Factorial growth and power corrections

INSPIRE

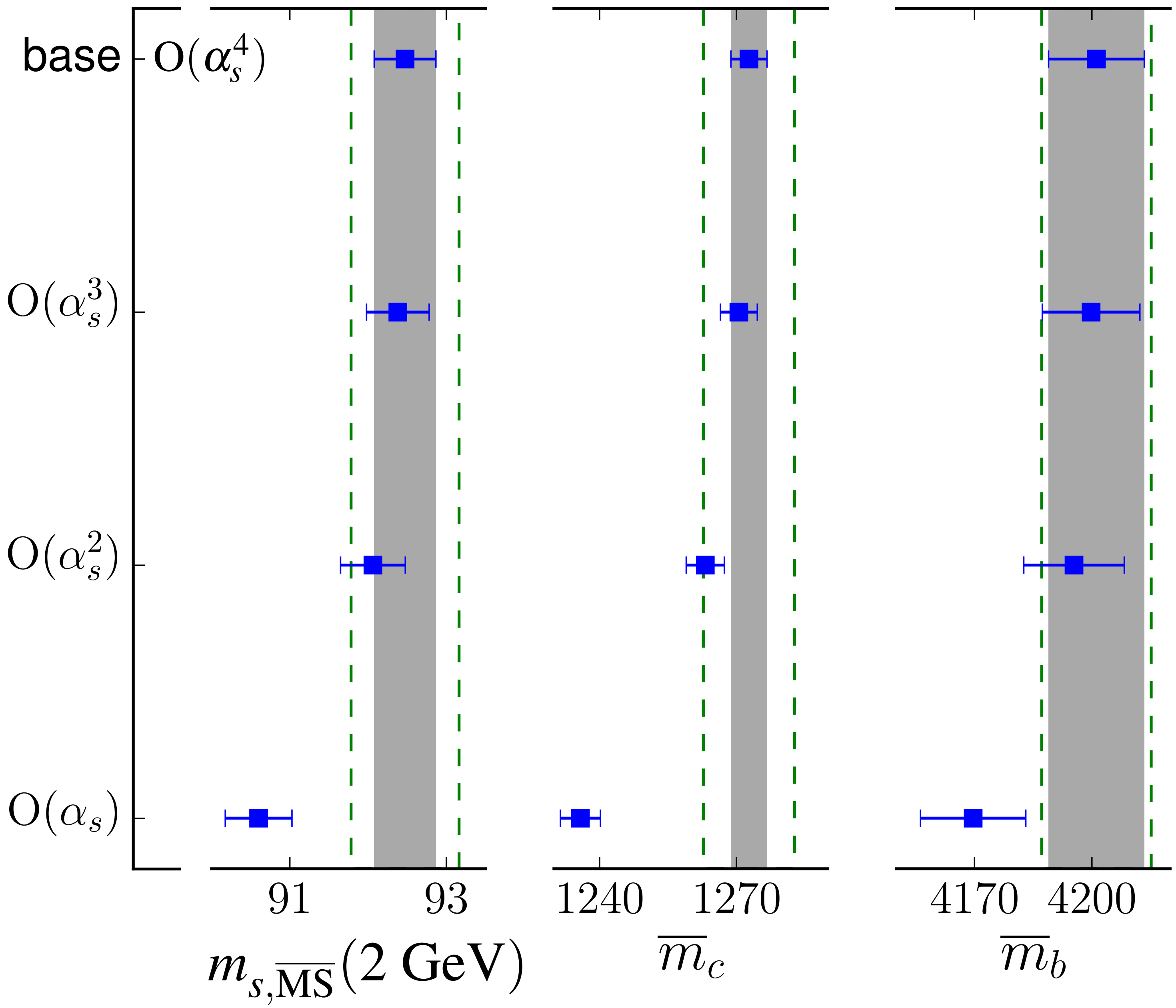
Andreas S. Kronfeld
IAS TU München

TUM/MPP Collider Phenomenology Seminar
28 October 2025

No Perturbative Truncation Uncertainty!?!

- Quark masses in $\overline{\text{MS}}$ scheme with small uncertainties:
    - total < 1% for bottom, charm, strange;
    - and 1–2% for up and down. [arXiv:1802.04248].
- Negligible uncertainty for truncating perturbation theory:
    - order α_s^4 “matching”, but still 🤨;
    - could whatever wizardry was used be generalized?

Perturbative Stability



excerpt of Fig. 4

$\overline{m} = m_{\overline{\rm MS}}(\overline{m})$

Fermilab Lattice, MILC, & TUMQCD [arXiv:1802.04248]

Relation Between Meson Mass and Quark Mass

meson mass

"brown muck"

$$M_B = m_b + \bar{\Lambda} + \mathrm{O}(1/m_b)$$

heavy quark "pole" mass

$\bar{m}_b = m_{b,\overline{\mathrm{MS}}}(\bar{m}_b)$

$$m_b = \bar{m}_b \left(1 + \sum_{l=0} r_l \alpha_s^{l+1}(\bar{m}) \right)$$

$$r_l = \{0.42, 1.03, 3.69, 17.4\}$$

Relation Between Meson Mass and Quark Mass

meson mass

"brown muck"

$$M_B = m_b + \bar{\Lambda} + \mathrm{O}(1/m_b)$$

heavy quark "pole" mass

$\bar{m}_b = m_{b,\overline{\mathrm{MS}}}(\bar{m}_b)$

$$m_b = \bar{m}_b \left(1 + \sum_{l=0} r_l \alpha_s^{l+1}(\bar{m}) \right)$$

$$r_l = \{0.42, 1.03, 3.69, 17.4\}$$

Yikes!

Relation Between Meson Mass and Quark Mass

meson mass

"brown muck"

$$M_B = m_b + \bar{\Lambda} + \mathrm{O}(1/m_b)$$

heavy quark "pole" mass

$\bar{m}_b = m_{b,\overline{\mathrm{MS}}}(\bar{m}_b)$

$$m_b = \bar{m}_b \left(1 + \sum_{l=0} r_l \alpha_s^{l+1}(\bar{m}) \right) \longrightarrow m_{b,\mathrm{MRS}}$$

$$r_l = \{0.42, 1.03, 3.69, 17.4\}$$

Yikes!

