Towards NNLO Matching with Local Subtraction Schemes

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TUM-MPP Collider Phenomenology Seminar

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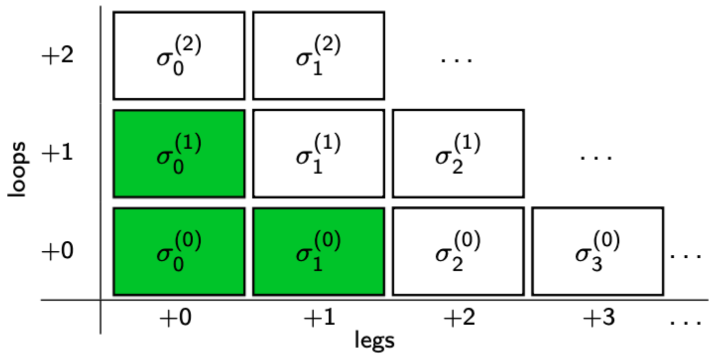
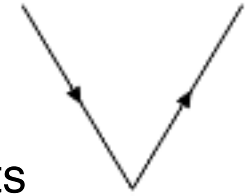
Fixed-order calculations vs parton showers

NLO

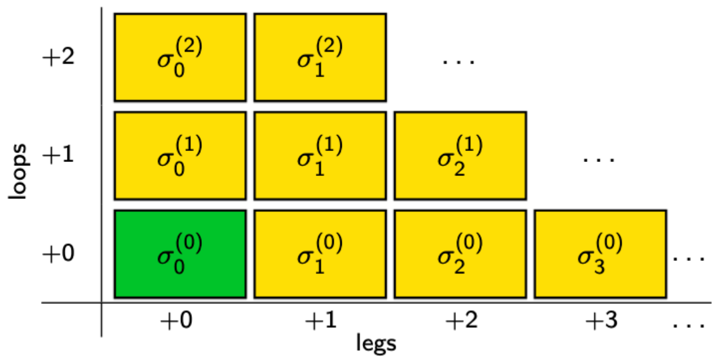


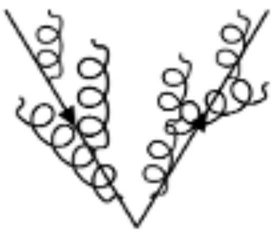


Fixed-order calculations → hard jets

- reliable at high scales, without scale hierarchies
- accurate for limited number of legs (+ loops)
- **perturbative accuracy** (LO, NLO, NNLO, …)

LO+PS





Parton showers → jet substructure

- reliable at small scales, with scale hierarchies
- approximate predictions for many particles
- **logarithmic accuracy** (LL, NLL, NNLL, …)

⇒ ideally combine (match) the two approaches!

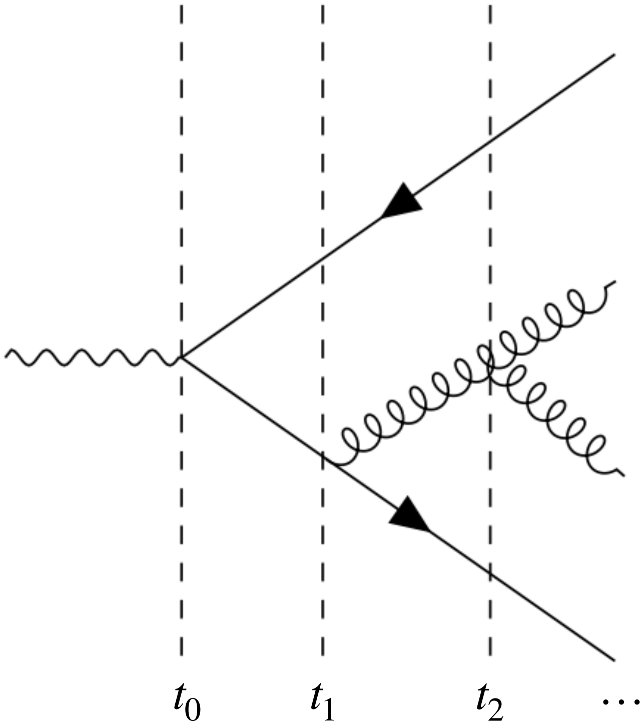
Starting from a hard scale t_0, parton showers model radiation under the assumption that it is **soft/collinear** and **ordered** ($t_0 > t_1 > t_2 > \ldots > t_c$)

Assume that parton branching occurs independently and that parton evolution conserves probability (unitarity).

The probability for n emissions is then given by Poisson statistics

$$P(n, \lambda) = \frac{\lambda^n e^{-\lambda}}{n!}$$

with "decay probability" λ and $\sum_n P(n, \lambda) = 1$



The shower prediction for the expected value of an (IRC-safe) observable is

$$\langle O \rangle_{\mathrm{PS}} = \int \mathrm{d}\Phi_n \, \mathrm{B}(\Phi_n) \, \mathcal{S}_n(\Phi_n, t_n; O)$$

The shower operator is defined recursively

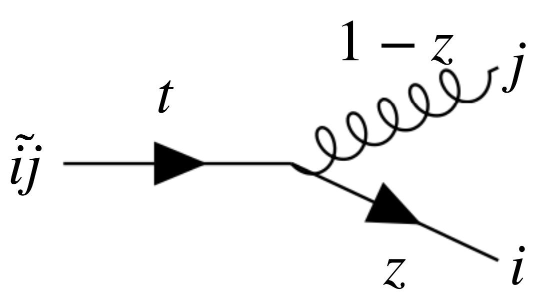
$$\mathcal{S}_n(\Phi_n, t_n; O) = \Delta(t_n, t_{\mathrm{c}}) \, O(\Phi_n) + \sum_{i,j} \int_{t_{\mathrm{c}}}^{t_n} \mathrm{d}t \int \mathrm{d}z \, \Delta(t_n, t) \frac{\alpha_{\mathrm{s}}}{2\pi} \frac{P_{ij}(z)}{t} \mathcal{S}_{n+1}(\Phi_{n+1}, t; O)$$

no branching between t_0 and t_{c}

branching at scale t

The no-branching probability is expressed in terms of splitting functions

$$\Delta_n(t_n, t) = \exp\left\{ -\sum_{i,j} \int_{t}^{t_n} \frac{\mathrm{d}t'}{t'} \int \mathrm{d}z \, \frac{\alpha_{\mathrm{s}}}{2\pi} P_{ij}(z) \right\}$$



To compare showers and fixed-order calculations, the shower has to be expanded in the strong coupling.

The first-order expansion of the shower operator is given by

$$\mathcal{S}_n^{(1)}(\Phi_n, t_n; O) = \left[1 - \Delta^{[1]}(t_n, t_c)\right] O(\Phi_n) + \sum_{i,j} \int_{t_c}^{t_n} \mathrm{d}t \int \mathrm{d}z \, \frac{\alpha_s}{2\pi} \frac{P_{ij}(z)}{t} O(\Phi_{n+1})$$

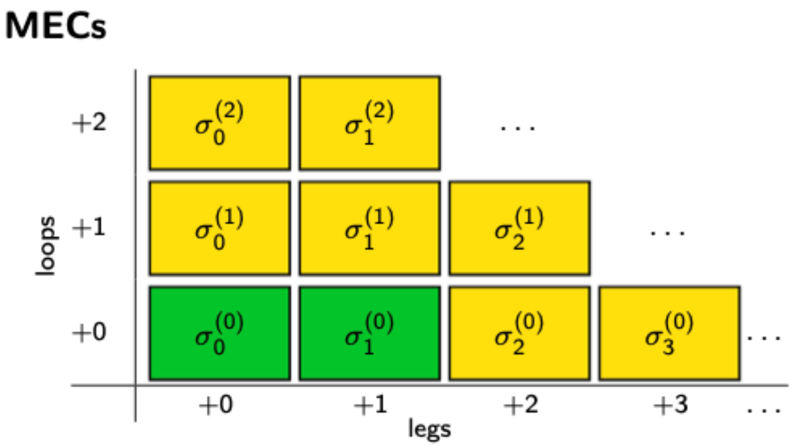
with $\Delta^{[1]}(t_n, t_c) = \sum_{i,j} \int_{t_c}^{t_n} \mathrm{d}t \int \mathrm{d}z \, \frac{\alpha_s}{2\pi} \frac{P_{ij}(z)}{t}$.

