Dissecting Exclusive Multijet Cross Sections

An analysis of factorization properties and perturbative ingredients in direct QCD.

Based on 2509.06612, 2508.19226, 2512.03954, in collaboration with L. Buonocore, M. Grazzini, F. Guadagni, S. Kallweit, and L. Rottoli.

Jürg Haag, München, 16 Dec 2025

Outline

Introduction
Jet cross section and the N-jet limit

Part 1
A direct QCD **framework** to describe the N-jet limit
through factorized hard, soft and collinear ingredients

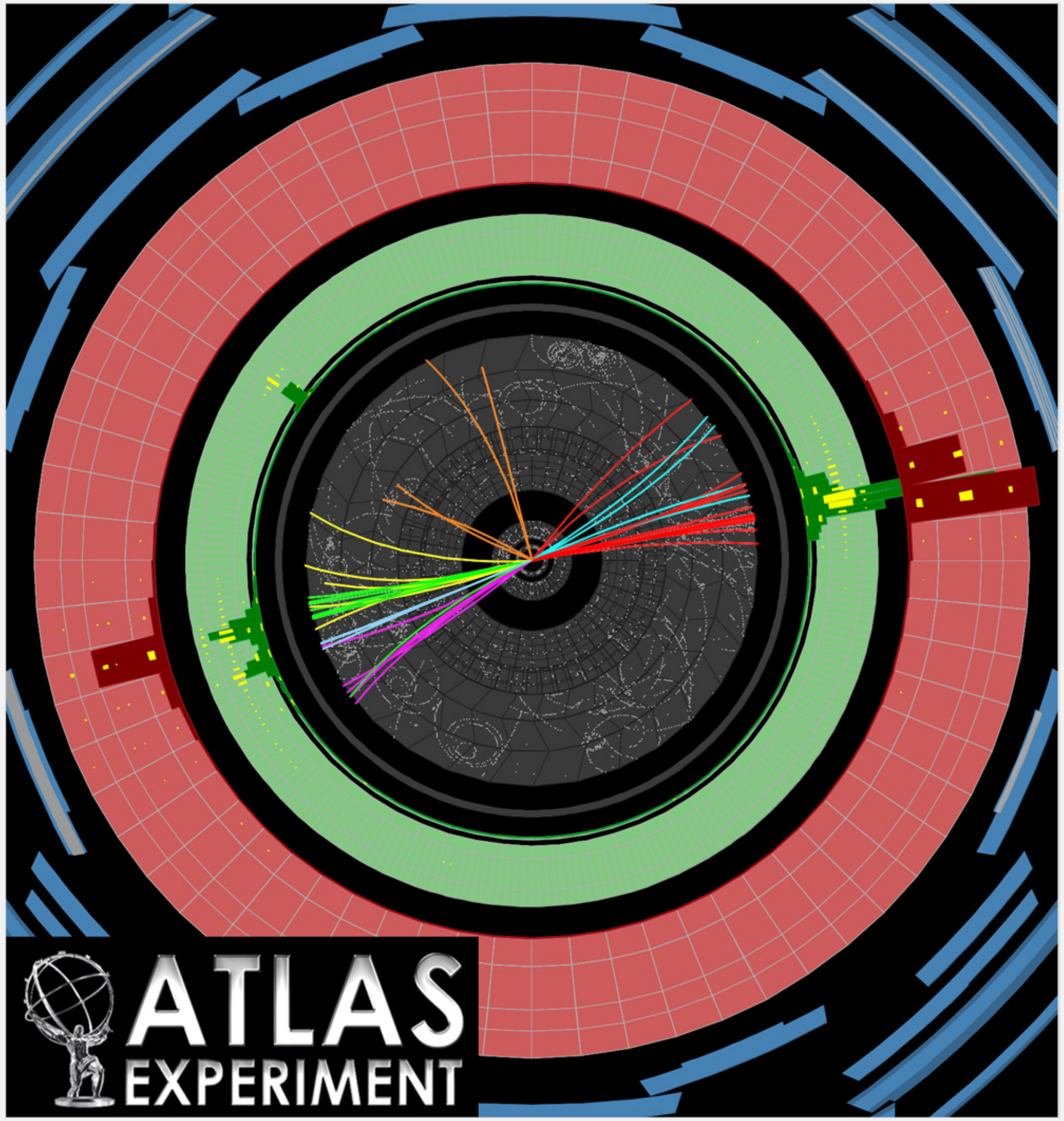
Part 2
Quark jet functions for kt-like observables

Part 3
Testing the framework and the perturbative ingredients in an NNLO slicing calculation

Introduction

Jet cross sections

- Quarks and gluons are not free.
- They fragment and hadronize into collinear sprays of hadrons.
- To compare theory and experiments, one needs an infrared (IR) safe definition of jets.



The Durham algorithm for ll collisions

Based on the two-particle distance $d_{ij} = \min(E_i, E_j)\frac{R_{ij}}{R}$, where $R_{ij}^2 = 2(1 - \cos\vartheta_{ij})$.

Exclusive Durham algorithm for n jets: On a collection of particle momenta $\{p_i\}$

1. If only n momenta remain, stop the algorithm
2. Calculate all distances $\{d_{ij}\}$ and find the minimum.
3. If the minimum is d_{ij} replace momenta p_i and p_j with p_{ij}.

Repeat

E-scheme:

$$p_{ij} = p_i + p_j$$

WTA scheme:

$$E_{ij} = E_i + E_j$$

$$\theta_{ij} = \theta_{\text{harder}}$$

$$\phi_{ij} = \phi_{\text{harder}}$$

The distance selected in the very last step of the algorithm is called $y_{n(n+1)}$

Calculating jet cross sections

- Jet cross sections can be calculated in pQCD:

$$\boxed{\sigma = \sum_{b_1,b_2} \int \mathrm{d}x_1 \mathrm{d}x_2 f_{b_1/h_1}(x_1) f_{b_2/h_2}(x_2) \hat{\sigma}_{b_1 b_2}(x_1, x_2) + \mathcal{O}\left[\left(\frac{\Lambda_{\mathrm{QCD}}}{\mathfrak{q}_{\mathrm{cut}}}\right)^{p_\Lambda}\right]}$$

The lowest scale introduced in the measurement and process

- This formula is **assumed to be correct** to all orders in this talk, and we **ignore power corrections**.

- The cross section can then be expanded as $\sigma = \underbrace{\sigma^{(0)}}_{\text{LO}} + \underbrace{\alpha_S \sigma^{(1)}}_{\text{NLO}} + \underbrace{\alpha_S^2 \sigma^{(2)}}_{\text{NNLO}} + \ldots$

The n-jet limit

- The n-jet limit of a jet process is approached when all partons are either collinear to one of n directions, or soft.
- *n-jet resolution variables* are *IR safe* observables that are 0 in the n-jet limit and positive in any other phase space configuration.
- Lepton collider examples: 2-jet: 1-Thrust,Broadening,... n-jet: $y_{n(n+1)}$
- Hadron collider examples: 0-jet: q_T n-jet: n-jettiness (τ_n), n-k_t^{ness}
- The cross section $\sigma_{\mathfrak{q}<\mathfrak{q}_{\text{cut}}}$ behaves like $\sigma_{\mathfrak{q}<\mathfrak{q}_{\text{cut}}} = \sigma_B \left(1 + \alpha_S \left(\Sigma_2^{(1)} L^2 + \Sigma_1^{(1)} L + \Sigma_0^{(1)}\right) + \ldots\right)$ where $L = \log \frac{\mathfrak{q}_{\text{cut}}}{Q}$.
- If $\alpha_S L \sim 1$this converges badly, and the higher logs should be resummed.
- Only radiation that is **collinear** or **soft** can contribute to the vetoed cross section $\sigma_{\mathfrak{q}<\mathfrak{q}_{\text{cut}}}$

Part 1: The framework

Questions to adress:

1. How does the phase space factorize if all emissions are collinear or soft?
2. Which soft/collinear modes contribute?
3. How does the matrix element factorize?
4. How do you combine all of this into a factorization theorem?
5. How to deal with rapidity divergences?

All of these questions are addressed in my paper 2509.06612.