Factorial Growth

- Even in quantum mechanics, high orders of perturbation theory grow factorially [*e.g.*, Bender & Wu 1971, 1973].
- Also in QFT [*e.g.*, Gross & Neveu 1974, Lautrup 1977].
- Quark-mass r_l grow factorially (known for a long time):

$$r_l \sim R_0 (2\beta_0)^l \frac{\Gamma(l+1+b)}{\Gamma(1+b)} \equiv R_l$$

for $l \gg 1$. Here $b = \beta_1 / 2\beta_0^2 \overset{n_f=3}{=} 32/81 \approx 0.4$.

- Does $r_l = \{0.42, 1.03, 3.69, 17.4\}$ start growing by $l = 3$?

Factorial Growth

- Even in quantum mechanics, high orders of perturbation theory grow factorially [*e.g.*, Bender & Wu 1971, 1973].

- Also in QFT [*e.g.*, Gross & Neveu 1974, Lautrup 1977].

- Quark-mass r_l grow factorially (known for a long time):

$$r_l \sim R_0 (2\beta_0)^l \frac{\Gamma(l+1+b)}{\Gamma(1+b)} \equiv R_l$$

for $l \gg 1$. Here $b = \beta_1/2\beta_0^2 \overset{n_f=3}{=} 32/81 \approx 0.4$.

- Does $r_l = \{0.42, 1.03, 3.69, 17.4\}$ start growing by $l = 3$?

Normalization Factor R_0

- R_0 less well understood:
    - expressions with complicated derivations in the literature [T. Lee 1998, 1999, Pineda 2001; Hoang, Jain, Scimemi, Stewart 2008; Komijani 2017].
- Komijani [arXiv:1701.00347]:

$$R_0 = \sum_{k=0}^{\infty} (k+1) \frac{\Gamma(1+b)}{\Gamma(k+2+b)} (2\beta_0)^{-k} f_k$$

f_k obtained from r_j, β_j, $j \leq k$; see below

Minimal Renormalon Subtraction

- Brambilla, Komijani, & ASK, Vairo [arXiv:1712.04983] proposed adding and subtracting the R_l series:

$$m_b = \bar{m} + \bar{m}\sum_{l=0}^{\infty}[r_l - R_l]\alpha_s^{l+1}(\bar{m}) + \bar{m}\sum_{l=0}^{\infty} R_l\alpha_s^{l+1}(\bar{m})$$

- First sum is truncated at some order: $\infty \to L-1$.
- Second sum can be carried out via Borel procedure.
- Dubbed "minimal renormalon subtraction" (MRS).
- Are medium orders approximated well?

Questions

- When does factorial growth set in?
- Simpler derivation of normalization possible?
- Generalizations:
  - Λ^p instead of single power $\overline{\Lambda}$;
  - subtract subleading factorial growth, i.e., series with more than one power correction;
  - scale dependence $\alpha_s(Q) \to \alpha_s(sQ)$.

Main Outcomes

- Generalizations to arbitrary power corrections Λ^p / Q^p and different scale choices $\alpha_s(sQ)$ straightforward.
- Simple argument—
  - reproduces Komijani's normalization (in practice);
  - demonstrates factorial behavior already at low order;
  - shows how to treat more than one power correction—
  - relyies only on renormalization group.

Outline

- Introduction
- Power Corrections and Factorial Growth
- New Approximation for Perturbative Series
- Borel Summation
- Worked Cases: Static Energy, Pole Mass, Bj Sum Rule
- Two or More Power Corrections
- Conclusions & Outlook

Power Corrections and Factorial Growth

Notation & Setup

- Consider (dimensionless)

$$\mathscr{R}(Q) = r_{-1} + R(Q) + C_p \frac{\Lambda^p}{Q^p} \qquad R(Q) = \sum_{l=0} r_l(\mu/Q)\,\alpha_s(\mu)^{l+1}$$

$\overline{\mathrm{MS}}$ perturbative series

- RGE: coefficients' μ dependence must cancel that of α_s;
  - $\therefore$ RGE constrains Q dependence of $R(Q)$.

Notation & Setup

- Consider (dimensionless)

physical quantity

$$\mathscr{R}(Q) = r_{-1} + R(Q) + C_p \frac{\Lambda^p}{Q^p} \qquad R(Q) = \sum_{l=0} r_l(\mu/Q)\alpha_s(\mu)^{l+1}$$

$\overline{\text{MS}}$ perturbative series

- RGE: coefficients' μ dependence must cancel that of α_s;
  - $\therefore$ RGE constrains Q dependence of $R(Q)$.

Notation & Setup

- Consider (dimensionless)

physical quantity

$$\mathscr{R}(Q) = r_{-1} + R(Q) + C_p \frac{\Lambda^p}{Q^p} \qquad R(Q) = \sum_{l=0} r_l(\mu/Q)\alpha_s(\mu)^{l+1}$$

not QCD

$\overline{\text{MS}}$ perturbative series

- RGE: coefficients' μ dependence must cancel that of α_s;
  - $\therefore$ RGE constrains Q dependence of $R(Q)$.

Notation & Setup

- Consider (dimensionless)

physical quantity

$$\mathscr{R}(Q) = r_{-1} + R(Q) + C_p \frac{\Lambda^p}{Q^p} \qquad R(Q) = \sum_{l=0} r_l(\mu/Q)\alpha_s(\mu)^{l+1}$$

not QCD

"perturbative" part

$\overline{\text{MS}}$ perturbative series

- RGE: coefficients' μ dependence must cancel that of α_s;
  - $\therefore$ RGE constrains Q dependence of $R(Q)$.

Notation & Setup

- Consider (dimensionless)

physical quantity

power correction

$$\mathscr{R}(Q) = r_{-1} + R(Q) + C_p \frac{\Lambda^p}{Q^p} \qquad R(Q) = \sum_{l=0} r_l(\mu/Q)\alpha_s(\mu)^{l+1}$$

not QCD

"perturbative" part

$\overline{\text{MS}}$ perturbative series

- RGE: coefficients' μ dependence must cancel that of α_s;
  - $\therefore$ RGE constrains Q dependence of $R(Q)$.