The shower approximation to an $n+1$- parton observable* is given by

$$\langle O_{n+1} \rangle_{\mathrm{PS}} = \int \mathrm{d}\Phi_n \, \mathrm{B}(\Phi_n) \sum_{i,j} \int_{t_c}^{t_n} \mathrm{d}t \int \mathrm{d}z \, \frac{\alpha_s}{2\pi} \frac{P_{ij}(z)}{t} O_{n+1}(\Phi_{n+1})$$

* i.e. $O_{n+1}(\Phi_m) = 0$ for all $m < n+1$

Schematically the simplest way to combine parton showers with LO calculations: Matrix-element corrections (MECs)



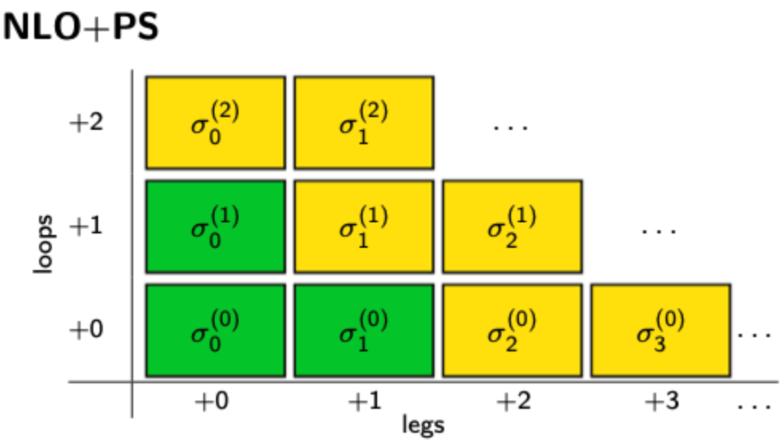
Replace splitting functions (for first branching) by full matrix element

$$\frac{P_{ij}(z)}{t} \to \frac{\mathrm{R}(\Phi_{n+1})}{\frac{\alpha_s}{2\pi}\sum_{i,j}\frac{1}{t}P_{ij}(z)\mathrm{B}(\Phi_n)}\frac{P_{ij}(z)}{t}$$

The shower approximation to an $n+1$- parton observable becomes

$$\langle O_{n+1}\rangle_{\mathrm{MEC}} = \int \mathrm{d}\Phi_n\,\mathrm{B}(\Phi_n)\sum_{i,j}\int_{t_c}^{t_n}\mathrm{d}t\int\mathrm{d}z\,\frac{\alpha_s}{2\pi}\frac{\mathrm{R}(\Phi_{n+1})}{\frac{\alpha_s}{2\pi}\sum_{i,j}\frac{1}{t}P_{ij}(z)\mathrm{B}(\Phi_n)}\frac{P_{ij}(z)}{t}\,O_{n+1}(\Phi_{n+1})$$

$$= \int \mathrm{d}\Phi_{n+1}\,\mathrm{R}(\Phi_{n+1})\,O_{n+1}(\Phi_{n+1}) \qquad \text{(above shower cut off } t_c\text{)}$$



Augmenting a MEC prediction by virtual corrections yields a simple (multiplicative) NLO+PS matching scheme.

Strategy developed ~ 25 years ago
[Norrbin, Sjöstrand NPB603(2001)297-342]

Nowadays known as POWHEG matching
[Nason JHEP11(2004)040]

Idea: first-order expansion of matrix-element-corrected shower reproduces NLO calculation if Born-level event is weighted by NLO K-factor:

$$\langle O\rangle^{\mathrm{POWHEG}}_{\mathrm{NLO+PS}} = \int \mathrm{d}\Phi_n\, \mathrm{B}(\Phi_n)\, K_{\mathrm{NLO}}(\Phi_n)\, \mathcal{S}_n(\Phi_n, t_n; O)$$

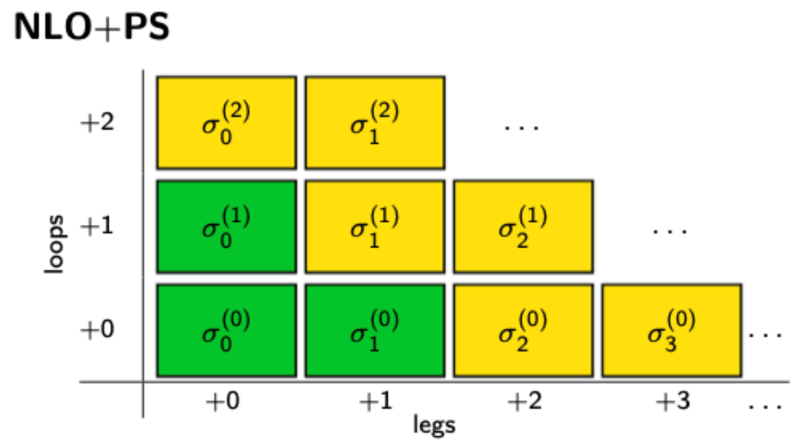
Schematically, $K_{\mathrm{NLO}}(\Phi_n) = 1 + \frac{\mathrm{V}(\Phi_n)}{\mathrm{B}(\Phi_n)} + \int \mathrm{d}\Phi_{+1} \frac{\mathrm{R}(\Phi_{n+1})}{\mathrm{B}(\Phi_{\mathrm{n}})}$

Schematically:

$$
\begin{aligned}
\mathrm{B}_{\mathrm{NLO}}(\Phi_n)\, K_{\mathrm{NLO}}(\Phi_n)\, \mathcal{S}_n(\Phi_n, t_n; O) \approx{}& \left(\mathrm{B}(\Phi_n) + \mathrm{V}(\Phi_n) + \int \mathrm{d}\Phi_{+1}\, \mathrm{R}(\Phi_{n+1})\right) \left[1 - \Delta^{[1]}_{\mathrm{MEC}}(t_n, t_c)\right] O(\Phi_n) \\
&+ \mathrm{B}(\Phi_n) \sum_{i,j} \int_{t_c}^{t_n} \mathrm{d}t \int \mathrm{d}z\, \frac{\alpha_s}{2\pi} \frac{\mathrm{R}(\Phi_{n+1})}{\frac{\alpha_s}{2\pi}\sum_{i,j}\frac{1}{t}P_{ij}(z)\mathrm{B}(\Phi_n)} \frac{P_{ij}(z)}{t}\, O(\Phi_{n+1}) \\
\approx{}& \left(\mathrm{B}(\Phi_n) + \mathrm{V}(\Phi_n) + \int \mathrm{d}\Phi_{+1}\, \mathrm{R}(\Phi_{n+1})\right) O(\Phi_n) \\
&- \int_{t_c}^{t_n} \mathrm{d}\Phi_{+1}\, \mathrm{R}(\Phi_{n+1})\, O(\Phi_n) + \int_{t_c}^{t_n} \mathrm{d}\Phi_{+1}\, \mathrm{R}(\Phi_{n+1})\, O(\Phi_{n+1}) \\
={}& \left(\mathrm{B}(\Phi_n) + \mathrm{V}(\Phi_n) + \int \mathrm{d}\Phi_{+1}\, \mathrm{R}(\Phi_{n+1})\right) O(\Phi_n) \\
&+ \int_{t_c}^{t_n} \mathrm{d}\Phi_{+1}\, \mathrm{R}(\Phi_{n+1}) \left[O(\Phi_{n+1}) - O(\Phi_n)\right] + \mathcal{O}(\alpha_s^2)
\end{aligned}
$$

$\Rightarrow$ first-order expansion yields projection-to-Born result (with cut off t_c)

Alternatively, use shower kernels as subtraction terms → MC@NLO matching
[Frixione, Webber JHEP 06 (2002) 029]



Idea: modify subtraction scheme so that first-order shower expansion reproduces NLO calculation:

$$\langle O \rangle^{\text{MC@NLO}}_{\text{NLO+PS}} = \int \mathrm{d}\Phi_n \left[\mathrm{B}(\Phi_n) + \mathrm{V}(\Phi_n) + \sum_{i,j} \mathscr{C}_{ij}(\Phi_n) \right] \mathcal{S}_n(\Phi_n, t_n; O)$$

$$+ \int \mathrm{d}\Phi_{n+1} \left[\mathrm{R}(\Phi_{n+1}) - \sum_{i,j} C_{ij}(\Phi_{n+1}) \right] \mathcal{S}_{n+1}(\Phi_{n+1}, t_{n+1}; O)$$

with $\sum_{i,j} \int \mathrm{d}\Phi_{+1} C_{ij}(\Phi_{n+1}) = \sum_{i,j} \mathscr{C}_{ij}(\Phi_n)$

Alternatively, use shower kernels as subtraction terms:

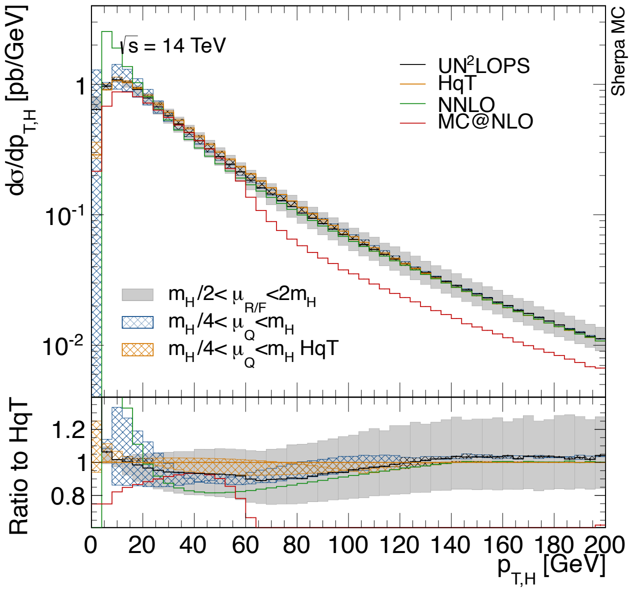
$$\begin{aligned}
\langle O\rangle^{\mathrm{MC@NLO}}_{\mathrm{NLO+PS}} \approx & \int \mathrm{d}\Phi_n \left[\mathrm{B}(\Phi_n) + \mathrm{V}(\Phi_n) + \sum_{i,j} \mathscr{C}_{ij}(\Phi_n)\right] \left[-\Delta^{[1]}(t_n, t_c)\, O(\Phi_n) + \sum_{i,j} \int_{t_c}^{t_n} \mathrm{d}t \int \mathrm{d}z\, \frac{\alpha_s}{2\pi} \frac{P_{ij}(z)}{t}\, O(\Phi_{n+1}) \right] \\
& + \int \mathrm{d}\Phi_{n+1} \left[\mathrm{R}(\Phi_{n+1}) - \sum_{i,j} C_{ij}(\Phi_{n+1})\right] O(\Phi_{n+1}) \\
= & \int \mathrm{d}\Phi_n \left[\mathrm{B}(\Phi_n) + \mathrm{V}(\Phi_n) + \sum_{i,j} \mathscr{C}_{ij}(\Phi_n)\right] \\
& + \int \mathrm{d}\Phi_{n+1} \left[\mathrm{R}(\Phi_{n+1})\, O(\Phi_{n+1}) - \sum_{i,j} \frac{\alpha_s}{2\pi} \frac{P_{ij}(z)}{t} \mathrm{B}(\Phi_n)\, O(\Phi_n)\right] \\
& + \int \mathrm{d}\Phi_n\, \mathrm{B}(\Phi_n) \sum_{i,j} \int_0^{t_c} \mathrm{d}t \int \mathrm{d}z\, \frac{\alpha_s}{2\pi} \frac{P_{ij}(z)}{t} \left[O(\Phi_{n+1}) - O(\Phi_n)\right] + \mathcal{O}(\alpha_s^2)
\end{aligned}$$

