	$41.522 - 41.969\,i$	$-15.990 + 15.739\,i$	$6.2964 - 6.4584\,i$

Helicity	$R^{(2),[1]}/R^{(0),[1]}$	$R^{(2),[2]}/R^{(0),[1]}$	$R^{(2),[3]}/R^{(0),[1]}$	$R^{(2),[4]}/R^{(0),[1]}$
$+++$	$882.48 - 91.619\,i$	$2489.7 - 266.72\,i$	$-492.28 + 8.1003\,i$	$593.35 - 87.569\,i$
$+-+$	$414.16 - 206.87\,i$	$-171.78 + 189.69\,i$	$25.226 - 1.5639\,i$	$-54.820 + 95.716\,i$
$++-$	$332.97 - 646.02\,i$	$623.01 - 1325.1\,i$	$-259.14 + 512.33\,i$	$89.185 - 198.65\,i$

Check out our paper arXiv:2412.13876 😊

Beyond numerical evaluation

Is it possible to decrease the maximal polynomial degree of the coefficients?

Univariate partial fraction decomposition

Step 1 Look for **linear relations** among the coefficients

$$\sum_i c_i \, r_i(\vec{x}) = 0 \quad \forall \vec{x}$$

$\longrightarrow$ Express the more complicated coefficients in terms of the simpler ones

Univariate partial fraction decomposition

Step 1 Look for **linear relations** among the coefficients

$$\sum_i c_i \, r_i(\vec{x}) = 0 \quad \forall \vec{x}$$

Step 2 Make the ansatz

$$r_i(\vec{x}) = \frac{N_i(\vec{x})}{\prod_j D(\vec{x})_j^{q_{ij}}}, \qquad q_{ij} \in \mathbb{Z}$$

Denominators known from the study of Feynman integrals contributing to the amplitude

They can be determined by **reconstructing** the coefficients **on a random phase-space slice**

$$r_i(\vec{x}(t)) = r_i(t)$$

[Abreu, Cordero, Page '18]

Univariate partial fraction decomposition

Step 1 Look for **linear relations** among the coefficients

$$\sum_i c_i\, r_i(\vec{x}) = 0 \quad \forall \vec{x}$$

Step 2 Make the ansatz

$$r_i(\vec{x}) = \frac{N_i(\vec{x})}{\prod_j D(\vec{x})_j^{q_{ij}}}, \qquad q_{ij} \in \mathcal{Z}$$

Step 3 Perform **univariate partial fraction decomposition** [Badger, Hartanto, Zoia '21]

Example

$$r_i(\vec{x}) = \frac{x^3 + 2x^2 y + 2xy^2 + 4y^3}{y^2(x^2+y^2)} \longrightarrow \frac{\overset{x}{q_1(x)}}{y^2} + \frac{\overset{2}{q_2(x)}}{y} + \frac{\overset{x}{q_3(x)} + \overset{2}{q_4(x)}y}{x^2+y^2}$$

max. numerator|denominator degrees 4|5 $\longrightarrow$ max. numerator|denominator degrees 1|0

Analytic reconstruction

Univariate partial fraction decomposition
+
Momentum twistor variables
=
feasible analytic reconstruction

$$s_{34} \to 2d_{34}\,, \quad t_{45} \to \frac{d_{45}}{d_{34}}\,, \quad t_{12} \to \frac{d_{12}+m_t^2}{d_{34}}\,, \quad t_{23} \to \frac{d_{23}}{d_{34}}\,, \quad t_{51} \to \frac{d_{15}}{d_{34}}\,,$$

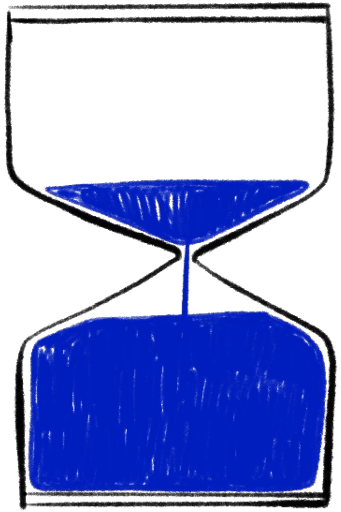
$$x_{5123} \to \frac{-4d_{12}(d_{15}+d_{23}) + 4d_{15}d_{45} - 4d_{34}d_{45} + 4d_{23}(d_{34}-m_t^2) - 4d_{15}m_t^2 + \mathrm{tr}_5}{8(d_{12}+m_t^2)(d_{12}-d_{34}-d_{45}+m_t^2)}$$