Notation & Setup

- Consider (dimensionless)

power $p \leftrightarrow$ factorial growth

physical quantity

power correction

$$\mathscr{R}(Q) = r_{-1} + R(Q) + C_p \frac{\Lambda^p}{Q^p} \qquad R(Q) = \sum_{l=0} r_l(\mu/Q)\alpha_s(\mu)^{l+1}$$

not QCD

"perturbative" part

$\overline{\text{MS}}$ perturbative series

- RGE: coefficients' μ dependence must cancel that of α_s;
  - $\therefore$ RGE constrains Q dependence of $R(Q)$.

Power-Term Removal

- Start with $\mathscr{R}(Q) = r_{-1} + R(Q) + C_p \dfrac{\Lambda^p}{Q^p}$.
- To eliminate Λ^p / Q^p, multiply by Q^p and differentiate:

$$r_{-1} + F(Q) \equiv \frac{1}{pQ^{p-1}} \frac{\mathrm{d}\, Q^p \mathscr{R}}{\mathrm{d}Q} \equiv \hat{Q}^{(p)} \mathscr{R}$$

- As a series $F(Q) = \sum_{k=0} f_k \alpha_s^{k+1}(Q) \Rightarrow f(\alpha) = \sum_{k=0} f_k \alpha^{k+1}$.

$$f_k = r_k - \frac{2}{p} \sum_{j=0}^{k-1} (j+1) \beta_{k-1-j} r_j$$

- Differential equation $r(\alpha) + \frac{2}{p} \beta(\alpha) r'(\alpha) = f(\alpha)$.

Differential Equation

- Differential equation $r(\alpha) + \frac{2}{p}\beta(\alpha)r'(\alpha) = f(\alpha)$.
- Take $f(\alpha)$ as given and solve for $r(\alpha)$:
  - Komijani's solution reproduces R_l's growth, yields R_0.
- Here, use only the elementary feature—
  - general solution is any particular solution plus a solution of the homogeneous equation (0 on RHS);
  - solution to homogeneous equation is $\propto \Lambda^p$.

My Solution

- The relation between the coefficients is a matrix equation

$$f_k^{(p)} = r_k - \frac{2}{p}\sum_{j=0}^{k-1}(j+1)\beta_{k-1-j}r_j$$

$$\boldsymbol{f}^{(p)} = \left[\mathbf{1} - \frac{2}{p}\mathbf{D}\right] \cdot \boldsymbol{r} \equiv \mathbf{Q}^{(p)} \cdot \boldsymbol{r}$$

and $\mathbf{D}$ is on the lower triangle.

- Matrix is infinite, but the lower triangular form makes a row-by-row solution straightforward.

- Scheme for α_s is chosen to simplify algebra ("geometric"):

$$\beta(\alpha_g) = -\frac{\beta_0 \alpha_g^2}{1-(\beta_1/\beta_0)\alpha_g}$$

- Notation to make the expressions compact: $\tau \equiv 2\beta_0/p$.

$$\mathbf{Q}_g^{(p)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -\tau & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -\tau^2 pb & -2\tau & 1 & 0 & 0 & 0 & 0 & \cdots \\ -\tau(\tau pb)^2 & -2\tau^2 pb & -3\tau & 1 & 0 & 0 & 0 & \cdots \\ -\tau(\tau pb)^3 & -2\tau(\tau pb)^2 & -3\tau^2 pb & -4\tau & 1 & 0 & 0 & \cdots \\ -\tau(\tau pb)^4 & -2\tau(\tau pb)^3 & -3\tau(\tau pb)^2 & -4\tau^2 pb & -5\tau & 1 & 0 & \cdots \\ -\tau(\tau pb)^5 & -2\tau(\tau pb)^4 & -3\tau(\tau pb)^3 & -4\tau(\tau pb)^2 & -5\tau^2 pb & -6\tau & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

- As before $b = \beta_1/2\beta_0^2 \overset{n_f=3}{=} 32/81 \approx 0.4$.

- Inverse reveals that factorial growth begins at low orders:

$$\mathbf{Q}_{\mathrm{g}}^{(p)\,-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \tau & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \tau^2\frac{\Gamma(3+pb)}{\Gamma(2+pb)} & 2\tau & 1 & 0 & 0 & 0 & 0 & \cdots \\ \tau^3\frac{\Gamma(4+pb)}{\Gamma(2+pb)} & 2\tau^2\frac{\Gamma(4+pb)}{\Gamma(3+pb)} & 3\tau & 1 & 0 & 0 & 0 & \cdots \\ \tau^4\frac{\Gamma(5+pb)}{\Gamma(2+pb)} & 2\tau^3\frac{\Gamma(5+pb)}{\Gamma(3+pb)} & 3\tau^2\frac{\Gamma(5+pb)}{\Gamma(4+pb)} & 4\tau & 1 & 0 & 0 & \cdots \\ \tau^5\frac{\Gamma(6+pb)}{\Gamma(2+pb)} & 2\tau^4\frac{\Gamma(6+pb)}{\Gamma(3+pb)} & 3\tau^3\frac{\Gamma(6+pb)}{\Gamma(4+pb)} & 4\tau^2\frac{\Gamma(6+pb)}{\Gamma(5+pb)} & 5\tau & 1 & 0 & \cdots \\ \tau^6\frac{\Gamma(7+pb)}{\Gamma(2+pb)} & 2\tau^5\frac{\Gamma(7+pb)}{\Gamma(3+pb)} & 3\tau^4\frac{\Gamma(7+pb)}{\Gamma(4+pb)} & 4\tau^3\frac{\Gamma(7+pb)}{\Gamma(5+pb)} & 5\tau^2\frac{\Gamma(7+pb)}{\Gamma(6+pb)} & 6\tau & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

$$\boldsymbol{r} = \mathbf{Q}_{\mathrm{g}}^{(p)\,-1} \cdot \boldsymbol{f}^{(p)}$$

- Inverse reveals that factorial growth begins at low orders:

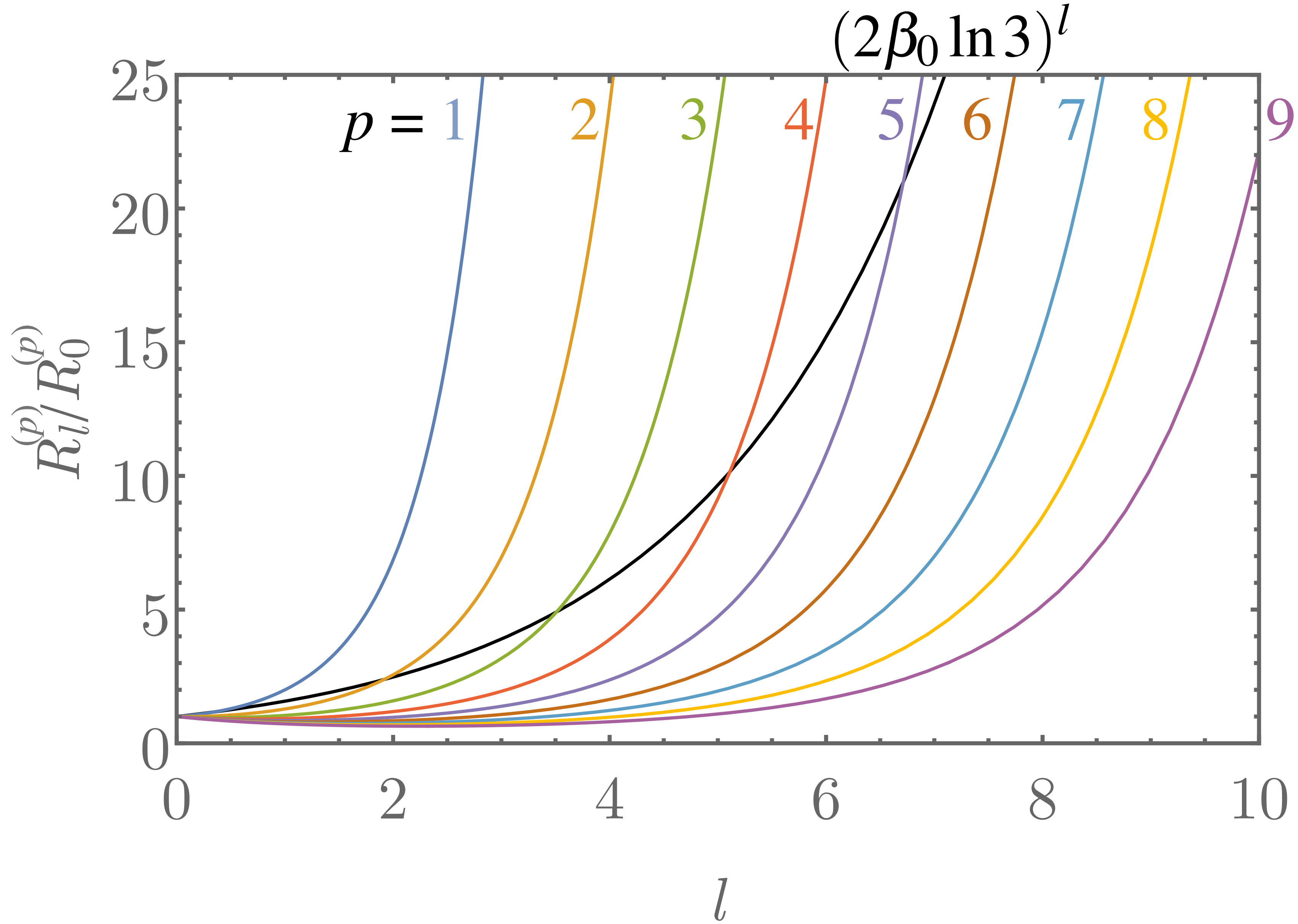
$$\mathbf{Q}_{\mathrm{g}}^{(p)\,-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \tau & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \tau^2\frac{\Gamma(3+pb)}{\Gamma(2+pb)} & 2\tau & 1 & 0 & 0 & 0 & 0 & \cdots \\ \tau^3\frac{\Gamma(4+pb)}{\Gamma(2+pb)} & 2\tau^2\frac{\Gamma(4+pb)}{\Gamma(3+pb)} & 3\tau & 1 & 0 & 0 & 0 & \cdots \\ \tau^4\frac{\Gamma(5+pb)}{\Gamma(2+pb)} & 2\tau^3\frac{\Gamma(5+pb)}{\Gamma(3+pb)} & 3\tau^2\frac{\Gamma(5+pb)}{\Gamma(4+pb)} & 4\tau & 1 & 0 & 0 & \cdots \\ \tau^5\frac{\Gamma(6+pb)}{\Gamma(2+pb)} & 2\tau^4\frac{\Gamma(6+pb)}{\Gamma(3+pb)} & 3\tau^3\frac{\Gamma(6+pb)}{\Gamma(4+pb)} & 4\tau^2\frac{\Gamma(6+pb)}{\Gamma(5+pb)} & 5\tau & 1 & 0 & \cdots \\ \tau^6\frac{\Gamma(7+pb)}{\Gamma(2+pb)} & 2\tau^5\frac{\Gamma(7+pb)}{\Gamma(3+pb)} & 3\tau^4\frac{\Gamma(7+pb)}{\Gamma(4+pb)} & 4\tau^3\frac{\Gamma(7+pb)}{\Gamma(5+pb)} & 5\tau^2\frac{\Gamma(7+pb)}{\Gamma(6+pb)} & 6\tau & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

$$\boldsymbol{r} = \mathbf{Q}_{\mathrm{g}}^{(p)\,-1} \cdot \boldsymbol{f}^{(p)}$$

$$r_l = \underbrace{\boxed{\left(\frac{2\beta_0}{p}\right)^l \frac{\Gamma(l+1+pb)}{\Gamma(1+pb)}}}_{\text{well-known growth}} \underbrace{\boxed{\sum_{k=0}^{l-1}(k+1)\frac{\Gamma(1+pb)}{\Gamma(k+2+pb)}\left(\frac{p}{2\beta_0}\right)^k f_k^{(p)}}}_{\text{Komijani's } R_0 \text{ (truncated)}} \underbrace{\boxed{+ f_l^{(p)}}}_{\text{extra}}$$

Growth ↔ Power

- Larger $p \Rightarrow$ growth takes over at larger l.