wrong observable dependence in subtraction term

⇒ first-order expansion yields NLO calculation if $\frac{\alpha_s}{2\pi} \frac{P_{ij}(z)}{t} \mathrm{B}(\Phi_n) = C_{ij}(\Phi_{n+1})$

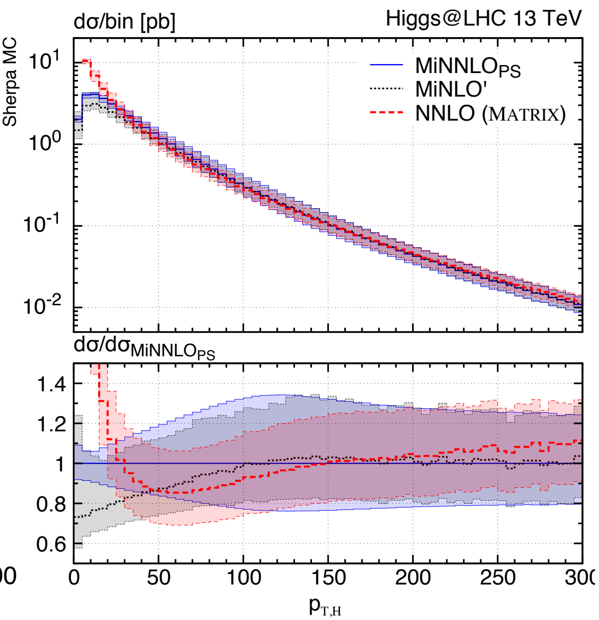
(with power correction in t_c)

UN^{2}LOPS



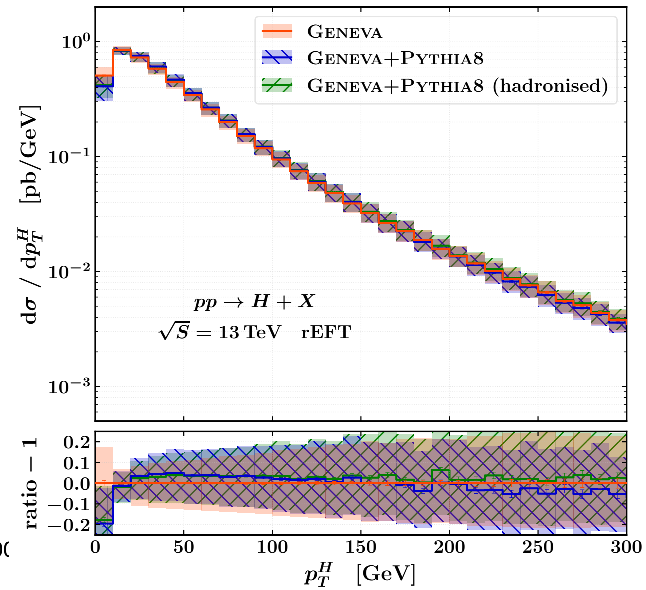
[Prestel, Li, Höche 1407.3773]

MiNNLO$_{PS}$



[Monni et al. 1908.06987]

GENEVA



[Alioli et al. 2301.11875]

All of the above rely on conventional (LO) parton-shower evolution.

⇒ does not reflect singular structure of the fixed-order calculation

⇒ requires regularisation via auxiliary scales / analytic resummation

⇒ second-order shower corrections improve existing + enable new methods

The second-order expansion of the shower operator is given by

$$
\begin{aligned}
\mathcal{S}_n^{(2)}(\Phi_n, t_n; O) &= \left[1 - \Delta_n^{[1]}(t_n, t_c) + \frac{1}{2}\Delta_n^{[1]}(t_n, t_c)^2\right] O(\Phi_n) \\
&\quad + \sum_{i,j} \int_{t_c}^{t_n} \mathrm{d}\Phi_{+1} \left(1 - \Delta_n^{[1]}(t_n, t_{n+1})\right) C_{ij}(\Phi_{n+1})\, \mathcal{S}_{n+1}^{(1)}(\Phi_{n+1}, t_{n+1}; O) \\
&= \left[1 - \Delta_n^{[1]}(t_n, t_c) + \frac{1}{2}\Delta_n^{[1]}(t_n, t_c)^2\right] O(\Phi_n) \\
&\quad + \sum_{i,j} \int_{t_c}^{t_n} \mathrm{d}\Phi_{+1} \left(1 - \Delta_n^{[1]}(t_n, t_{n+1})\right) C_{ij}(\Phi_{n+1}) \\
&\qquad \times \left[\left(1 - \Delta_{n+1}^{[1]}(t_{n+1}, t_c)\right) O(\Phi_{n+1}) + \sum_{k,l} \int_{t_c}^{t_{n+1}} \mathrm{d}\Phi_{+1}\, C_{kl}(\Phi_{n+2})\, O(\Phi_{n+2})\right]
\end{aligned}
$$

The second-order expansion of the shower operator is given by

$$\mathcal{S}_n^{(2)}(\Phi_n, t_n; O) = \left[1 - \Delta_n^{[1]}(t_n, t_\mathrm{c}) + \frac{1}{2}\Delta_n^{[1]}(t_n, t_\mathrm{c})^2\right] O(\Phi_n)$$

$$+ \sum_{i,j} \int_{t_\mathrm{c}}^{t_n} \mathrm{d}\Phi_{+1} \left(1 - \Delta_n^{[1]}(t_n, t_{n+1})\right) C_{ij}(\Phi_{n+1})\, O(\Phi_{n+1})$$

$$+ \sum_{i,j} \int_{t_c}^{t_n} \mathrm{d}\Phi_{+1}\, C_{ij}(\Phi_{n+1}) \left[- \Delta_{n+1}^{[1]}(t_{n+1}, t_c)\, O(\Phi_{n+1}) + \sum_{k,l} \int_{t_c}^{t_{n+1}} \mathrm{d}\Phi_{+1}\, C_{kl}(\Phi_{n+2})\, O(\Phi_{n+2}) \right]$$

I. First steps towards NNLO matching with local subtraction schemes
based on arXiv:2108.07133 and arXiv:2412.14242

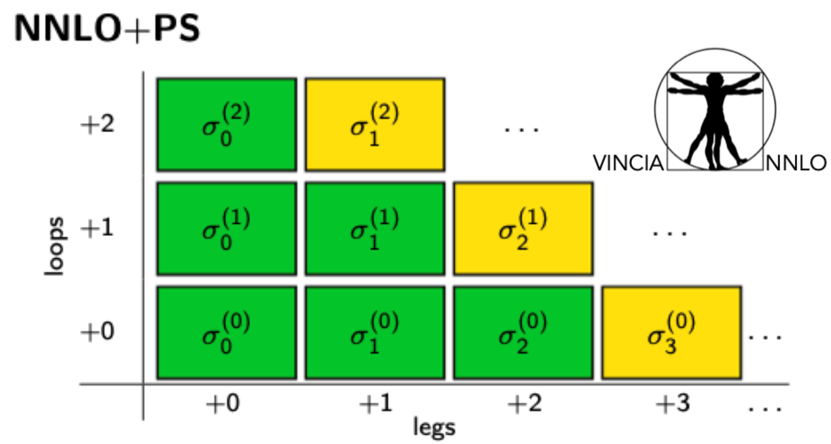
II. Second-order shower evolution

III. Decomposition of splitting functions
based on arXiv:2505.10408

First steps towards NNLO matching with local subtraction schemes

Multiplicative NNLO+PS

Extend multiplicative matching idea to NNLO via matrix-element corrections.
[Campbell et al. PLB836(2023)137614]



Idea: fully-differential multiplicative NNLO matching scheme ("POWHEG at NNLO")
(focus here on $e^+e^- \to 2j$ and the like)

$$\langle O \rangle^{\text{VINCIA}}_{\text{NNLO+PS}} = \int \mathrm{d}\Phi_2\, \mathrm{B}(\Phi_2)\, K_{\text{NNLO}}(\Phi_2)\, \mathcal{S}_2(\Phi_2, t_2; O)$$

Need:
(1) Born-local NNLO K-factor
(2) NLO Matrix-element correction in first branching
(3) LO Matrix-element correction in second iterated branching
(4) Direct $2 \mapsto 4$ branching with Matrix-element correction