1. Phase space factorization

- Assume that final-state momenta $\{k_i\}$ split into soft and collinear subsets as

$$\boxed{\{k_i\} = I_1 \cup I_2 \cup S \bigcup_{j=3}^{n_J+2} F_j}$$

- Then the leading-power phase space factorizes as

$$\begin{aligned}\frac{1}{2\hat{s}}\, \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}\Pi_n \sim\ & \frac{\mathrm{d}\tilde{x}_1\, \mathrm{d}\tilde{x}_2}{2Q^2} \prod_{i=3}^{n_J+2} [\,\mathrm{d}p_i]\, \delta^d \left[\sum_{i=3}^{n_J+2} p_i - q\right] \quad \text{Born/Hard} \qquad \text{Soft} \\ \times\ & \frac{\mathrm{d}\mathrm{z}_1\, \mathrm{d}\mathrm{z}_2}{\mathrm{z}_1 \mathrm{z}_2}\, \mathrm{d}^d k_{I_1}\, \mathrm{d}\Pi_{I_1} \mathrm{z}_1 \delta\left(\bar{\mathrm{z}}_1 - \frac{k_{I_1}\cdot P_2}{x_1 P_1 \cdot P_2}\right) \mathrm{d}^d k_{I_2}\, \mathrm{d}\Pi_{I_2} \mathrm{z}_2 \delta\left(\bar{\mathrm{z}}_2 - \frac{k_{I_2}\cdot P_1}{x_2 P_1 \cdot P_2}\right) \mathrm{d}^d k_S\, \mathrm{d}\Pi_S \\ \times\ & \prod_{i=3}^{n_J+2} 2(2\pi)^{d-1} \prod_{j\in F_i} \left[\,\mathrm{d}\tilde{k}_j\right] \delta\left(1 - \sum_j z_j\right) \delta^{d-2}\left(\sum_j \tilde{k}_{j,\perp}\right) \qquad \text{IS collinear} \quad \text{FS collinear}\end{aligned}$$

$$\begin{aligned} k_j^\mu &= \frac{z_j}{\mathrm{z}_1} p_1^\mu + k_{t,j}^\mu + \frac{\mathrm{z}_1 |k_{t,j}|^2}{z_j Q^2} p_2^\mu \\ k_j^\mu &= z_j p_i^\mu + k_{\perp,j}^\mu + \frac{Q^2 |k_{\perp,j}|^2}{4 z_j\, (p_i\cdot q)^2} \bar{p}_i^\mu \\ k_{\perp,j} &= \tilde{k}_{\perp,j} - z_j \frac{p_i \cdot q}{Q^2} k^\mu_{\mathrm{rec},\perp} \\ k_{\mathrm{rec}} &= k_{t,I_1} + k_{t,I_2} + \sum_{j\in S} k_j \end{aligned}$$

Note the **recoil** in the jet sectors!

$$\boxed{\sum_j k_{j,\perp} \neq 0} \quad \text{but} \quad \boxed{\sum_j \tilde{k}_{j,\perp} = 0}$$

- The radiation phase space has factorized. We can now assign power scaling to the variables $k_{t,j}^\mu$, $\tilde{k}_{\perp,j}$, and k_j, and use the method of regions.

2. Which soft/collinear modes contribute exactly?

- Exactly how soft or collinear are our momenta?
- The theta function $\theta(\mathfrak{q}_{\text{cut}} - \mathfrak{q}(\{k_i\}))$ selects the possible modes:
- Define $\lambda = \dfrac{\mathfrak{q}_{\text{cut}}}{Q} \ll 1$, then the variables $k_{t,j}^{\mu}$, $\tilde{k}_{\perp,j}$, and k_j need to scale such that $\mathfrak{q}(\{k_i\}) \sim \lambda$
- What are the possible scalings?

$$\boxed{\begin{aligned} k_{t,j}^{\mu} &\sim \lambda^{a_j(\{p_i,\mathbf{z}_i\},\{\#_i\},\{z_i,\phi_i,\dots\})} k_{t,j}^{\mu} \\ \tilde{k}_{\perp,j} &\sim \lambda^{b_j(\{p_i,\mathbf{z}_i\},\{\#_i\},\{z_i,\phi_i,\dots\})} \tilde{k}_{\perp,j} \\ k_j &\sim \lambda^{c_j(\{p_i,\mathbf{z}_i\},\{\#_i\},\{z_i,\phi_i,\dots\})} k_j \end{aligned}}$$

What do we require?

1. IR safety: $a_j(\dots) = a(\dots) > 0$, $b_j(\dots) = b(\dots) > 0$, $c_j(\dots) = c(\dots) > 0$

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2. Which soft/collinear modes contribute exactly?

$$\frac{1}{\tilde{a}_\ell} = c \qquad \text{Ceasar scaling}$$

$$\frac{1}{\tilde{a}_\ell + \tilde{b}_\ell} = \begin{cases} a & \text{if } \ell \text{ is an initial-state leg} \\ b & \text{if } \ell \text{ is a final-state leg.} \end{cases}$$

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3. Factorization of the squared matrix elements

- Given sets of momeneta $\boxed{\{k_i\} = I_1 \cup I_2 \cup S \bigcup_{j=3}^{n_J+2} F_j}$, with the momenta scaling as $\boxed{\begin{array}{c} k_{t,j}^{\mu} \sim \lambda^a k_{t,j}^{\mu} \\ \tilde{k}_{\perp,j} \sim \lambda^b \tilde{k}_{\perp,j} \\ k_j \sim \lambda^c k_j \end{array}}$

  we assume that the squared matrix elements factorize as

All-order squared matrix element

All-order amplitude in colour/spin space on the hard configuration

$$\left|\mathcal{M}_{b_1,b_2,\mathfrak{A}}\left(\{k_i\}\right)\right|^2 \simeq \mathrm{Tr}\Big[\left|\mathcal{M}_{\mathcal{A}}\left(p_1,\ldots,p_{2+n_J}\right)\right\rangle \left\langle\mathcal{M}_{\mathcal{A}}\left(p_1,\ldots,p_{2+n_J}\right)\right|$$

Incoming flavours

Outgoing flavours

Hard flavours

$$\times\, \hat{\mathcal{P}}_{a_1 b_1;\mathfrak{A}_1}\left(\{k_i\}_{I_1}, z_1\right) \hat{\mathcal{P}}_{a_2 b_2;\mathfrak{A}_2}\left(\{k_i\}_{I_2}, z_2\right) \mathcal{J}_{\mathfrak{A}_S}\left(\{k_i\}_S\right) \prod_{j=3}^{n_J+2} \hat{\mathcal{P}}_{a_j \to \mathfrak{A}_j}\left(\{k_i\}_{F_j}\right)\Big]$$

SL splitting function

ISC flavours

Squard soft currents

TL splitting function

FSC flavours

- **Important!** This formula is known to break down at $\mathrm{N^3LO}$ loops due to *Glauber* exchanges! I will neglect Glauber contributions.

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All-order squared matrix element

Hard function

$$|\mathcal{M}_{b_1,b_2,\mathfrak{A}}(\{k_i\})|^2 \simeq \mathrm{Tr}\Big[\mathcal{H}_{\mathcal{A}}(\{p_i\})$$

Incoming flavours   Outgoing flavours

$$\times\, \hat{\mathcal{P}}_{a_1 b_1;\mathfrak{A}_1}(\{k_i\}_{I_1}, z_1)\hat{\mathcal{P}}_{a_2 b_2;\mathfrak{A}_2}(\{k_i\}_{I_2}, z_2)\mathcal{J}_{\mathfrak{A}_S}(\{k_i\}_S) \prod_{j=3}^{n_J+2} \hat{\mathcal{P}}_{a_j \to \mathfrak{A}_j}(\{k_i\}_{F_j})\Big]$$

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4. How do you combine all of this into a factorization theorem?

We have identified **all regions** that contribute, and know how the **matrix elements** and **phase space** factorize in these regions. Putting everything together, we find:

Any IR safe observable

$$\mathrm{d}x_1\mathrm{d}x_2\hat{\sigma}_{b_1b_2}(\mathfrak{q}_{\mathrm{cut}}) = \frac{\mathrm{d}\mathrm{z}_1\mathrm{d}\mathrm{z}_2}{\mathrm{z}_1\mathrm{z}_2}\int\frac{\mathrm{d}\tilde{x}_1\mathrm{d}\tilde{x}_2}{2Q^2}\prod_{i=3}^{n_J+2}[\mathrm{d}p_i]\,\delta^d\Big[\sum_{i=3}^{n_J+2}p_i - q\Big]\mathcal{F}(\{p_i\})$$

Hard function

$$\times\sum_{\mathcal{A}}\mathrm{Tr}\Bigg\{\frac{1}{S_{\mathcal{A}}}\mathcal{H}_{\mathcal{A}}(\{p_i\})\Bigg[\sum_{n=n_J}^{\infty}\sum_{\mathfrak{p}_n\in\mathfrak{P}_n}\int\mathrm{d}\boldsymbol{\mathcal{B}}_{a_1b_1}(I_1,z_1)\mathrm{d}\boldsymbol{\mathcal{B}}_{a_2b_2}(I_2,z_2)$$

Hard partonic configurations

Sum over possible regions

Sym. Fact.

$$\times\,\mathrm{d}\boldsymbol{\mathcal{S}}(S,n_1,n_2,\{n_i\})\prod_{i=1}^{n_J}\mathrm{d}\mathcal{J}_{a_i}(F_i)\theta\left(\mathfrak{q}_{\mathrm{cut}}-\tilde{\mathfrak{q}}_{\mathfrak{p}_n}(\{k\})\right)\Bigg]\Bigg\}$$

Resolution variable approximated in the region $\mathfrak{p}_n$

With the **maximally differential, bare** beam, jet and, soft functions

$$\mathrm{d}\boldsymbol{\mathcal{B}}_{a_1b_1}(I_1,\mathrm{z}_1) = \sum_{\mathfrak{A}_1}\frac{1}{S_{\mathfrak{A}_1}}\mathrm{z}_1\mathrm{d}^dk_{I_1}\mathrm{d}\Pi_{I_1}\delta\left(\bar{\mathrm{z}}_1-\frac{k_{I_1}\cdot P_2}{x_1P_1\cdot P_2}\right)\hat{\boldsymbol{\mathcal{P}}}_{a_1b_1;\mathfrak{A}_1}(I_1,\mathrm{z}_1)$$