New Approximation for Perturbative Series

Perturbative Series

- We must be back where we started, right? $\boldsymbol{r} = \mathbf{Q}^{-1} \cdot \boldsymbol{f} = \mathbf{Q}^{-1} \cdot \mathbf{Q} \cdot \boldsymbol{r}$
    - In practice, we know r_l and, hence, f_l for $l < L$.
      The formula returns these r_l (as it must).
- For $l \geq L$, the formula tells us (formally) the largest part:
    - truncate on f_l, not r_l; evaluate $\sum_{l=0}^{\infty} r_l \alpha_{\mathrm{s}}^{l+1}$ by—
        - taking exact r_l from the literature for $l < L$;
        - approximating $r_l \approx R_l$ for $l \geq L$.

Perturbative Series

$$r_l = \underbrace{\boxed{\left(\frac{2\beta_0}{p}\right)^l \frac{\Gamma(l+1+pb)}{\Gamma(1+pb)}}}_{\text{well-known growth}} \underbrace{\boxed{\sum_{k=0}^{l-1}(k+1)\frac{\Gamma(1+pb)}{\Gamma(k+2+pb)}\left(\frac{p}{2\beta_0}\right)^k f_k^{(p)}}}_{\text{Komijani } R_0 \text{ (truncated)}} \underbrace{\boxed{+ f_l^{(p)}}}_{\text{drop}}$$

- For $l \geq L$, the formula tells us (formally) the largest part:
  - truncate on f_l, not r_l; evaluate $\sum_{l=0}^{\infty} r_l \alpha_s^{l+1}$ by—
    - taking exact r_l from the literature for $l < L$;
    - approximating $r_l \approx R_l$ for $l \geq L$.

Perturbative Series

$$r_l \approx \underbrace{\boxed{\left(\frac{2\beta_0}{p}\right)^l \frac{\Gamma(l+1+pb)}{\Gamma(1+pb)}}}_{\text{well-known growth}} \underbrace{\boxed{\sum_{k=0}^{l-1}(k+1)\frac{\Gamma(1+pb)}{\Gamma(k+2+pb)}\left(\frac{p}{2\beta_0}\right)^k f_k^{(p)}}}_{\text{Komijani } R_0 \text{ (truncated)}}, \; l \geq L$$

- For $l \geq L$, the formula tells us (formally) the largest part:
  - truncate on f_l, not r_l; evaluate $\sum_{l=0}^{\infty} r_l \alpha_s^{l+1}$ by—
    - taking exact r_l from the literature for $l < L$;
    - approximating $r_l \approx R_l$ for $l \geq L$.

Perturbative Series

$$r_l \approx \underbrace{\left(\frac{2\beta_0}{p}\right)^l \frac{\Gamma(l+1+pb)}{\Gamma(1+pb)}}_{\text{well-known growth}} \underbrace{\sum_{k=0}^{L-1} (k+1) \frac{\Gamma(1+pb)}{\Gamma(k+2+pb)} \left(\frac{p}{2\beta_0}\right)^k f_k^{(p)}}_{\text{Komijani } R_0 \text{ (truncated)}}, \; l \geq L$$

- For $l \geq L$, the formula tells us (formally) the largest part:
  - truncate on f_l, not r_l; evaluate $\sum_{l=0}^{\infty} r_l \alpha_s^{l+1}$ by—
    - taking exact r_l from the literature for $l < L$;
    - approximating $r_l \approx R_l$ for $l \geq L$.

Recap & Compendium

- That means $\sum_{l=0}^{\infty} r_l \alpha_s^{l+1} \to \sum_{l=0}^{L-1} r_l \alpha_s^{l+1} + \sum_{l=L}^{\infty} R_l^{(p)} \alpha_s^{l+1}$

  with

$$R_l^{(p)} \equiv R_0^{(p)} \left(\frac{2\beta_0}{p}\right)^l \frac{\Gamma(l+1+pb)}{\Gamma(1+pb)}$$

$$R_0^{(p)} \equiv \sum_{k=0}^{L-1} (k+1) \frac{\Gamma(1+pb)}{\Gamma(k+2+pb)} \left(\frac{p}{2\beta_0}\right)^k f_k^{(p)}$$

- Systematic approximation because the retained terms are formally larger than the ones omitted.

Borel Summation

Rearrange and React

- We have

$$R(Q) = \sum_{l=0}^{\infty} r_l \alpha_s^{l+1} \to \sum_{l=0}^{L-1} r_l \alpha_s^{l+1} + \sum_{l=L}^{\infty} R_l^{(p)} \alpha_s^{l+1}$$

$$= \underbrace{\sum_{l=0}^{L-1} \left(r_l - R_l^{(p)} \right) \alpha_s^{l+1}}_{R_{\mathrm{RS}}^{(p)}(Q)} + \underbrace{\sum_{l=0}^{\infty} R_l^{(p)} \alpha_s^{l+1}}_{R_{\mathrm{B}}^{(p)}(Q)}$$

- The "renormalon subtracted" part and the "Borel" part.

- The R_l from above yield divergent sum for R_{B}, but we're not done yet: use Borel summation to assign meaning.

Borel Summation

- Using the integral representation of $\Gamma(l{+}1)$:

$$R_{\mathrm{B}}^{(p)}(Q) = R_0^{(p)} \sum_{l=0}^{\infty} \left[\frac{\Gamma(l+1+pb)}{\Gamma(1+pb)\Gamma(l+1)} \int_0^{\infty} \left(\frac{2\beta_0 t}{p} \right)^l \mathrm{e}^{-t/\alpha_{\mathrm{g}}(Q)} \mathrm{d}t \right]$$

$$\to R_0^{(p)} \int_0^{\infty} \frac{\mathrm{e}^{-t/\alpha_{\mathrm{g}}(Q)}}{(1-2\beta_0 t/p)^{1+pb}} \mathrm{d}t$$

Mathematica knows the sum

where 2nd line comes from (illegally) swapping Σ and $\int$.

- Branch point in integrand at $t = p/2\beta_0$, dubbed "renormalon singularity" ['t Hooft 1979].