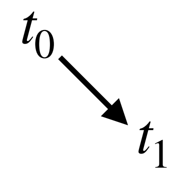
$\Rightarrow$ for two-parton processes, everything can be calculated using antenna subtraction

Key aspect: [Campbell et al. PLB836(2023)137614]

up to matched order, include process-specific NLO corrections into shower evolution

(2) correct first branching to exclusive NLO rate

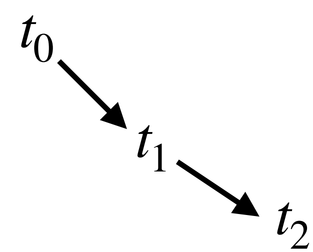
$$\Delta^{\mathrm{NLO}}_{2\mapsto 3}(t_0, t_1) = \exp\left\{ -\int_{t_1}^{t_0} \mathrm{d}\Phi_{+1} \frac{\mathrm{R}(\Phi_3)}{\mathrm{B}(\Phi_2)} \left(1 + v^{\mathrm{NLO}}\right) \right\}$$



RV

(3) correct second branching to LO matrix element

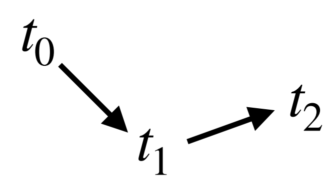
$$\Delta^{\mathrm{LO}}_{3\mapsto 4}(t_1, t_2) = \exp\left\{ -\int_{t_2}^{t_1} \mathrm{d}\Phi_{+1} \frac{\mathrm{RR}(\Phi_4)}{\mathrm{R}(\Phi_3)} \right\}$$



RR

(4) add $2 \mapsto 4$ branching and correct it to LO matrix element

$$\Delta^{\mathrm{LO}}_{2\mapsto 4}(t_0, t_1) = \exp\left\{ -\int_{t_2}^{t_0} \mathrm{d}\Phi_{+2} \frac{\mathrm{RR}(\Phi_4)}{\mathrm{B}(\Phi_2)} \Theta(t_2 - t_1) \right\}$$

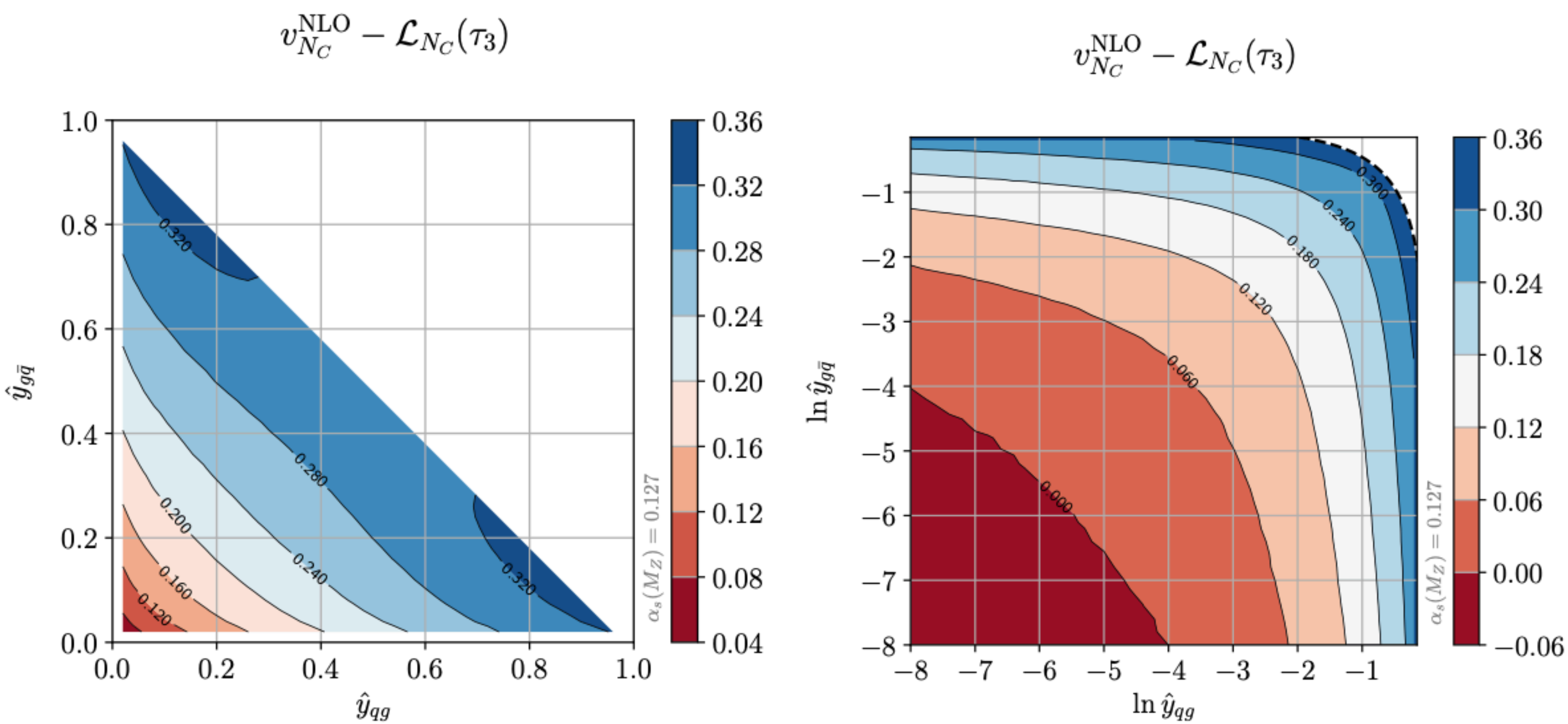


RR

[El-Menoufi, CTP, Scyboz, Skands 2412.14242]

Virtual correction to first branching evaluated numerically using antenna subtraction ("POWHEG in the exponent")

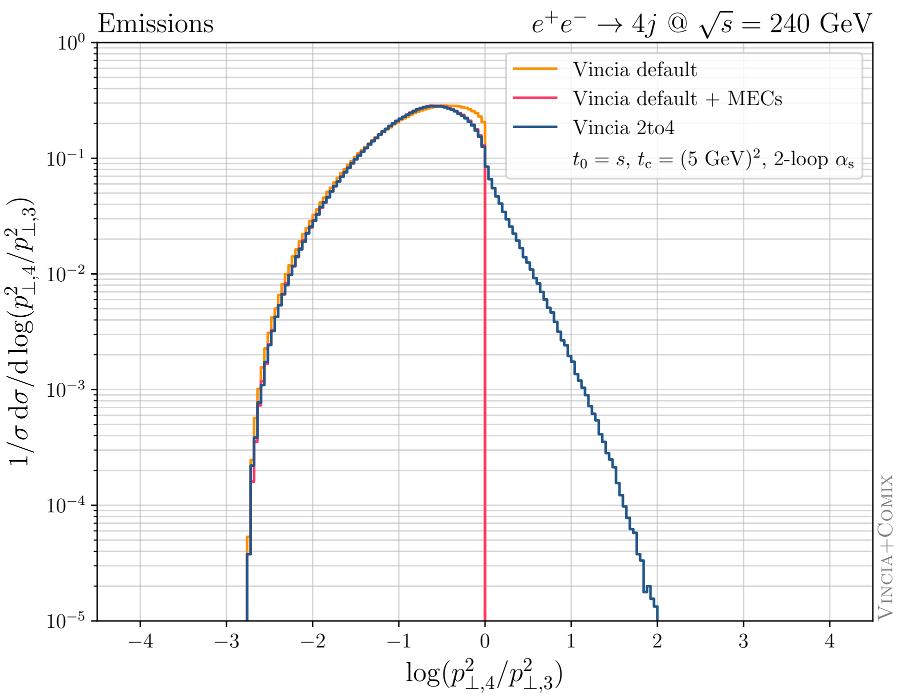
$$\Delta^{\mathrm{NLO}}_{2\mapsto 3}(t_0,t_1) = \exp\left\{ -\int_{t_1}^{t_0} \mathrm{d}\Phi_{+1}\,\frac{\mathrm{R}(\Phi_3)}{\mathrm{B}(\Phi_2)}\left(1+v^{\mathrm{NLO}}_{N_C}+v^{\mathrm{NLO}}_{N_F}+v^{\mathrm{NLO}}_{1/N_C}\right)\right\}$$