Generalization of Catani, Dhani 2208.05840

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\times\sum_{\mathcal{A}}\mathrm{Tr}\Big\{\frac{1}{S_{\mathcal{A}}}\mathcal{H}_{\mathcal{A}}(\{p_i\})\Big[\sum_{n=n_J}^{\infty}\sum_{\mathfrak{p}_n\in\mathfrak{P}_n}\int\mathrm{d}\mathcal{B}_{a_1b_1}(I_1,z_1)\mathrm{d}\mathcal{B}_{a_2b_2}(I_2,z_2) \\
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\mathrm{d}\mathcal{J}_{a_i}(F_i) = \sum_{\mathfrak{A}_i}\frac{1}{S_{\mathfrak{A}_i}}2(2\pi)^{d-1}\prod_{j\in F_i}[\mathrm{d}\tilde{k}_j]\delta(1-\sum_j z_j)\delta^{d-2}\left(\tilde{k}_{j,\perp}\right)\hat{\mathcal{P}}_{\mathfrak{A}_i}(F_i)
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&\times\sum_{\mathcal{A}}\mathrm{Tr}\Big\{\frac{1}{S_{\mathcal{A}}}\mathcal{H}_{\mathcal{A}}(\{p_i\})\Big[\sum_{n=n_J}^{\infty}\sum_{\mathfrak{p}_n\in\mathfrak{P}_n}\int\mathrm{d}\boldsymbol{\mathcal{B}}_{a_1b_1}(I_1,z_1)\mathrm{d}\boldsymbol{\mathcal{B}}_{a_2b_2}(I_2,z_2)\\
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$$

Special Cases: Cumulant factorization

If the region expanded resolution variable simplifies to

$$\boxed{\tilde{\mathfrak{q}}_{\mathfrak{p}_n}(\{k\}) = \max\left(\tilde{\mathfrak{q}}_{C_1}(I_1), \tilde{\mathfrak{q}}_{C_1}(I_2), \tilde{\mathfrak{q}}_{C_3}(F_3), \ldots, \tilde{\mathfrak{q}}_S(S)\right)}$$

then, the θ-function simplifies as

$$\boxed{\theta\left(\mathfrak{q}_{\rm cut} - \tilde{\mathfrak{q}}_{\mathfrak{p}_n}(\{k\})\right) = \theta\left(\mathfrak{q}_{\rm cut} - \tilde{\mathfrak{q}}_{C_1}(I_1)\right)\theta\left(\mathfrak{q}_{\rm cut} - \tilde{\mathfrak{q}}_{C_2}(I_2)\right)\theta\left(\mathfrak{q}_{\rm cut} - \tilde{\mathfrak{q}}_{C_3}(F_3)\right)\ldots\theta\left(\mathfrak{q}_{\rm cut} - \tilde{\mathfrak{q}}_S(S)\right)}$$

We can define the cumulant beam, jet, and soft functions

$$\boldsymbol{\mathcal{B}}_{a_1 b_1}(\mathfrak{q}_{\rm cut}, \mathrm{z}_1) = \sum_{n=0}^{\infty}\int \mathrm{d}\boldsymbol{\mathcal{B}}_{a_1 b_1}(I_1, \mathrm{z}_1)\theta(\mathfrak{q}_{\rm cut} - \tilde{\mathfrak{q}}_{C_1}(I_1)) \qquad \mathcal{J}_{a_i}(\mathfrak{q}_{\rm cut}) = \sum_{n=1}^{\infty}\int \mathrm{d}\mathcal{J}_{a_i}(F_i)\theta(\mathfrak{q}_{\rm cut} - \tilde{\mathfrak{q}}_{C_i}(F_i))$$

$$\boldsymbol{\mathcal{S}}(\mathfrak{q}_{\rm cut}, n_1, n_2, \{n_i\}) = \sum_{n=0}^{\infty}\int \mathrm{d}\boldsymbol{\mathcal{S}}(S, n_1, n_2, \{n_i\})\theta(\mathfrak{q}_{\rm cut} - \tilde{\mathfrak{q}}_S(S))$$

and the factorization theorem takes product form:

$$\boxed{\Big[\ldots\Big] = \hat{\boldsymbol{\mathcal{B}}}_{a_1 b_1}(\mathfrak{q}_{\rm cut}, \mathrm{z}_1)\hat{\boldsymbol{\mathcal{B}}}_{a_2 b_2}(\mathfrak{q}_{\rm cut}, \mathrm{z}_2)\prod_{i=3}^{n_J+2}\hat{\mathcal{J}}_{a_i}(\mathfrak{q}_{\rm cut})\hat{\boldsymbol{\mathcal{S}}}(\mathfrak{q}_{\rm cut})}$$

5. How to deal with rapidity divergences?

- The bare factorization theorem is derived with the MoR and thus is free of double counting!

$$\boxed{\mathrm{d}x_1\mathrm{d}x_2\hat{\sigma}(\mathfrak{q}_{\mathrm{cut}}) = \frac{\mathrm{d}\mathrm{z}_1\mathrm{d}\mathrm{z}_2}{\mathrm{z}_1\mathrm{z}_2}\,\mathrm{Tr}\left\{\int \mathrm{d}\mathcal{H}\left[\mathrm{d}\boldsymbol{\mathcal{B}}\otimes\mathrm{d}\boldsymbol{\mathcal{B}}\otimes\mathrm{d}\boldsymbol{\mathcal{S}}\otimes\mathrm{d}\mathcal{J}\right]\right\}}$$

- However, for some observables, the beam, jet, and soft functions can become ill-defined. This happens if $a = c$ and/or $b = c$. (SCETII scenario).

- This is due to *rapidity divergences*.

- This framework can be used with any rapidity regulator.

The z_N-prescription

- We regularize rapidity divergences by replacing **some** momentum fractions in the splitting function with energy fractions (first suggested by Catani and Dhani 2208.05840)

$$\boxed{z = \frac{k \cdot \bar{n}}{p \cdot \bar{n}} \to z_N = \frac{k \cdot N}{p \cdot N}}$$

Similar to using timelike Wilson lines in the operator matrix elements

- N can be any timelike linear combination of n, and $\bar{n}$:

$$\boxed{N = \sqrt{N^2}\left(e^{\eta}\frac{n}{2} + e^{-\eta}\frac{\bar{n}}{2}\right)}$$

- The simplest and most convenient choice for fixed order calculations is $N = q$.

- The introduction of the scale N^2 in the collinear functions leads to non-vanishing zero bins.

The z_N-prescription

- We group the zero bins with the soft function to form a **subtracted soft function** that only contains wide-angle radiation:

- E.g., a cumulant factorization theorem would become

$$\boxed{\left[\ldots\right] = \hat{\mathcal{B}}_{N,a_1 b_1}(\mathfrak{q}_{\mathrm{cut}}, z_1)\hat{\mathcal{B}}_{N,a_2 b_2}(\mathfrak{q}_{\mathrm{cut}}, z_2)\prod_{i=3}^{n_J+2}\hat{\mathcal{J}}_{N,a_i}(\mathfrak{q}_{\mathrm{cut}})\hat{\mathcal{S}}_{\mathrm{sub}}(\mathfrak{q}_{\mathrm{cut}})}$$

Beam function with modified splitting kernel

Jet function with modified splitting kernel

Soft function minus zero bins

- The z_N -prescription can be used even if there are no rapidity divergences. This leads to universal ϵ-poles/anomalous dimensions in collinear and soft functions. (See also 2012.09213)

Part 2: The jet function

The quark jet function for kt-like observables

- We are interested in observables that, in the two-particle collinear limit, simplify to $\tilde{\mathfrak{q}} = k_\perp$
- The cumulant quark jet function is defined as

$$\boxed{\begin{aligned}\mathcal{J}_{N,q}(\mathfrak{q}_{\text{cut}}) = \theta(\mathfrak{q}_{\text{cut}}) + \sum_{n=2}^{\infty}\sum_{\mathfrak{A}_n} 2(2\pi)^{d-1}\int\prod_{j=1}^{n}[dk_j]\delta\Big(1-\sum_{j=1}^{n} z_j\Big)\delta^{(d-2)}\Big(\sum_{j=1}^{n} k_{j,\perp}\Big) \\ \times\frac{\hat{\mathcal{P}}_{N,\mathfrak{A}_n}(k_1,k_2\ldots k_n)}{S_{\mathfrak{A}_n}}\theta\Big(\mathfrak{q}_{\text{cut}}-\tilde{\mathfrak{q}}\Big),\end{aligned}}$$

- We can expand the result in powers or the strong coupling:

$$\boxed{\mathcal{J}_{N,q}(\mathfrak{q}_{\text{cut}}) = \theta(\mathfrak{q}_{\text{cut}}) + \sum_{n=1}^{\infty}\left(\frac{\alpha_0}{\pi}\right)^n\left(\frac{\mu^2}{\mathfrak{q}_{\text{cut}}^2}\right)^{n\epsilon}\mathcal{J}_{N,q}^{(n)}(\mathfrak{q}_{\text{cut}})}$$

$$\boxed{\alpha_S^u\mu_0^{2\epsilon}(4\pi)^\epsilon e^{-\epsilon\gamma_E} = \alpha_0(\mu)\mu^{2\epsilon}}$$