Borel Summation

- Split integration in two [BKKV, arXiv:1712.04983]:

Mathematica knows
the integrals

$$R_{\mathrm{B}}^{(p)}(Q) = R_0^{(p)} \int_0^{p/2\beta_0} \frac{\mathrm{e}^{-t/\alpha_{\mathrm{g}}(Q)}}{(1-2\beta_0 t/p)^{1+pb}} \mathrm{d}t$$

$$+ R_0^{(p)} \int_{p/2\beta_0}^{\infty} \frac{\mathrm{e}^{-t/\alpha_{\mathrm{g}}(Q)}}{(1-2\beta_0 t/p)^{1+pb}} \mathrm{d}t$$

Borel Summation

- Split integration in two [BKKV, arXiv:1712.04983]:

Mathematica knows the integrals

$$R_{\mathrm{B}}^{(p)}(Q) = R_0^{(p)}\,\frac{p}{2\beta_0}\,\mathscr{J}\left(pb, 1/2\beta_0\alpha_{\mathrm{g}}(Q)\right)$$

$$\pm R_0^{(p)}\,\mathrm{e}^{\pm \mathrm{i}pb\pi}\,\frac{p^{1+pb}}{2^{1+pb}\beta_0}\,\Gamma(-pb)\left[\frac{\mathrm{e}^{-1/[2\beta_0\alpha_{\mathrm{g}}(Q)]}}{[\beta_0\alpha_{\mathrm{g}}(Q)]^b}\right]^p$$

Borel Summation

- Split integration in two [BKKV, arXiv:1712.04983]:

Mathematica knows the integrals

$$R_{\mathrm{B}}^{(p)}(Q) = R_0^{(p)} \frac{p}{2\beta_0} \mathscr{I}\left(pb, 1/2\beta_0 \alpha_{\mathrm{g}}(Q)\right)$$

$$\pm R_0^{(p)} \mathrm{e}^{\pm \mathrm{i} pb\pi} \frac{p^{1+pb}}{2^{1+pb}\beta_0} \Gamma(-pb) \left[\frac{\Lambda_{\overline{\mathrm{MS}}}}{Q} \right]^p$$

Borel Summation

- Split integration in two [BKKV, arXiv:1712.04983]:

Mathematica knows the integrals

$$R_{\mathrm{B}}^{(p)}(Q) = R_0^{(p)} \frac{p}{2\beta_0} \mathscr{J}\left(pb, 1/2\beta_0 \alpha_{\mathrm{g}}(Q)\right)$$

absorb into power correction $\dfrac{\Lambda^p_{\overline{\mathrm{MS}}}}{Q^p}$

Definition and Properties of $\mathscr{J}$

- Thus, we now define

$$R_{\mathrm{B}}^{(p)}(Q) = R_0^{(p)} \frac{p}{2\beta_0} \mathscr{J}(pb, 1/2\beta_0\alpha_{\mathrm{g}}(Q))$$

$$\mathscr{J}(c,y) = \mathrm{e}^{-y}\Gamma(-c)\gamma^\star(-c,-y)$$

where $\gamma^\star(a,x)$ is an analytic function of both a and x:

limiting function of the incomplete gamma function

- convergent expansion in $x = -1/2\beta_0\alpha_{\mathrm{g}}$;

- asymptotic expansion in α_{g} regenerates the starting point; the dropped term is $\mathrm{O}(\mathrm{e}^{-p/2\beta_0\alpha_{\mathrm{g}}})$.

Alternative Borel Summation

- Using the integral representation of $\Gamma(l+1+pb)$:

$$R_{\mathrm{B}}^{(p)}(Q) = \frac{\alpha_{\mathrm{g}}(Q) R_0^{(p)}}{\Gamma(1+pb)} \sum_{l=0}^{\infty} \left[\int_0^{\infty} \left(\frac{2\beta_0 \alpha_{\mathrm{g}}(Q) t}{p} \right)^l t^{pb} \mathrm{e}^{-t} \mathrm{d}t \right]$$

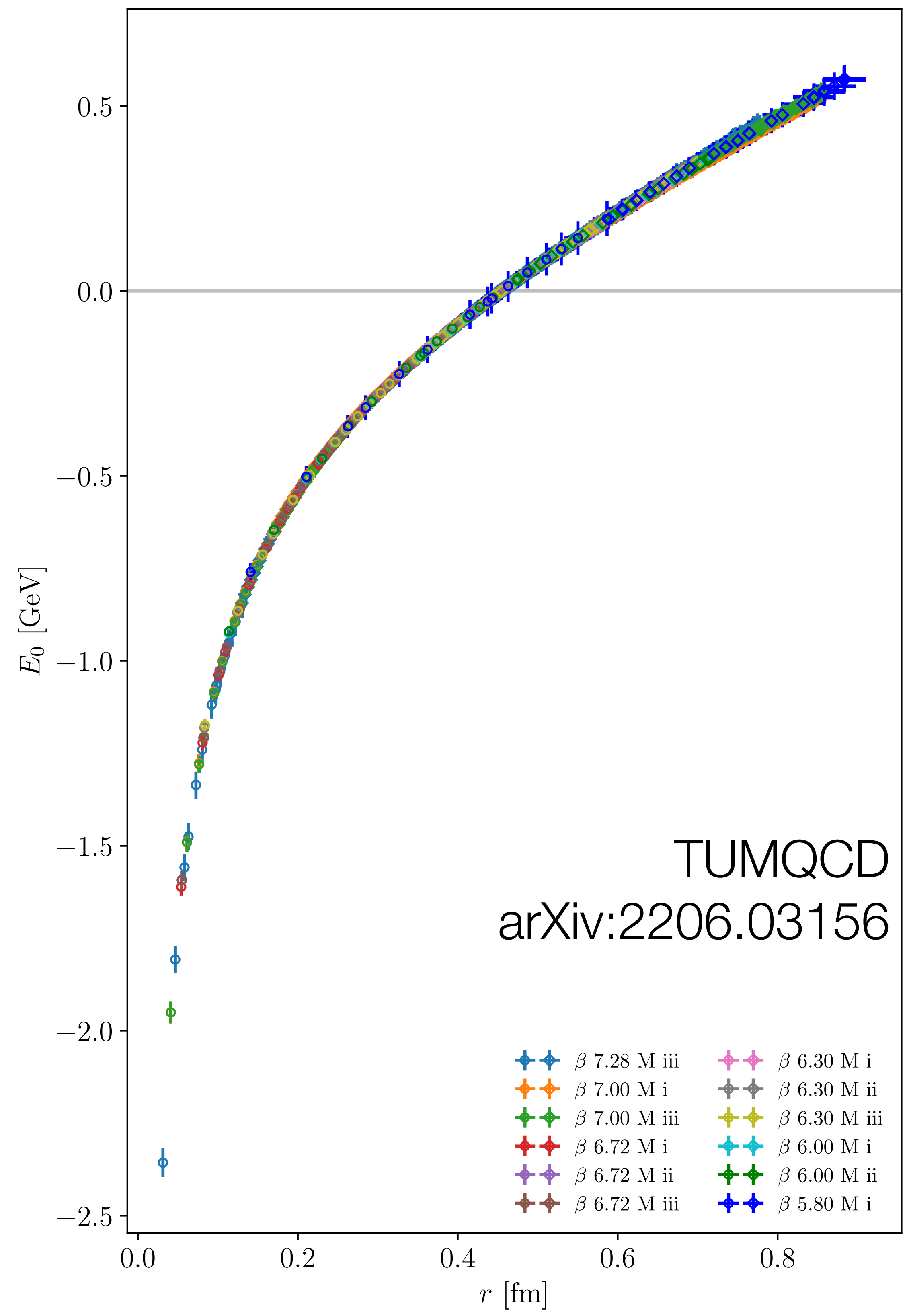
$$\to \frac{\alpha_{\mathrm{g}}(Q) R_0^{(p)}}{\Gamma(1+pb)} \int_0^{\infty} \frac{p t^{pb} \mathrm{e}^{-t}}{p - 2\beta_0 \alpha_{\mathrm{g}}(Q) t} \mathrm{d}t$$