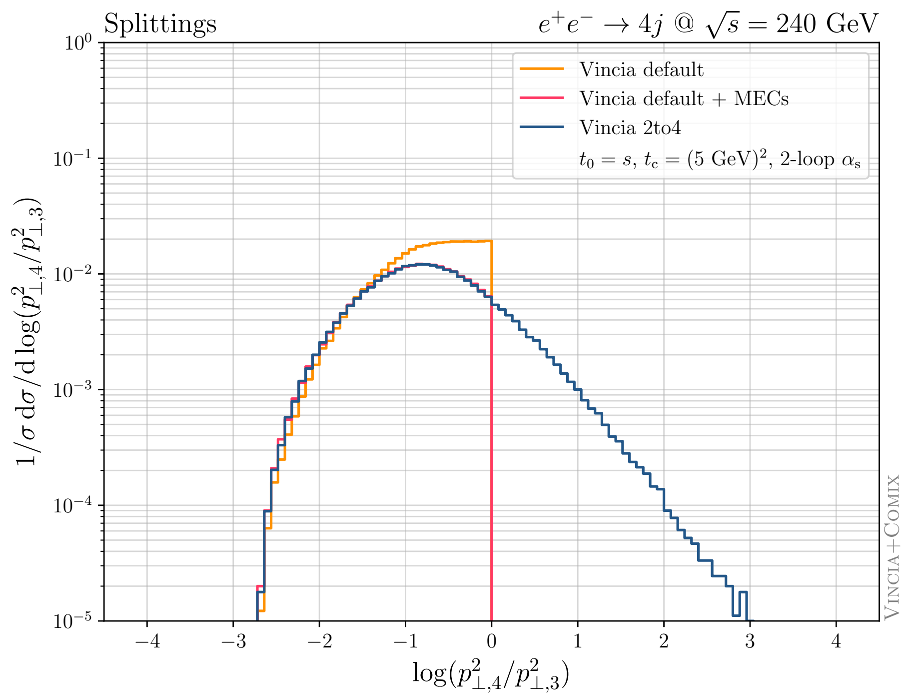


Double-real corrections

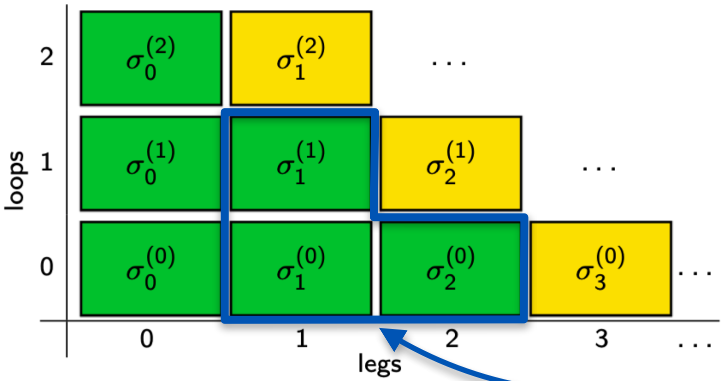
[Campbell et al. PLB836(2023)137614]

Direct $2 \mapsto 4$ shower fills unordered region of phase space with $p^2_{\perp,4} > p^2_{\perp,3}$





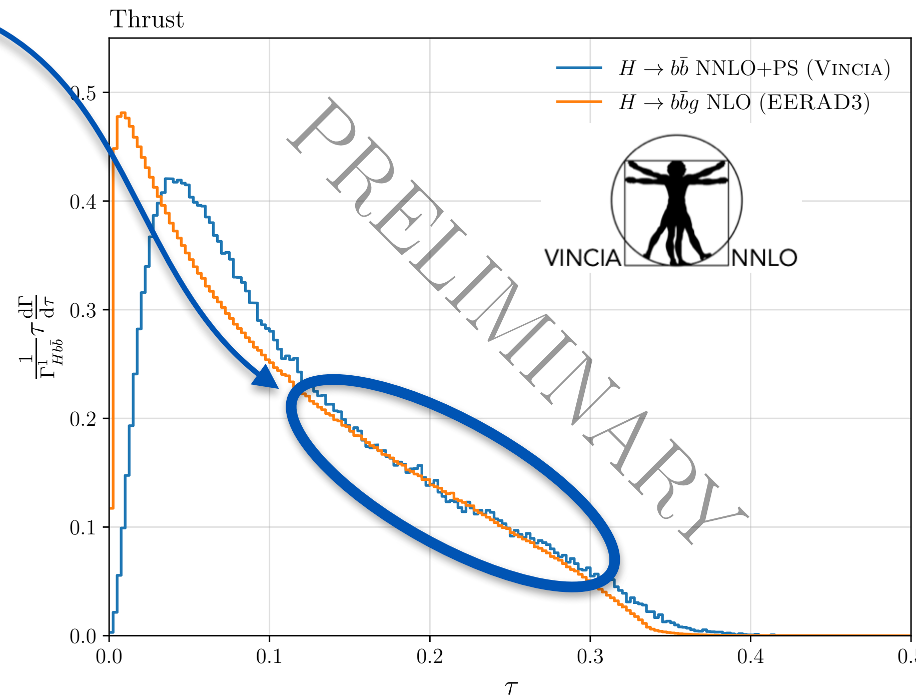
Sectorisation enforces a strict cutoff at $p^2_{\perp,4} = p^2_{\perp,3}$. No recoil effects!



By **construction** decay width is NNLO accurate.

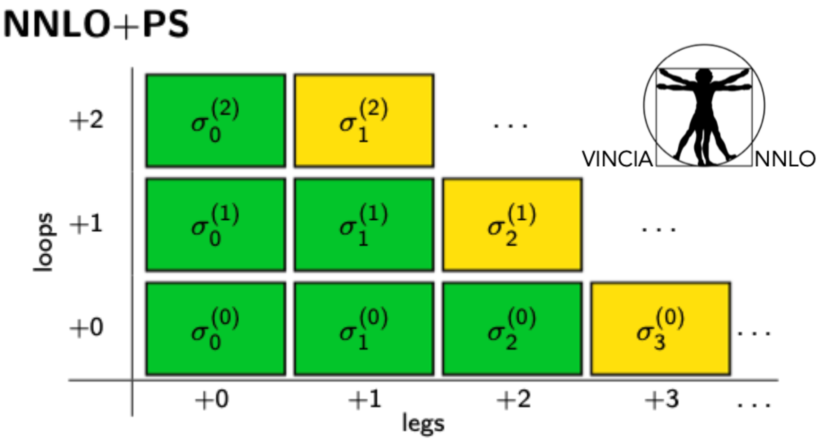
NNLO Born accuracy also implies
NLO accuracy in first branching and
LO accuracy in second branching.

E.g. H → b$\bar{\text{b}}$j at parton level vs. NLO
[Coloretti, Gehrmann-De Ridder, CTP 2202.07333]
(massless b-quarks, non-zero Yukawa)



General multiplicative NNLO+PS?

Extend multiplicative matching idea to NNLO via matrix-element corrections.
[Campbell et al. PLB836(2023)137614]



In principle, **the idea can be generalised** to arbitrary processes

$$\langle O \rangle^{\mathrm{VINCIA}}_{\mathrm{NNLO+PS}} = \int \mathrm{d}\Phi_n \, \mathrm{B}(\Phi_n) \, K_{\mathrm{NNLO}}(\Phi_n) \, \mathcal{S}_n(\Phi_n, t_n; O)$$

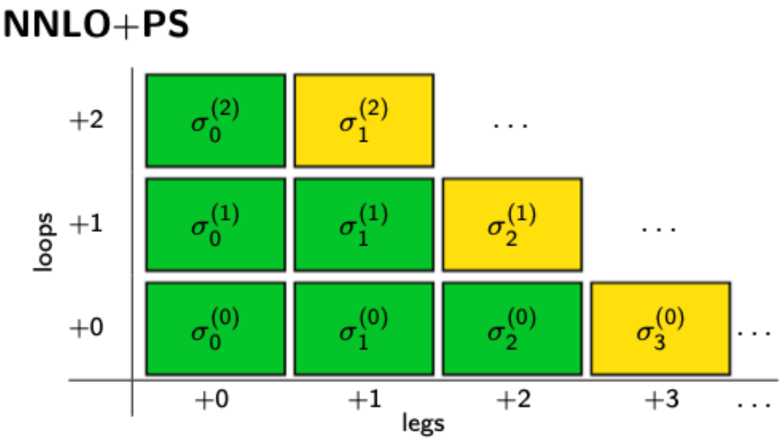
but requires:

(1) ~~Born-local NNLO K-factor~~ General Born-local NNLO subtraction scheme
(2) Efficient NLO Matrix-element correction in first branching at full colour
(3) LO Matrix-element correction in second iterated branching at full colour
(4) Direct $2 \mapsto 4$ branching with Matrix-element correction at full colour

$\Rightarrow$ antenna framework not ideal for **general** multiplicative matching

Use shower kernels as subtraction terms
→ MC@NNLO

Idea: modify subtraction scheme so that second-order shower expansion reproduces NNLO calculation.



Schematically:

$$\begin{aligned}\langle O\rangle^{\text{MC@NNLO}}_{\text{NNLO+PS}} = &\int \mathrm{d}\Phi_n \left[\mathrm{B}(\Phi_n) + \mathrm{V}(\Phi_n) + \sum_{i,j} \mathscr{C}_{ij}(\Phi_n) + \mathrm{VV}(\Phi_n) + \sum_{i,j} \mathscr{C}^{(1)}_{ij} + \sum_{i,j,k} \mathscr{C}_{ijk}\right] \mathscr{S}_n(\Phi_n, t_n; O) \\ &+ \int \mathrm{d}\Phi_{n+1} \Bigg[\mathrm{R}(\Phi_{n+1}) - \sum_{i,j} C_{ij}(\Phi_{n+1}) \\ &\qquad + \mathrm{RV}(\Phi_{n+1}) + \sum_{k,l} \mathscr{C}_{kl}(\Phi_{n+1}) - \sum_{i,j} C^{(1)}_{ij}(\Phi_{n+1})\Bigg] \mathscr{S}_{n+1}(\Phi_{n+1}, t_{n+1}; O) \\ &+ \int \mathrm{d}\Phi_{n+2} \left[\mathrm{RR}(\Phi_{n+2}) - \sum_{i,j,k} C_{ijk}(\Phi_{n+2}) - \sum_{k,l} C_{kl}(\Phi_{n+2})\right] \mathscr{S}_{n+2}(\Phi_{n+2}, t_{n+2}; O)\end{aligned}$$