The quark jet function at NLO

The NLO coefficient of the jet function is defined as

$$\mathcal{J}^{(1)}_{N,q}(\mathfrak{q}_{\mathrm{cut}}) = 4\,\mathfrak{q}_{\mathrm{cut}}^{2\epsilon}\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\int d\Phi_2^{(c)}\,\frac{\hat{P}^{(0)}_{N,gq}(z_1)}{2k_1\cdot k_2}\theta(\mathfrak{q}_{\mathrm{cut}}-k_\perp)$$

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$$= \mathfrak{q}_{\text{cut}}^{2\epsilon}\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\int_0^1 dz\int_0^{\mathfrak{q}_{\text{cut}}} dk_\perp k_\perp^{-1-2\epsilon}\hat{P}^{(0)}_{N,gq}(z)$$

Handling z_N:

$$z_N = \frac{k\cdot N}{p\cdot N} = z + \frac{N^2k_\perp^2}{(2p\cdot N)^2 z}$$

$$\Rightarrow \frac{1}{z_N} = -\frac{1}{2}\delta(z)\ln\frac{N^2k_\perp^2}{(2p\cdot N)^2} + \left(\frac{1}{z}\right)_+ + \mathcal{O}\left(\frac{k_\perp\sqrt{N^2}}{2p\cdot N}\right)$$

The z_N-prescription reintroduces the hard scale $\frac{2p\cdot N}{\sqrt{N^2}} = 2E_J$

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$$= \mathfrak{q}_{\rm cut}^{2\epsilon}\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\int_0^1 dz\int_0^{\mathfrak{q}_{\rm cut}} dk_\perp k_\perp^{-1-2\epsilon}\hat{P}^{(0)}_{N,gq}(z)$$
$$= \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}C_F\left[\frac{1}{2\epsilon^2}+\frac{\ln\left(\frac{\sqrt{N^2}\mathfrak{q}_{\rm cut}}{2p\cdot N}\right)}{\epsilon}+\frac{3}{4\epsilon}+\frac{1}{4}\right]$$

The *collinear anomaly* logarithm $\ln\left(\frac{\sqrt{N^2}\mathfrak{q}_{\rm cut}}{2p\cdot N}\right)$

vanishes, if we choose N very forward, such that

$$E_J = \frac{\mathfrak{q}_{\rm cut}}{2}$$

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Uncorrelated emission of two gluons

Here, the triple-collinear region lives at the same scale as the double-soft and collinear-soft region. Thus, the structure of rapidity divergences is more involved.

We regularize the splitting kernel as

$$\boxed{\hat{P}^{(0)(\mathrm{ab})}_{N,g_1g_2q_3} = \frac{s_{123}}{s_{23}}\underbrace{\hat{P}^{(0)}_{N,gq}(z_1)}_{\text{Not so easy!}}\hat{P}^{(0)}_{N,gq}\left(\frac{z_2}{z_2+z_3}\right) + \underbrace{R^{(0)(\mathrm{ab})}_{g_1g_2q_3}}_{\text{easy!}} + (1 \leftrightarrow 2)}$$

To integrate the strongly ordered piece, we introduce the variables

$$\boxed{\tilde{z}_2 = \frac{z_2}{z_2+z_3}, \qquad \vec{k}_{23,\perp} = \frac{z_3\vec{k}_{2,\perp} - z_2\vec{k}_{3,\perp}}{z_2+z_3}}$$

The z_N-regularization is then encapsulated in the identities

$$\boxed{z_{N,1} = z_1 + \frac{N^2 k_{1,\perp}^2}{(2p\cdot N)^2 z_1} \qquad \tilde{z}_{N,2} = \tilde{z}_2 + \frac{N^2 k_{23,\perp}^2}{(2p\cdot N)^2\,(1-z_1)^2\,\tilde{z}_2} + \dots}$$

Now just do the NLO trick of writing both z_N-denominators as distributions!

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~~Now just do the NLO trick of writing both z_N-denominators as distributions!~~

NO!

Uncorrelated emission of two gluons

To see what goes wrong, we split the integral into the regions $k_{1,\perp} \lessgtr k_{23,\perp}$, and we use the parametrization

$$\boxed{z_1, \quad \tilde{z}_2, \quad y \equiv \min\left\{\frac{k_{1,\perp}}{k_{23,\perp}}, \frac{k_{23,\perp}}{k_{1,\perp}}\right\}, \quad k_\perp \equiv \max\{k_{1,\perp}, k_{23,\perp}\}, \quad \cos\varphi \equiv \frac{\vec{k}_{1,\perp}\cdot\vec{k}_{23,\perp}}{k_{1,\perp}k_{23,\perp}}}$$

Then, the integral takes the form

$$\boxed{\theta(\mathfrak{q}_{\rm cut}^2 - \tilde{\mathfrak{q}}^2) = \theta(\mathfrak{q}_{\rm cut}^2 - k_\perp^2 F(z_1, \tilde{z}_2, y, \cos\varphi))}$$

Problem 1: Overlapping singularity in $(y, \tilde{z}_2) \to (0{,}0)$

$$\mathcal{J}^{(2)}_{N,q}(\mathfrak{q}_{\rm cut})\Big|^{s.o.}_{\rm ab} = \int {\rm d}k_\perp {\rm d}z_1 {\rm d}\tilde{z}_2 {\rm d}y \ldots \frac{1}{z_1 + \frac{N^2 k_\perp^2}{(2p\cdot N)^2 z_1}} \frac{1}{\tilde{z}_2 + \frac{N^2 y^2 k_\perp^2}{(2p\cdot N)^2 (1-z_1)^2 \tilde{z}_2}} \theta(\mathfrak{q}_{\rm cut}^2 - k_\perp^2 F(z_1, \tilde{z}_2, y, \cos\varphi)) + \ldots$$

Problem 2: Limits $z_1 \to 0$, and $\tilde{z}_2 \to 0$ do not commute

Our solution: Split the calculation into two pieces as

$$\int d\Phi_3^{(c)} \frac{P^{(0),(ab)S.O.}_{N,\,g_1g_2q_3}}{s_{123}^2} \theta(\mathfrak{q}_{\rm cut} - \tilde{\mathfrak{q}}) = \underbrace{\int d\Phi_3^{(c)} \frac{P^{(0),(ab)S.O.}_{N,\,g_1g_2q_3}}{s_{123}^2} \theta(\mathfrak{q}_{\rm cut} - k_\perp)}_{\text{Endpoint term}} + \underbrace{\int d\Phi_3^{(c)} \frac{P^{(0),(ab)S.O.}_{N,\,g_1g_2q_3}}{s_{123}^2} \left(\theta(\mathfrak{q}_{\rm cut} - \tilde{\mathfrak{q}}) - \theta(\mathfrak{q}_{\rm cut} - k_\perp)\right)}_{\text{Subtracted term}}$$

The endpoint term

The endpoint term

$$\mathcal{J}^{(2)}_{N,q}(\mathfrak{q}_{\rm cut})\Big|^{\rm EP}_{\rm ab} = C\int d\Phi_3^{(c)}\,\frac{P^{(0),(ab)S.O.}_{N,\,g_1g_2q_3}}{s_{123}^2}\theta(\mathfrak{q}_{\rm cut}-k_\perp)$$

Is the same for all k_t-like observables, and we can calculate it once and for all (e.g., with the MoR).

We find:

$$\mathcal{J}^{(2)}_{N,q}(\mathfrak{q}_{\rm cut})\Big|^{\rm EP}_{\rm ab} = L_N^2\sum_{k=0}^{2}\frac{D_k^{\rm EP}}{\epsilon^k} + L_N\sum_{k=0}^{3}\frac{A_k^{\rm EP}}{\epsilon^k} + \sum_{k=0}^{4}\frac{B_k^{\rm EP}}{\epsilon^k}$$

$$D_2^{\rm EP} = \frac{1}{2}C_F^2\,,\quad D_1^{\rm EP} = 0\,,\quad D_0^{\rm EP} = -\frac{\pi^2}{12}C_F^2\,,\quad A_3^{\rm EP} = \frac{1}{2}C_F^2\,,\quad A_2^{\rm EP} = \frac{3}{4}C_F^2\,,$$
$$A_1^{\rm EP} = C_F^2\left(\frac{1}{4}-\frac{\pi^2}{12}\right)\,,\quad A_0^{\rm EP} = -1.6343868(8)\;C_F^2\,,\quad B_4^{\rm EP} = \frac{1}{8}C_F^2\,,\quad B_3^{\rm EP} = \frac{3}{8}C_F^2\,,$$
$$B_2^{\rm EP} = C_F^2\left(\frac{23}{32}-\frac{5\pi^2}{48}\right)\,,\quad B_1^{\rm EP} = 0.06315(3)\;C_F^2\,,\quad B_0^{\rm EP} = 0.4688(1)\;C_F^2$$