you know the sum

where 2nd line comes from (illegally) swapping $\sum$ and $\int$.

- Principal part yields $\mathscr{J}$ function; ambiguity of encircling the pole above/below yields discardable power term.

Worked Examples



Static Energy

- Quantity extracted from oblong Wilson loops:
  - perturbative potential has IR divergences starting at 3 loops [Appelquist, Dine, Muzinich 1978];
  - compensated by multipole term [Brambilla, Pineda, Soto, Vairo 1999, 2000].
- Perturbative series:

$$E_0(r) = -\frac{C_F}{r} \sum_{l=0} v_l(\mu r) \alpha_s(\mu)^{l+1} + \Lambda_0$$

- In notation used above, $Q \to 1/r$, $\mathscr{R}(1/r) = -rE_0(r)/C_F$.

Related Quantities

- Perturbation theory carried out in momentum space:

$$\tilde{R}(q) = \sum_{l=0} a_l(\mu/q)\alpha_s(\mu)^{l+1}$$

- Leading power/factorial comes from Fourier transform, so $\tilde{R}(q)$ has $p > 1$.

- The "static force"

$$\mathfrak{F}(r) = -\frac{\mathrm{d}E_0}{\mathrm{d}r} \qquad \mathscr{F}(r) = F^{(1)}(1/r) = -r^2\mathfrak{F}(r)/C_F$$

has no power corrections (until instantons at $p \geq 9$).

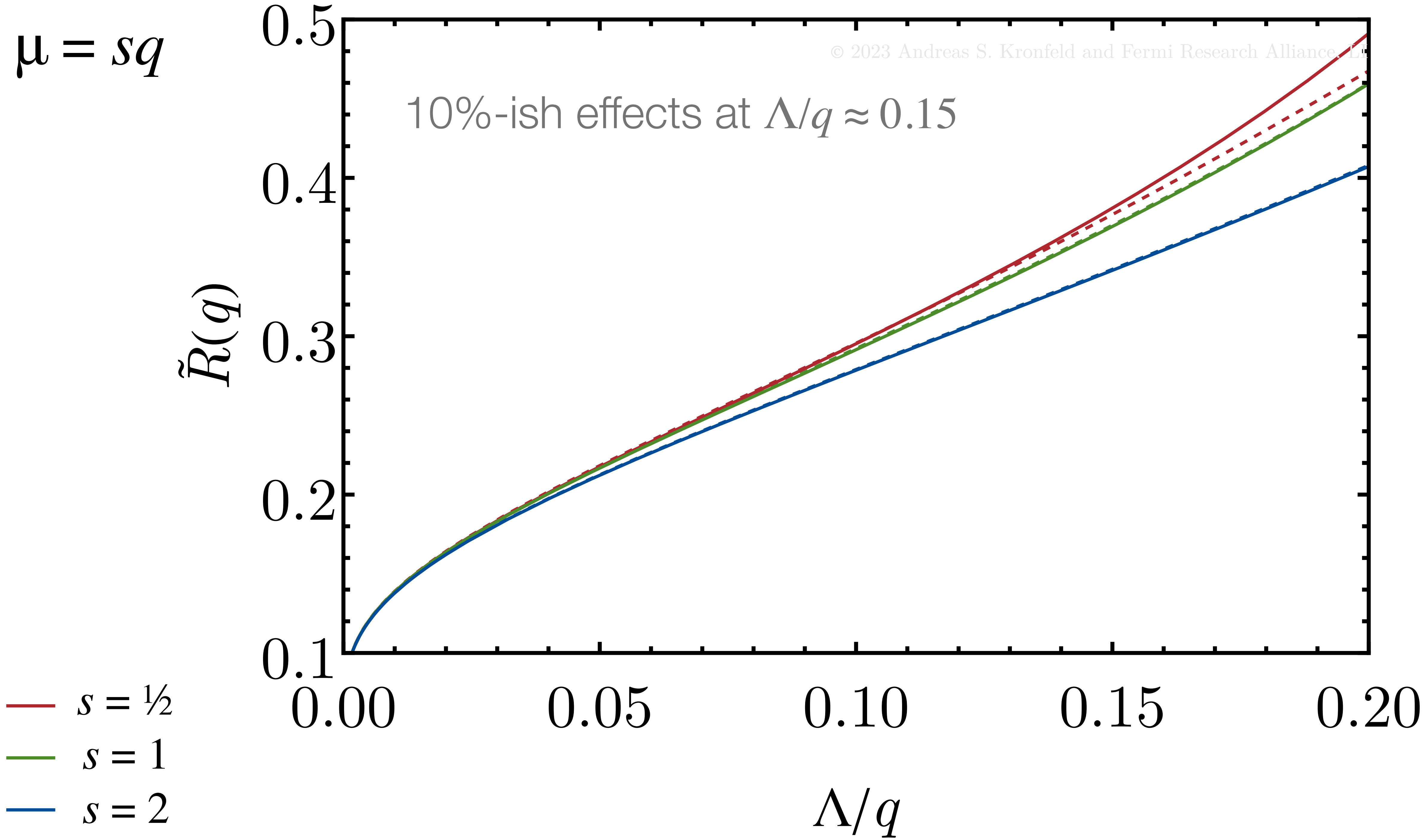
Coefficients at $\mu = 1/r$ or $\mu = q$