⇒ second-order shower expansion has to match NNLO subtraction terms

Second-order shower evolution

The second-order expansion of the shower operator is given by

$$\mathcal{S}_n^{(2)}(\Phi_n, t_n; O) = \left[1 - \Delta_n^{[1]}(t_n, t_\mathrm{c}) + \frac{1}{2}\Delta_n^{[1]}(t_n, t_\mathrm{c})^2\right] O(\Phi_n)$$

$$+ \sum_{i,j} \int_{t_\mathrm{c}}^{t_n} \mathrm{d}\Phi_{+1} \left(1 - \Delta_n^{[1]}(t_n, t_{n+1})\right) C_{ij}(\Phi_{n+1})\, O(\Phi_{n+1})$$

$$+ \sum_{i,j} \int_{t_c}^{t_n} \mathrm{d}\Phi_{+1}\, C_{ij}(\Phi_{n+1}) \left[- \Delta_{n+1}^{[1]}(t_{n+1}, t_c)\, O(\Phi_{n+1}) + \sum_{k,l} \int_{t_c}^{t_{n+1}} \mathrm{d}\Phi_{+1}\, C_{kl}(\Phi_{n+2})\, O(\Phi_{n+2}) \right]$$

The second-order expansion of the shower operator is given by

$$
\begin{aligned}
\mathcal{S}_n^{(2)}(\Phi_n, t_n; O) = {} & \left[1 + \frac{1}{2}\Delta_n^{[1]}(t_n, t_\mathrm{c})^2\right] O(\Phi_n) \\
& - \Delta_n^{[1]}(t_n, t_c)\, O(\Phi_n) + \sum_{i,j} \int_{t_\mathrm{c}}^{t_n} \mathrm{d}\Phi_{+1}\, C_{ij}(\Phi_{n+1})\, O(\Phi_{n+1}) \\
& - \sum_{i,j} \int_{t_\mathrm{c}}^{t_n} \mathrm{d}\Phi_{+1}\, \Delta_n^{[1]}(t_n, t_{n+1})\, C_{ij}(\Phi_{n+1})\, O(\Phi_{n+1}) \\
& + \sum_{i,j} \int_{t_c}^{t_n} \mathrm{d}\Phi_{+1}\, C_{ij}(\Phi_{n+1}) \left[- \Delta_{n+1}^{[1]}(t_{n+1}, t_c)\, O(\Phi_{n+1}) + \sum_{k,l} \int_{t_c}^{t_{n+1}} \mathrm{d}\Phi_{+1}\, C_{kl}(\Phi_{n+2})\, O(\Phi_{n+2}) \right]
\end{aligned}
$$

The second-order expansion of the shower operator is given by

$$\begin{aligned}\mathcal{S}_n^{(2)}(\Phi_n, t_n; O) = &\left[1 + \boxed{\frac{1}{2}\Delta_n^{[1]}(t_n, t_c)^2}\right] O(\Phi_n) \\ &\boxed{-\Delta_n^{[1]}(t_n, t_c)\, O(\Phi_n) + \sum_{i,j}\int_{t_c}^{t_n} \mathrm{d}\Phi_{+1}\, C_{ij}(\Phi_{n+1})\, O(\Phi_{n+1})} \\ &\boxed{-\sum_{i,j}\int_{t_c}^{t_n} \mathrm{d}\Phi_{+1}\, \Delta_n^{[1]}(t_n, t_{n+1})\, C_{ij}(\Phi_{n+1})}\, O(\Phi_{n+1}) \\ &+\sum_{i,j}\int_{t_c}^{t_n} \mathrm{d}\Phi_{+1}\, C_{ij}(\Phi_{n+1}) \boxed{\left[-\Delta_{n+1}^{[1]}(t_{n+1}, t_c)\, O(\Phi_{n+1}) + \sum_{k,l}\int_{t_c}^{t_{n+1}} \mathrm{d}\Phi_{+1}\, C_{kl}(\Phi_{n+2})\, O(\Phi_{n+2})\right]}\end{aligned}$$

NLO n-jet subtraction

"Sudakov on top"

NLO $n+1$-jet subtraction

Shower structure at second order (cont'd)

The second-order expansion of the shower operator **should be** given by

$$
\begin{aligned}
\mathcal{S}_n^{(2)}(\Phi_n, t_n; O) = & \left[1 + \boxed{\frac{1}{2}\Delta_n^{[1]}(t_n, t_c)^2} \right] O(\Phi_n) \\
& \boxed{- \Delta_n^{[1]}(t_n, t_c)\, O(\Phi_n) + \sum_{i,j} \int_{t_c}^{t_n} \mathrm{d}\Phi_{+1}\, C_{ij}(\Phi_{n+1})\, O(\Phi_{n+1})} \\
& \boxed{- \sum_{i,j} \int_{t_c}^{t_n} \mathrm{d}\Phi_{+1}\, \Delta_n^{[1]}(t_n, t_{n+1})\, C_{ij}(\Phi_{n+1})}\, O(\Phi_{n+1}) \\
& + \sum_{i,j} \int_{t_c}^{t_n} \mathrm{d}\Phi_{+1}\, C_{ij}(\Phi_{n+1}) \boxed{\left[- \Delta_{n+1}^{[1]}(t_{n+1}, t_c)\, O(\Phi_{n+1}) + \sum_{k,l} \int_{t_c}^{t_{n+1}} \mathrm{d}\Phi_{+1}\, C_{kl}(\Phi_{n+2})\, O(\Phi_{n+2}) \right]} \\
& + \sum_{i,j} \int_{t_c}^{t_n} \mathrm{d}\Phi_{+1} \left[\boxed{\Delta_n^{[1]}(t_n, t_{n+1}) C_{ij}(\Phi_{n+1})} + \boxed{C_{ij}^{(1)}(\Phi_{n+1})} \right] \left[O(\Phi_{n+1}) - O(\Phi_n) \right]
\end{aligned}
$$

NLO n-jet subtraction

"Sudakov on top"

NLO $n+1$-jet subtraction

real-virtual subtraction

Shower structure at second order (cont'd)

The second-order expansion of the shower operator **should be** given by

$$\mathcal{S}_n^{(2)}(\Phi_n, t_n; O) = \left[1 + \frac{1}{2}\Delta_n^{[1]}(t_n, t_c)^2\right] O(\Phi_n)$$

$$-\Delta_n^{[1]}(t_n, t_c)\, O(\Phi_n) + \sum_{i,j}\int_{t_c}^{t_n} \mathrm{d}\Phi_{+1}\, C_{ij}(\Phi_{n+1})\, O(\Phi_{n+1})$$

$$-\sum_{i,j}\int_{t_c}^{t_n} \mathrm{d}\Phi_{+1}\, \Delta_n^{[1]}(t_n, t_{n+1})\, C_{ij}(\Phi_{n+1})\, O(\Phi_{n+1})$$

$$+\sum_{i,j}\int_{t_c}^{t_n} \mathrm{d}\Phi_{+1}\, C_{ij}(\Phi_{n+1})\left[-\Delta_{n+1}^{[1]}(t_{n+1}, t_c)\, O(\Phi_{n+1}) + \sum_{k,l}\int_{t_c}^{t_{n+1}} \mathrm{d}\Phi_{+1}\, C_{kl}(\Phi_{n+2})\, O(\Phi_{n+2})\right]$$

$$+\sum_{i,j}\int_{t_c}^{t_n} \mathrm{d}\Phi_{+1}\left[\Delta_n^{[1]}(t_n, t_{n+1})C_{ij}(\Phi_{n+1}) + C_{ij}^{(1)}(\Phi_{n+1})\right]\left[O(\Phi_{n+1}) - O(\Phi_n)\right]$$

$$+\sum_{i,j,k}\int_{t_c}^{t_n} \mathrm{d}\Phi_{+2}\, C_{ijk}(\Phi_{n+2})\left[O(\Phi_{n+2}) - O(\Phi_n)\right]$$

"Sudakov on top"

NLO n-jet subtraction

NLO $n+1$-jet subtraction

real-virtual subtraction

double-real subtraction

Define no-branching probability for **double branchings** as:

$$\Delta_n^{\mathrm{RR}}(t_n,t) = \exp\left\{ -\sum_{i,j,k} \int_{t_c}^{t_n} \mathrm{d}\Phi_{+2}\, C_{ijk}(\Phi_{n+2}) \right\}$$

and for **one-loop branchings** as:

$$\Delta_n^{\mathrm{RV}}(t_n,t) = \exp\left\{ -\sum_{i,j} \int_{t_c}^{t_n} \mathrm{d}\Phi_{+1} \left[C_{ij}^{(1)}(\Phi_{n+1}) + \Delta_n^{[1]}(t_n,t_{n+1}) C_{ij}(\Phi_{n+1}) \right] \right\}$$

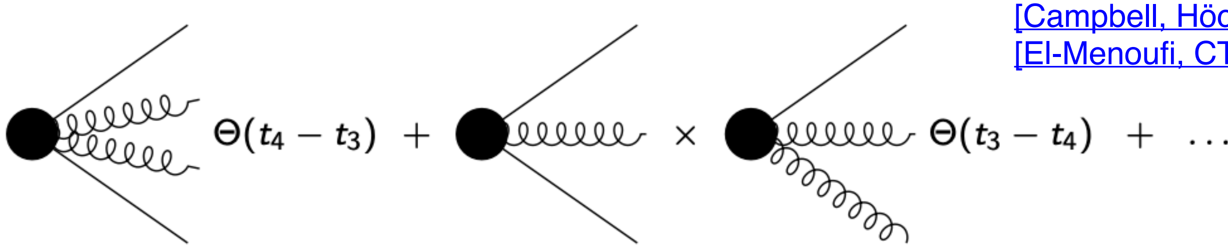
with $\Delta_n^{\mathrm{NLO}}(t_n,t) = \Delta_n(t_n,t)\Delta_n^{\mathrm{RV}}(t_n,t)\Delta_n^{\mathrm{RR}}(t_n,t)$.