- The poles are the ones predicted by the quark form factor
- The single log cannot be written with MZV and rational numbers only

The subtracted term

The subtracted term

$$\mathcal{J}^{(2)}_{N,q}(\mathfrak{q}_{\mathrm{cut}})\Big|^{\mathrm{sub}}_{\mathrm{ab}} = C\int d\Phi_3^{(c)}\frac{P^{(0),(ab)S.O.}_{N,g_1g_2q_3}}{s^2_{123}}\left(\theta(\mathfrak{q}_{\mathrm{cut}}-\tilde{\mathfrak{q}})-\theta(\mathfrak{q}_{\mathrm{cut}}-k_\perp)\right)$$

is now free of overlapping singularities. The non-commuting limits $\left(z_1 \to 0, \tilde{z}_2 \to 0\right)$ are handled with the distributional expansion

$$\int_0^1 dz_1 \int_0^1 dz_2 \frac{z_1}{z_1^2+y^2\lambda^2}\frac{z_2}{z_2^2+\frac{\lambda^2}{(1-z_1)^2}}p(z_1,z_2) =$$

$$\int_0^1 dz_1 \int_0^1 dz_2 \frac{1}{z_1 z_2}\left(p(z_1,z_2) - p_s\left(\frac{z_1}{z_2}\right) - p_{s_1}(z_2) - p_{s_2}(z_1) + p_{s_1,s_2} + p_{s_2,s_1}\right)$$

$$+\int_0^1 \frac{dz_1}{z_1}\log\left(\frac{1-z_1}{\lambda}\right)(p_{s_2}(z_1)-p_{s_2,s_1}) - \log(\lambda y)\int_0^1 \frac{dz_2}{z_2}(p_{s_1}(z_2)-p_{s_1,s_2})$$

$$+\int_0^1 \frac{dt}{t}\Bigg(\frac{y^2\log\left(\frac{y}{t}\right)(p_s(t)-p_{s_1,s_2})}{t^2-y^2} + \frac{\left(\log\left(\frac{1}{t}\right)-t^2y^2\log(y)\right)\left(p_s\left(\frac{1}{t}\right)-p_{s_2,s_1}\right)}{t^2y^2-1}$$

$$-\log(\lambda)\left(p_s\left(\frac{1}{t}\right)+p_s(t)-p_{s_1,s_2}-p_{s_2,s_1}\right)\Bigg) + \frac{1}{4}\mathrm{Li}_2\left(1-\frac{1}{y^2}\right)(p_{s_1,s_2}-p_{s_2,s_1})$$

$$-\frac{1}{2}\log(\lambda y)\left(\log(y)(p_{s_2,s_1}-p_{s_1,s_2}) - \log(\lambda)(p_{s_1,s_2}+p_{s_2,s_1})\right) - \frac{1}{6}\pi^2 p_{s_2,s_1} + \mathcal{O}(\lambda)$$

$$p_{s_1}(z_2) = \lim_{z_1\to 0} p(z_1,z_2) \qquad p_{s_2}(z_1) = \lim_{z_2\to 0} p(z_1,z_2)$$

$$p_s(t) = \lim_{\lambda\to 0} p(\lambda t,\lambda) \qquad p_{s_1,s_2} = \lim_{z_2\to 0}\lim_{z_1\to 0} p(z_1,z_2) = \lim_{t\to 0} p_s(t)$$

$$p_{s_2,s_1} = \lim_{z_1\to 0}\lim_{z_2\to 0} p(z_1,z_2) = \lim_{t\to 0} p_s\left(\frac{1}{t}\right)$$

The solution is free of ϵ-poles but contains logs and even double-logs for variables that break cumulant factorization (NLL contribution)!

Jet function results

We calculated the cumulant quark jet function with the z_N-prescription for y_{23} with E-scheme and WTA scheme:

$$\boxed{\mathcal{J}_{N,q}^{(2)} = L_N^2 \sum_{k=0}^{2} \frac{D_k}{\epsilon^k} + L_N \sum_{k=0}^{3} \frac{A_k}{\epsilon^k} + \sum_{k=0}^{4} \frac{B_k}{\epsilon^k}}$$

The quark form factor determines the poles, and the finite pieces are

E-scheme:

$$D_0^E = \left(\frac{\ln^2(2)}{2} - \frac{\pi^2}{6}\right) C_F^2,$$

$$A_0^E = -0.17976(1)\ C_F^2 - 2.20169(6)\ C_F C_A - 0.12794(2)\ C_F n_F T_R\,,$$

$$B_0^E = 4.514(1)\ C_F^2 - 0.2997(5)\ C_F C_A - 0.2210(1)\ C_F n_F T_R$$

WTA scheme:

$$D_0^{\mathrm{WTA}} = -\frac{\pi^2}{12} C_F^2,$$

Determined by EP contribution

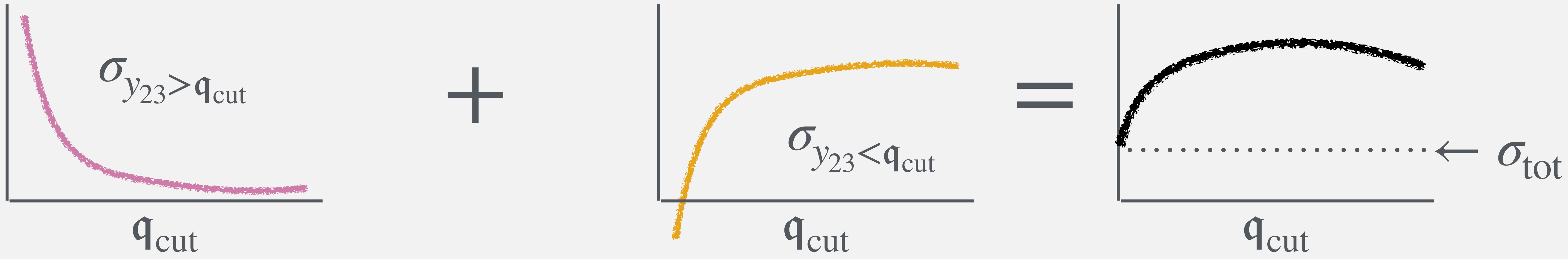
$$A_0^{\mathrm{WTA}} = 2.01322(9)\ C_F^2 - 2.64831(2)\ C_F C_A - 0.0766(1)\ C_F n_F T_R,$$

$$B_0^{\mathrm{WTA}} = 8.3346(8)\ C_F^2 - 1.7774(3)\ C_F C_A - 0.0735(1)\ C_F n_F T_R$$

Part 3: Application to slicing

Application in slicing at NNLO

- We used the framework and jet function to use y_{23} as a slicing variable for dijet production at lepton colliders ($e^+e^- \to jj$ and $\mu^-\mu^+ \to H \to b\bar{b}$)
- We calculate $\sigma_{y_{23}>\mathfrak{q}_{\rm cut}}$ with MATRIX and $\sigma_{y_{23}<\mathfrak{q}_{\rm cut}}$ (up to power corrections) with our framework



- The factorization theorems read

$$\boxed{d\sigma_{\mathfrak{q}<\mathfrak{q}_{\rm cut}} = d\sigma_{\rm B}\Big[H\,\mathcal{R}(\mathfrak{q}_{\rm cut})\Big]_{\epsilon\to 0}}$$

$$\boxed{\mathcal{R}(\mathfrak{q}_{\rm cut}) = \mathcal{J}_{N,q_1}(\mathfrak{q}_{\rm cut})\mathcal{J}_{N,\bar{q}_2}(\mathfrak{q}_{\rm cut})\mathcal{S}_{\rm sub}(\mathfrak{q}_{\rm cut}) + \mathcal{B}_{N,q_1}(\mathfrak{q}_{\rm cut}) + \mathcal{B}_{N,\bar{q}_2}(\mathfrak{q}_{\rm cut}) + \mathcal{O}(\alpha_S^3)}$$

$= 0$, for WTA

Subtracted soft contribution: Abelian emisions

The subtracted soft function for Abelian emissions takes the form

Honest soft

$$\mathcal{S}^{(2),(\mathrm{ab})}_{gg,\mathrm{sub}} = \mathfrak{q}_{\mathrm{cut}}^{4\epsilon}\left(\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\right)^2 2C_F^2 \int \frac{d^d k}{\Omega_{d-2}}\delta_+(k^2)\int \frac{d^d l}{\Omega_{d-2}}\delta_+(l^2)$$
$$\times\left\{\frac{(p_1\cdot p_2)^2}{p_1\cdot k\; p_2\cdot k\; p_1\cdot l\; p_2\cdot l}(\Theta_S - \Theta_{C_1C_2,S})\right.$$
$$-\left[\frac{(p_1\cdot p_2)^2}{p_1\cdot k\; q\cdot k\; p_1\cdot l\; q\cdot l}(\Theta_{C_1,S} - \Theta_{C_1,SC_1,S} - \Theta_{C_1,C_1S,S} + \Theta_{C_1C_2,S}) + (1\leftrightarrow 2)\right]$$
$$\left.-\left[\left(\frac{(p_1\cdot p_2)^2}{p_1\cdot k\, q\cdot k\, p_1\cdot l\, p_2\cdot l}(\Theta_{C_1S,S} - \Theta_{C_1C_2,S}) + (k\leftrightarrow l)\right) + (1\leftrightarrow 2)\right]\right\}$$

Zero bin of the double-collinear region

Zero bin of the z_N prescribed soft × collinear zero bin of the triple collinear region

Zero bin of the triple-collinear region

Zero bin of the soft × collinear region

Subtracted soft contribution: Abelian emisions

The subtracted soft function for Abelian emissions takes the form

While this was a mess to define, it is **finite in 4D** and trivial to integrate numerically!