with $d_{ij} = p_i \cdot p_j$ and $\mathrm{tr}_5 = 4\,\mathrm{i}\,\epsilon_{\mu\nu\rho\sigma}p_1^\mu p_2^\nu p_3^\rho p_4^\sigma$

no x_{5123} dependence

Sub-amplitude	max degrees original	max degrees UPFD
$A^{(2),[1]}(1_t^+, 2_{\bar{t}}^+, 3^{h_3}, 4^{h_4}, 5^{h_5})$	314/303	44/0
$A^{(2),[2]}(1_t^+, 2_{\bar{t}}^+, 3^{h_3}, 4^{h_4}, 5^{h_5})$	305/296	54/0
$A^{(2),[3]}(1_t^+, 2_{\bar{t}}^+, 3^{h_3}, 4^{h_4}, 5^{h_5})$	321/312	54/0
$A^{(2),[4]}(1_t^+, 2_{\bar{t}}^+, 3^{h_3}, 4^{h_4}, 5^{h_5})$	326/317	56/0

CONCLUSIONS



Conclusions

- **Numerical evaluation** of two-loop leading colour helicity finite remainders for gg→ttg
- Developed a **new strategy for master integrals** bypassing canonical form challenges
- Efficiently isolated **polylogarithmic and non-polylogarithmic components** for numerical evaluation

Outlook

- Completing the **analytic reconstruction** (including contributions from closed fermion loop)
- Optimize numerical evaluation of special functions
- Integration over phase space to deliver **cross-section** results (using available subtraction schemes)

Conclusions

- **Numerical evaluation** of two-loop leading colour helicity finite remainders for gg→ttg
- Developed a **new strategy for master integrals** bypassing canonical form challenges
- Efficiently isolated **polylogarithmic and non-polylogarithmic components** for numerical evaluation

Outlook

- Completing the **analytic reconstruction** (including contributions from closed fermion loop)
- Optimize numerical evaluation of special functions
- Integration over phase space to deliver **cross-section** results (using available subtraction schemes)

Thank You!

BACKUP

BACKUP

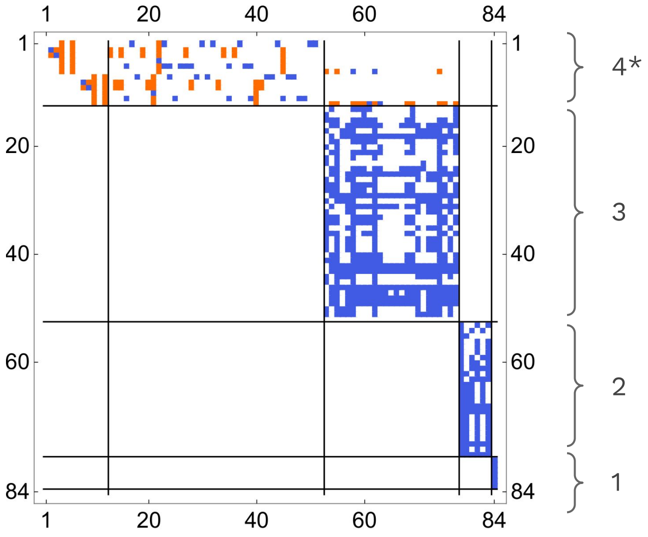
DEs for non-polylogarithmic special functions

The DEs for the non-polylogarithmic functions are

$$\mathrm{d}\vec{G}(\vec{x}) = M(\vec{x})\,\vec{G}(\vec{x})$$

$$\left\{F_i^{(4^*)}\right\}_{i=1}^{12}$$

where the connection matrix has the form



Non-polylogarithmic functions

BACKUP

Physical point

The point chosen in the **physical region** for the sub-amplitudes evaluation is

$$d_{12} = \frac{1617782845110651539}{1506833389797120000}, \quad d_{23} = \frac{335}{1232}, \quad d_{34} = -\frac{5}{32},$$
$$d_{45} = \frac{3665}{7328}, \quad d_{15} = -\frac{45}{1408}, \quad m_t^2 = \frac{376940175237098461}{1506833389797120000},$$

$$\text{with} \quad \mathrm{tr}_5 = \mathrm{i}\,\frac{\sqrt{582950030096630501}}{426229309440}.$$

In terms of **momentum twistor variables**

$$s_{34} = -\frac{5}{16}, \quad t_{45} = -\frac{733}{229}, \quad t_{12} = -\frac{61}{72},$$
$$t_{23} = -\frac{134}{77}, \quad t_{51} = \frac{9}{44}, \quad x_{5123} = \frac{11}{51} + \frac{1}{125}\mathrm{i}.$$