The exact form of C_{ijk} and $C_{ij}^{(1)}$ depends on the shower algorithm.

Two approaches to implement second-order corrections in the shower evolution:

Multiplicative (e.g. VINCIA)

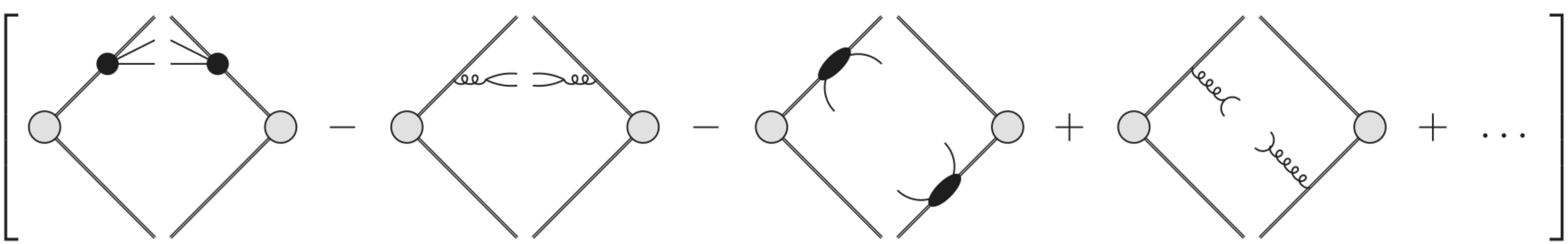
[Hartgring, Laenen, Skands 1303.4974],
[Li, Skands 1611.00013],
[Campbell, Höche, Li, CTP, Skands 2108.07133],
[El-Menoufi, CTP, Skands, Scyboz 2412.14242]



- sectorisation of phase space into ordered (LO) and direct (double-real) branchings
- real-virtual correction as MEC on LO branchings

[Höche, Prestel 1705.00742],
[Höche, Krauss, Prestel 1705.00982],
[Dulat, Höche, Prestel 1805.03757],
[Gellersen, Höche, Prestel 2110.05964]

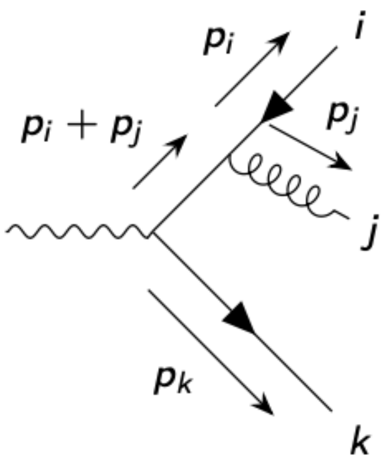
Additive (e.g. DIRE)



- explicit subtraction of overlaps between limits
- real-virtual correction added to LO branching

Decomposition of splitting functions

Factorisation in unresolved limits



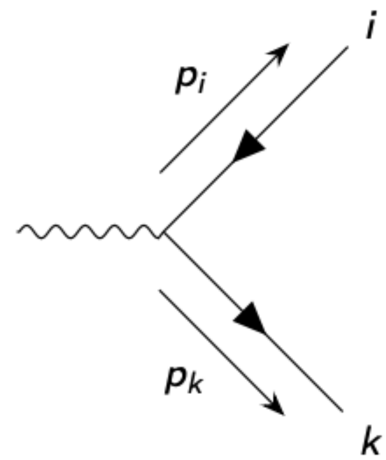
ij-coll

$$g_s^2 \frac{1}{s_{ij}} \left(\frac{2z_i}{1-z_i} + (1-z_i) \right)$$

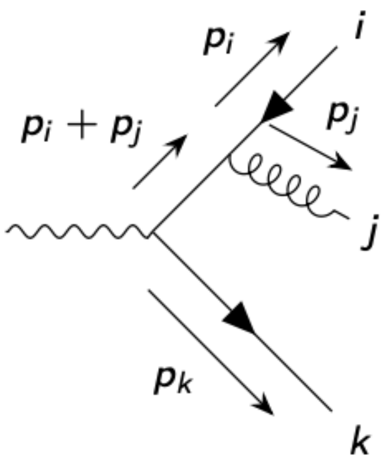
j-soft

$$g_s^2 \frac{2s_{ik}}{s_{ij}s_{jk}} \xrightarrow{i\|j} g_s^2 \frac{1}{s_{ij}} \frac{2z_i}{1-z_i}$$

$\times$



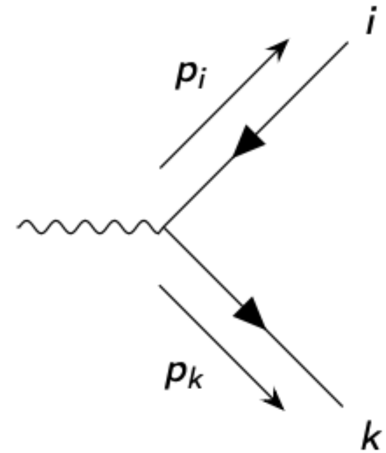
Factorisation in unresolved limits

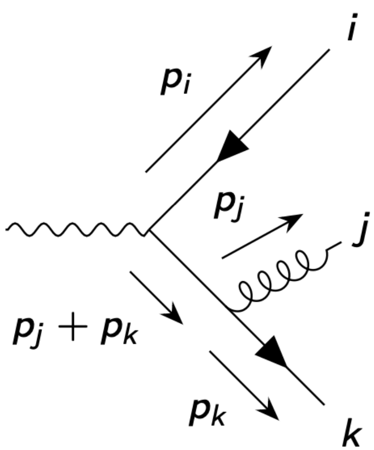


ij-coll: $$g_s^2 \frac{1}{s_{ij}} \left(\frac{2z_i}{1-z_i} + (1-z_i) \right)$$

j-soft: $$g_s^2 \frac{2s_{ik}}{s_{ij}s_{jk}} \xrightarrow{i\|j} g_s^2 \frac{1}{s_{ij}} \frac{2z_i}{1-z_i}$$

$\times$

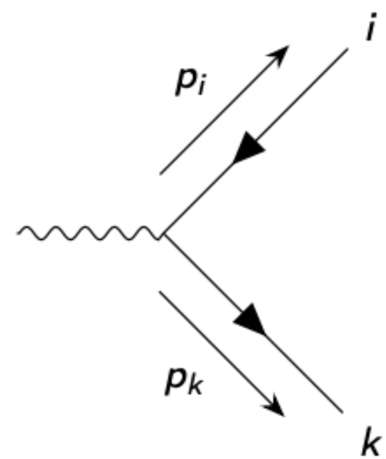




j-soft: $$g_s^2 \frac{2s_{ik}}{s_{ij}s_{jk}} \xrightarrow{j\|k} g_s^2 \frac{1}{s_{ij}} \frac{2z_k}{1-z_k}$$

jk-coll: $$g_s^2 \frac{1}{s_{ij}} \left(\frac{2z_k}{1-z_k} + (1-z_k) \right)$$

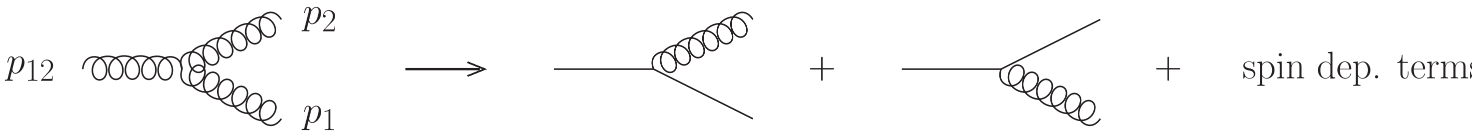
$\times$



Note: limits do not commute!

Idea: decompose vertices into scalar and spin-dependent terms

(a)



(b)

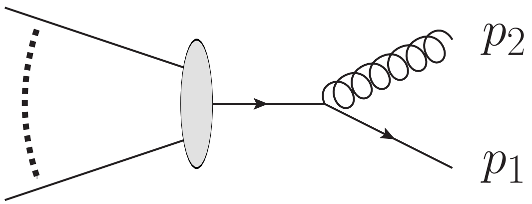
E.g. quark propagator + quark-gluon vertex:

$$\frac{\not{p}+\not{q}}{(p+q)^2} T^a_{ij} \gamma^\mu = T^a_{ij} \left[S^\mu(p,q) + \frac{\mathrm{i}\sigma^{\nu\mu}}{(p+q)^2} - \frac{\gamma^\mu \not{p}}{(p+q)^2} \right]$$

with **scalar current**:

$$S^\mu(p,q) = \frac{(2p+q)^\mu}{(p+q)^2}$$

Simple example: LO $q \to q$ splitting function (massless)