Honest soft

$$
\begin{aligned}
\mathcal{S}^{(2),(\mathrm{ab})}_{gg,\mathrm{sub}} = \mathfrak{q}_{\mathrm{cut}}^{4\epsilon}\left(\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\right)^2 2C_F^2 \int \frac{d^dk}{\Omega_{d-2}}\delta_+(k^2)\int\frac{d^dl}{\Omega_{d-2}}\delta_+(l^2) \\
\times\Bigg\{\frac{(p_1\cdot p_2)^2}{p_1\cdot k\; p_2\cdot k\; p_1\cdot l\; p_2\cdot l}(\Theta_S - \Theta_{C_1C_2,S}) \\
-\left[\frac{(p_1\cdot p_2)^2}{p_1\cdot k\; q\cdot k\; p_1\cdot l\; q\cdot l}(\Theta_{C_1,S} - \Theta_{C_1,SC_1,S} - \Theta_{C_1,C_1S,S} + \Theta_{C_1C_2,S}) + (1\leftrightarrow 2)\right] \\
-\left[\left(\frac{(p_1\cdot p_2)^2}{p_1\cdot k\, q\cdot k\, p_1\cdot l\, p_2\cdot l}(\Theta_{C_1S,S} - \Theta_{C_1C_2,S}) + (k\leftrightarrow l)\right) + (1\leftrightarrow 2)\right]\Bigg\}
\end{aligned}
$$

Zero bin of the double-collinear region

Zero bin of the z_N prescribed soft × collinear zero bin of the triple collinear region

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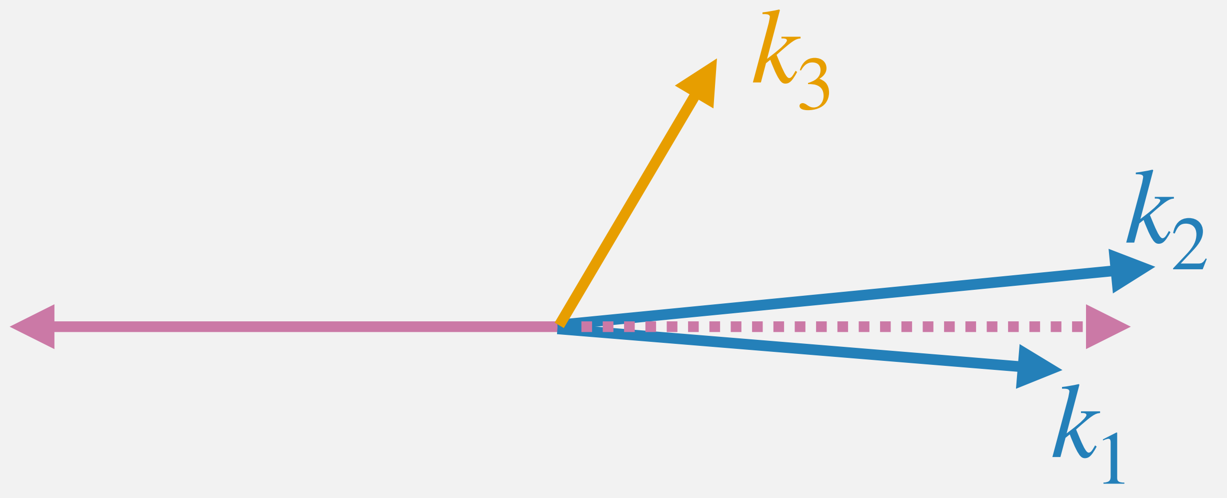
Zero bin of the soft × collinear region

Cumulant factorization breaking

In the WTA scheme, y_{23} factorizes as $y_{23} \sim \max\left(\tilde{\mathfrak{q}}_{C_1}(F_1), \tilde{\mathfrak{q}}_{C_2}(F_2), \tilde{\mathfrak{q}}_S(S)\right)$

In the E-scheme, this is **not true**. Consider a situation where we have two collinear and one soft particle:



Assume k_3 clusters first with k_2 to form k_{23}

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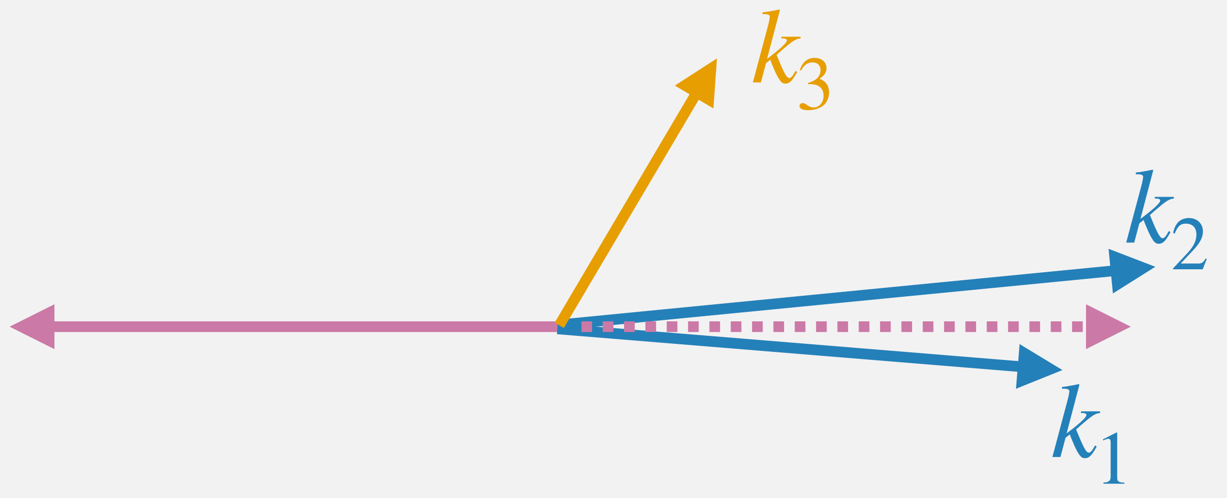

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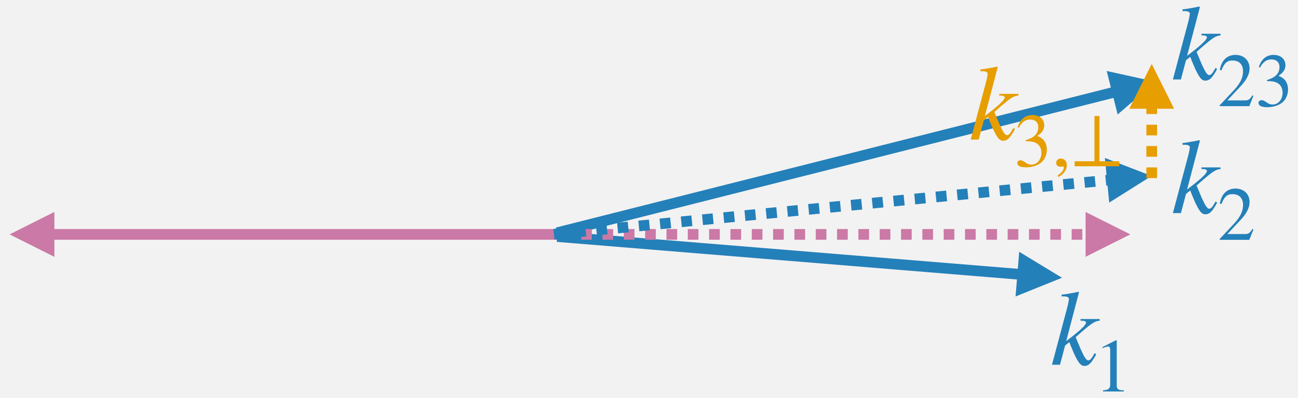

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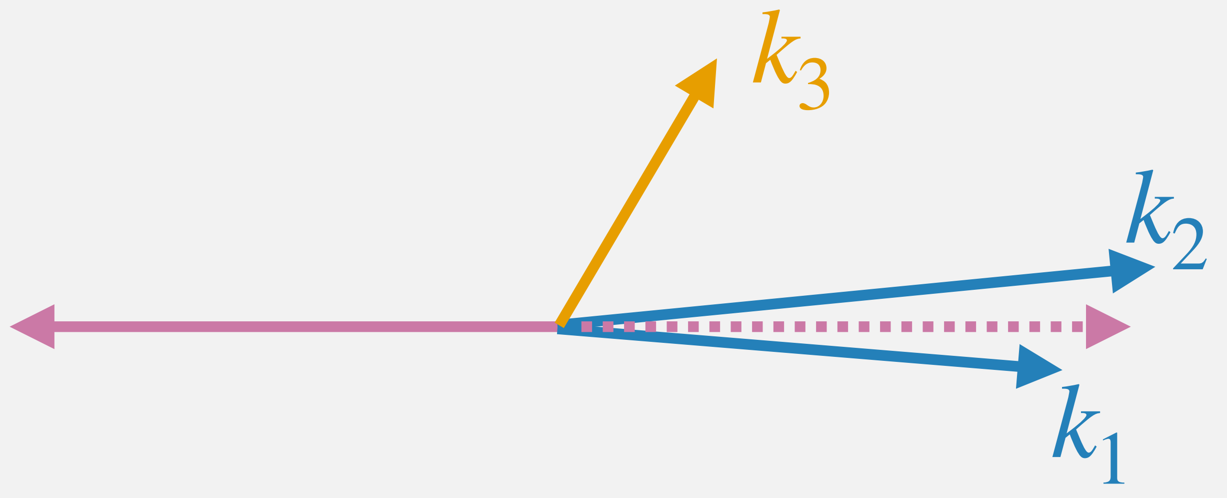

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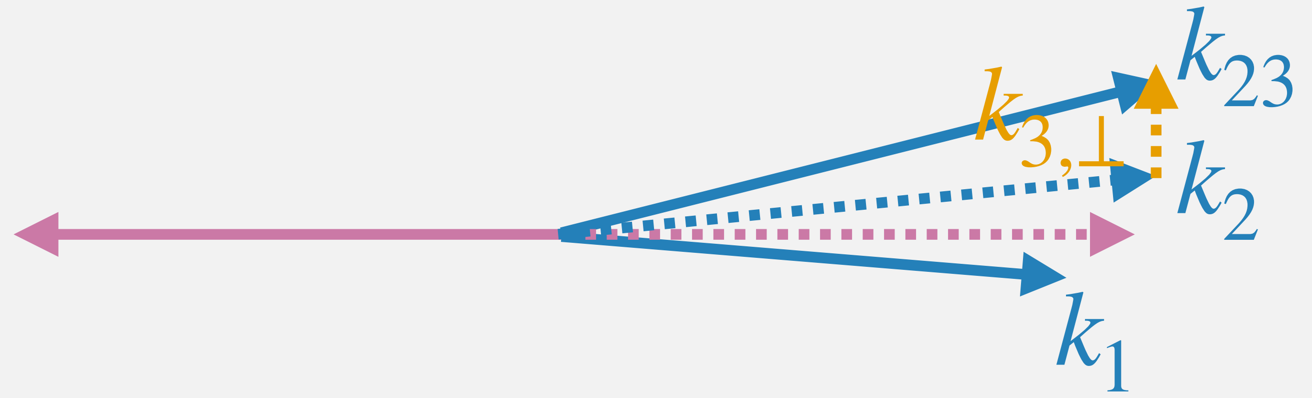

Then, $y_{23} = d_{1(23)}$ which is not the same as d_{12}! The recoil from the soft particle changed the distance between the collinear particles!

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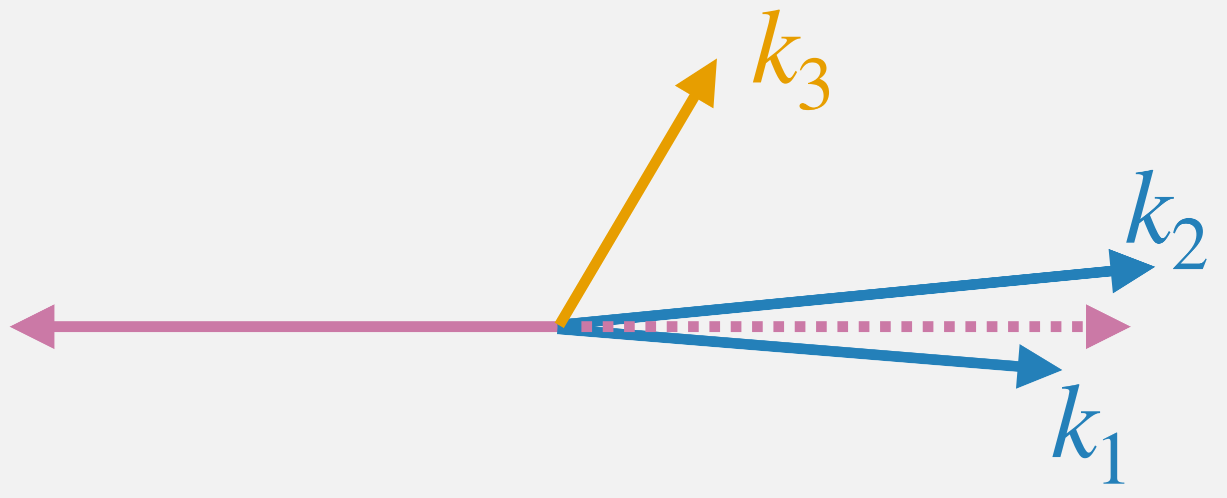
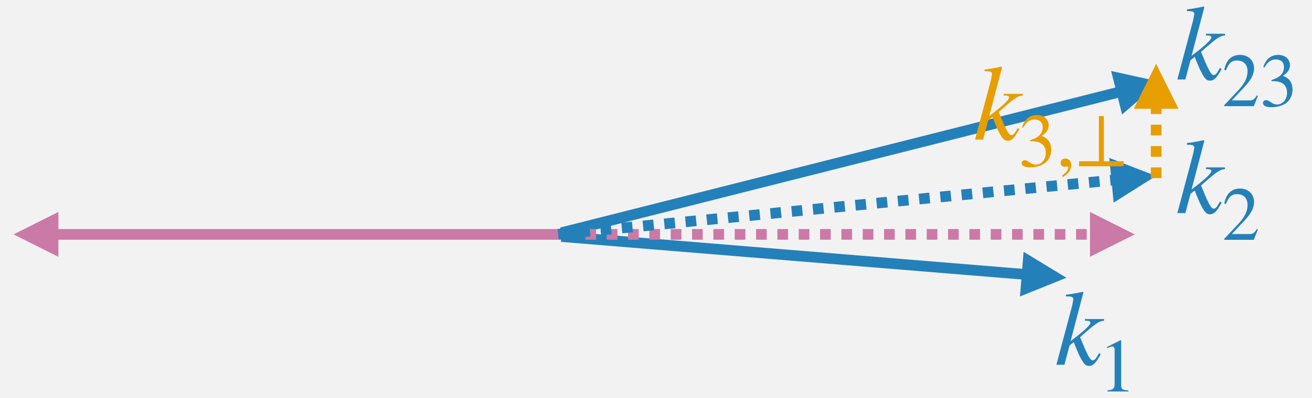


Assume k_3 clusters first with k_2 to form k_{23}



Then, $y_{23} = d_{1(23)}$ which is not the same as d_{12}! The recoil from the soft particle changed the distance between the collinear particles!

To make things worse, it matters whether k_3 clusters with k_1 or k_2 ($d_{1(23)} \neq d_{2(13)}$). However, to decide on the clustering history, one needs to compare d_{13} to d_{23}. They are the same at leading power, and one needs to expand them to subleading power!

Cumulant factorization breaking

These effects can be captured in the integral

Honest collinear × soft contribution

$$
\begin{aligned}
\mathcal{S}^{(2)}_{N,i} &= \mathfrak{q}_{\mathrm{cut}}^{4\epsilon}\left(\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\right)^2 \int_0^1 dz \frac{d^{d-2}\vec{k}_{1,\perp}}{\Omega_{d-2}} \frac{\hat{P}^{(0)}_{N,qg}(z)}{k^2_{1,\perp}} \int \frac{d^d k_3}{\Omega_{d-2}} \delta_+(k_3^2) \\
&\times \Big\{ \mathbb{J}^{(0)}(k_3) \Big[\theta(\mathfrak{q}_{\mathrm{cut}} - \mathfrak{q}_{C_i S}(\{k_1,k_2\},\{k_3\})) - \theta(\mathfrak{q}_{\mathrm{cut}} - \max(k_{1,\perp}, \mathfrak{q}_S(k_3)))\Big] \\
&\quad - 2C_F \frac{p_1 \cdot p_2}{(p_i \cdot k_3)(q \cdot k_3)} \Big[\theta(\mathfrak{q}_{\mathrm{cut}} - \mathfrak{q}_{C_i, C_i S}(\{k_1,k_2\},\{k_3\})) - \theta(\mathfrak{q}_{\mathrm{cut}} - \max(k_{1,\perp}, k_{3,\perp}))\Big]\Big\} \\
&= A^{(0)}_{\mathrm{F}B} L + B^{(0)}_{\mathrm{F}B} \,,
\end{aligned}
$$

Mistake made by cumulant factorization

collinear × soft zero bin of triple collinear region

NNLL effect also captured by treatment in ARES framework (1607.03111)