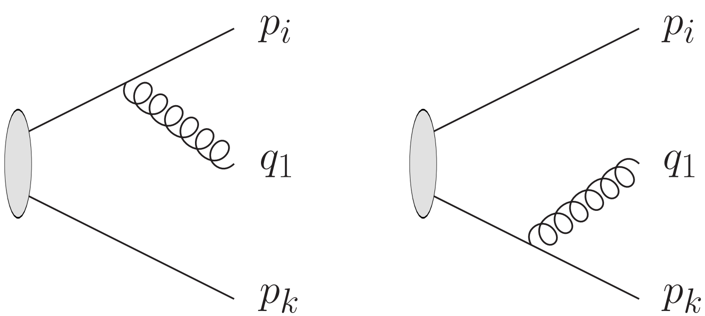


$$P_{q\to q}(p_1, p_2) = P_{\tilde{q}\to\tilde{q}}(p_1, p_2) + P^{(\mathrm{f})}_{q\to q}(p_1, p_2)$$

with **scalar** and **purely fermionic** splitting functions

$$P_{\tilde{q}\to\tilde{q}}(p_1, p_2) = C_F \frac{2z_1}{z_2} \qquad P^{(\mathrm{f})}_{q\to q}(p_1, p_2) = C_F(1-\varepsilon)z_2$$

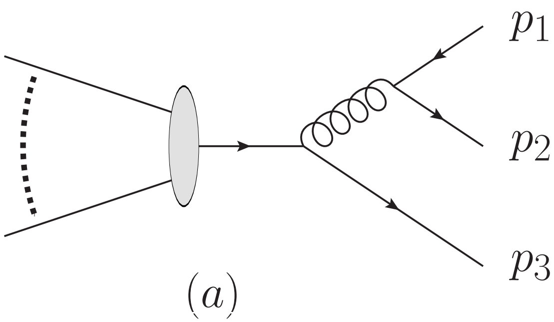
Simple combination with **scalar multipole**



$$\mathcal{S}_g(\{p\}, q_1) = -\sum_{i,k} \boldsymbol{T}_i \boldsymbol{T}_k \frac{p_i p_k}{(p_i q_1)(p_k q_1)}$$

⇒ same result as simplistic overlap removal

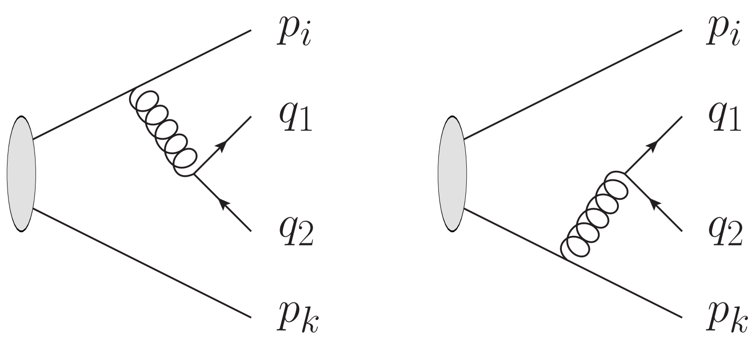
The same idea can be applied to $1 \to 3$ splitting functions
[Catani, Grazzini NPB 570 (2000) 287]



E.g. $q \to qq'\bar{q}'$ splitting function:

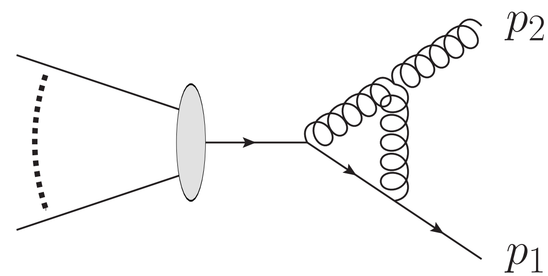
$$\langle P_{q\to\bar{q}'q'q}(p_1,p_2,p_3)\rangle = S_{q\bar{q}}(\{p\},p_1,p_2) + \frac{s_{123}}{s_{12}}\langle P^{(\mathrm{f}),\mu\nu}_{q\to q}(p_3,p_{12})\rangle P_{g\to q,\mu\nu}(p_1,p_2)$$

with **scalar multipole** contribution



$$S_{q\bar{q}}(\{p\},q_1,q_2) = 4T_R \sum_{i,k} \boldsymbol{T}_i \boldsymbol{T}_k \frac{s_{i1}s_{k2} + s_{i2}s_{k1} - s_{ik}s_{12}}{s_{12}^2 s_{i12} s_{k12}}$$

At one loop, the same exercise yields e.g.

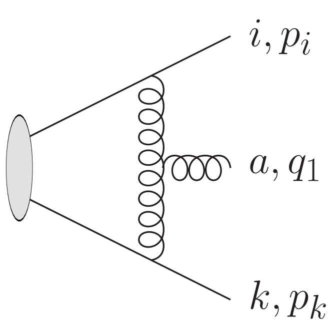


$$\langle P^{(1)}_{q\to q}(p_1,p_2)\rangle = P^{(1)}_{\tilde q\to\tilde q}(p_1,p_2) + \langle P^{(1,p)}_{q\to q}(p_1,p_2)\rangle$$

with pure **spin-dependent** remainder

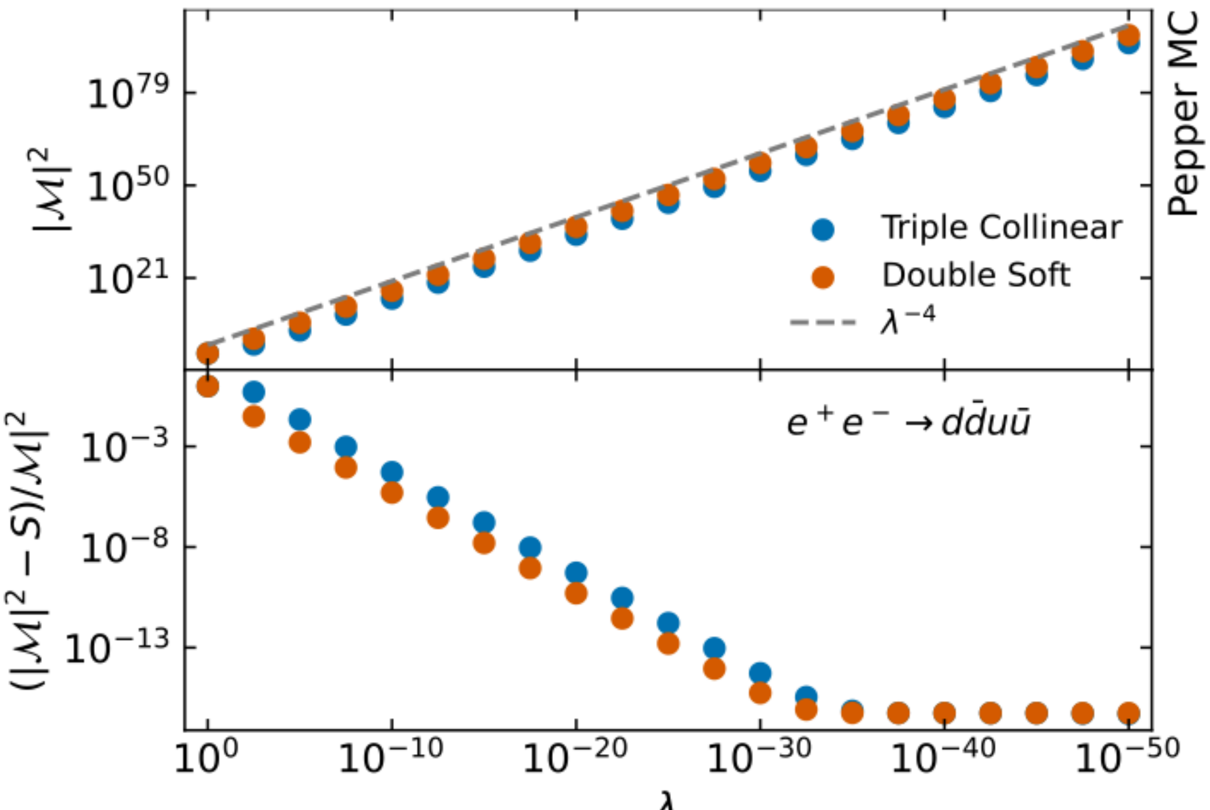
$$\langle P^{(1,p)}_{q\to q}(p_1,p_2)\rangle = C_A\left(-\frac{\mu^2}{s_{12}}\right)^{-\varepsilon}\left[(1-z)f_1(1-z) - \frac{1}{N_C^2}(zf_1(z)-2f_2)\right]\langle P^{(f)}_{q\to q}(p_1,p_2)\rangle$$
$$- C_A\left(1+\frac{1}{N_C^2}\right)\left(-\frac{\mu^2}{s_{12}}\right)^{-\varepsilon}\frac{\varepsilon^2}{1-2\varepsilon}f_2\langle P^{(f,1)}_{q\to q}(p_1,p_2)\rangle$$

Collinear limit of scalar one-loop multipoles yields **scalar contribution**



$$\lim_{t\to 0} I_{4,f}(zQ^2, t, (1-z)Q^2) = \frac{1}{t}P^{(1)}_{\tilde q\to\tilde q}(p_i, q_1)$$

First test of subtraction terms in simple process: $e^+e^- \to q\bar{q}$



Plot by M. Knobbe

Conclusions

Conclusions

At NLO, parton showers (typically) reproduce the fixed-order singularity structure.

At NNLO, this is not (yet) the case.

For simple processes (like $e^+e^- \to 2\mathrm{j}$, DY, ggH, ...) process-dependent NLO (matrix element) corrections can be incorporated into the shower evolution.

When Born-local NNLO K-factors are available, a simple multiplicative NNLO matching scheme can be constructed.

To capture the NNLO singularity structure in showers in general, detailed knowledge of the (overlapping) limits is required.