Novel NNLL′ contribution calculated here for the first time

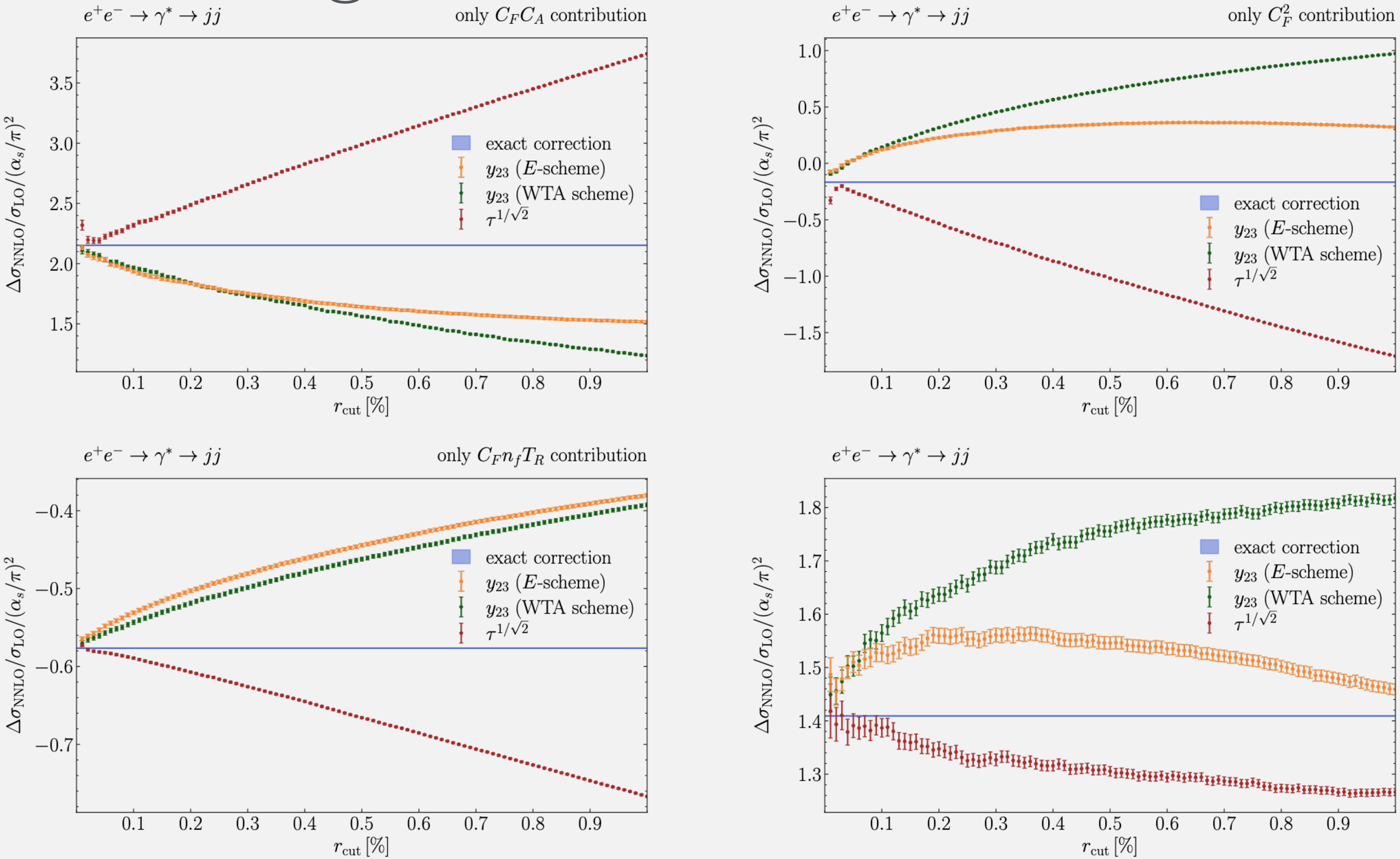
Full below-the-cut contribution

We can now put all ingredients together to find

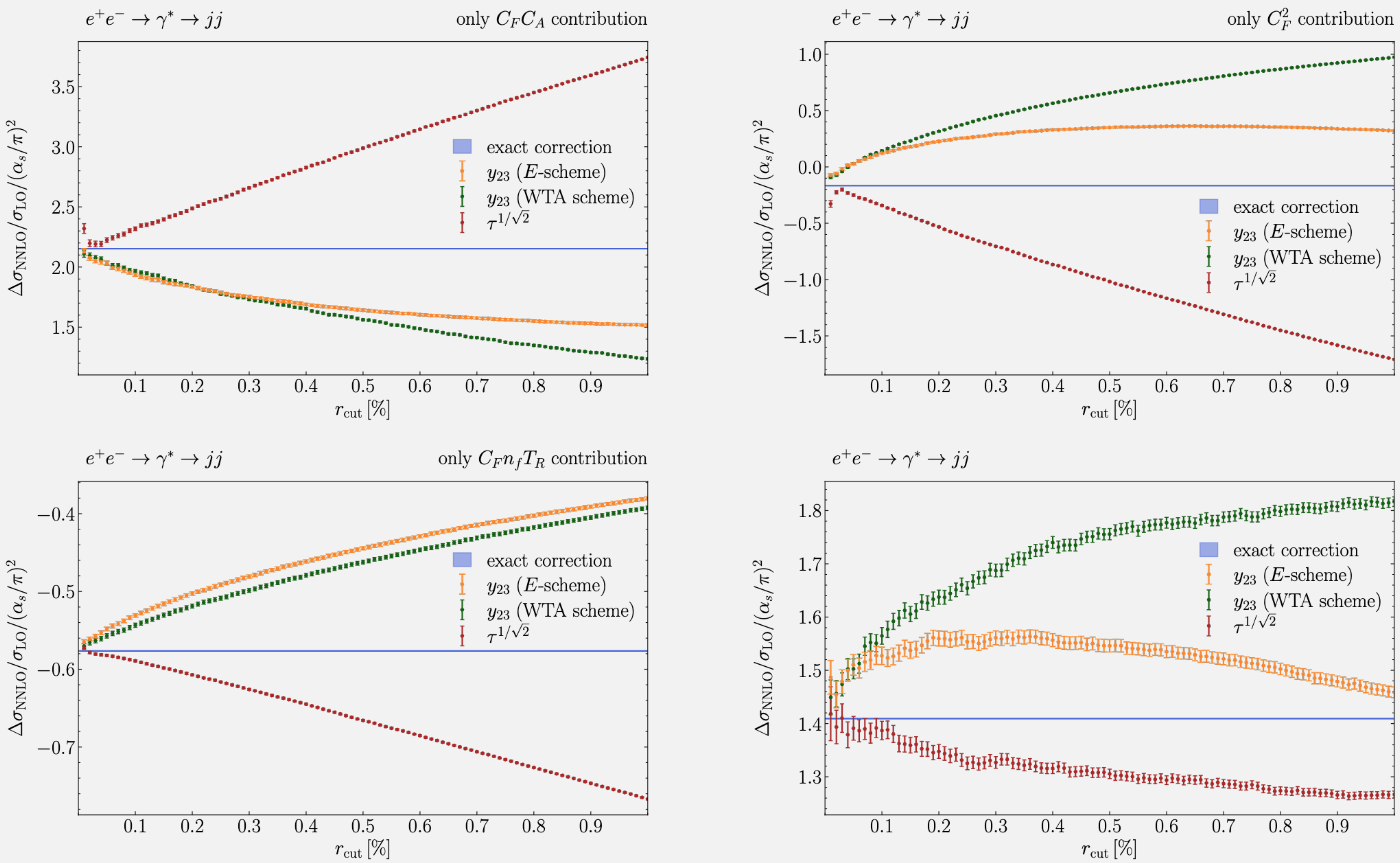
$$\boxed{\begin{aligned} &d\sigma_{\mathfrak{q}<\mathfrak{q}_{\mathrm{cut}}} = d\sigma_{\mathrm{B}}\left[1+\frac{\alpha_S}{\pi}\Sigma^{(1)}+\left(\frac{\alpha_S}{\pi}\right)^2\Sigma^{(2)}+\mathcal{O}(\alpha_S^3)\right]\,, \\ &\Sigma^{(1)} = \sum_{k=0}^{2}\Sigma^{(1,k)}L^k, \quad \Sigma^{(2)} = \sum_{k=0}^{4}\Sigma^{(2,k)}L^k\,. \end{aligned}}$$

All $\boldsymbol{\Sigma}$-coefficients are analytical, except the single-log coefficients and the finite piece.

Final slicing results



Final slicing results



- We compared y_{23} with E-scheme and WTA scheme with $\tau = 1 - T$, which can be viewed as a version of jettiness
- We achieve 1-2% precision on the NNLO correction (roughly 0.003% on the total cross section)
- Jettiness and y_{23} **perform similarly** with same computational resources of ~10^5 CPU hours per color structure.
- Excellent test of framework and perturbative ingredients

Outlook

- We can use the framework to generalize y_{23}-slicing to k_t^{ness}-slicing for general processes at hadron colliders at NNLO. To achieve this, we need the
  - Gluon jet function
  - Subtracted soft functions for any number of legs
  - Beam functions (if we use the WTA scheme)
  - Cumulant factorization breaking contributions (if we use the E-scheme)
- Rapidity evolution for fully differential jet and subtracted soft functions
- Semi-numerical NNLL′ resummation for general resolution variables
- Subleading power corrections

Thank You!