l	$\overline{\text{MS}}$ $a_l(1)$	$\overline{\text{MS}}$ $f_l(1)$	geometric $a_l(1)$	geometric $f_l(1)$	α_2 $a_l(1)$	α_2 $f_l(1)$
0	1	1	1	1	1	1
1	0.557042	−0.048552	0.557042	−0.048552	0.557042	−0.048552
2	1.70218	0.687291	1.83497	0.820079	1.83497	0.820079
3	2.43687	0.323257	2.83268	0.558242	3.01389	0.739452

l	$\overline{\text{MS}}$ $v_l(1)$	$\overline{\text{MS}}$ $v_l(1) - V_l(1)$	geometric $v_l(1)$	geometric $v_l(1) - V_l(1)$	α_2 $v_l(1)$	α_2 $v_l(1) - V_l(1)$
0	1	0.206061	1	0.182531	1	0.177584
1	1.38384	−0.202668	1.38384	−0.249689	1.38384	−0.259574
2	5.46228	0.019479	5.59507	−0.009046	5.59507	−0.042959
3	26.6880	0.219262	27.3034	0.050179	27.4846	0.066468

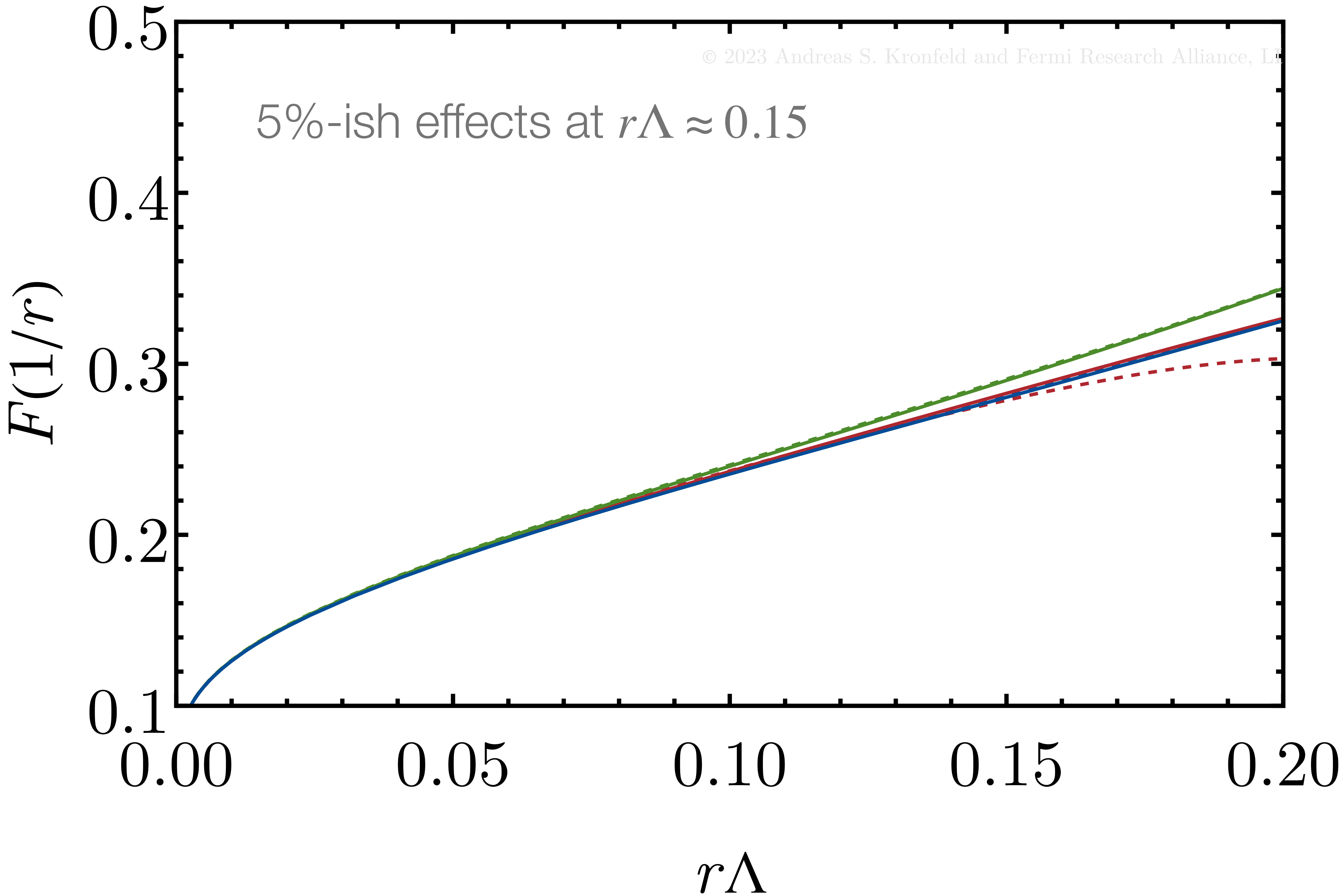
Good Series (at most $p > 1$ growth)

$\mu = sq$



Great Series (instanton power $p \geq 9$)

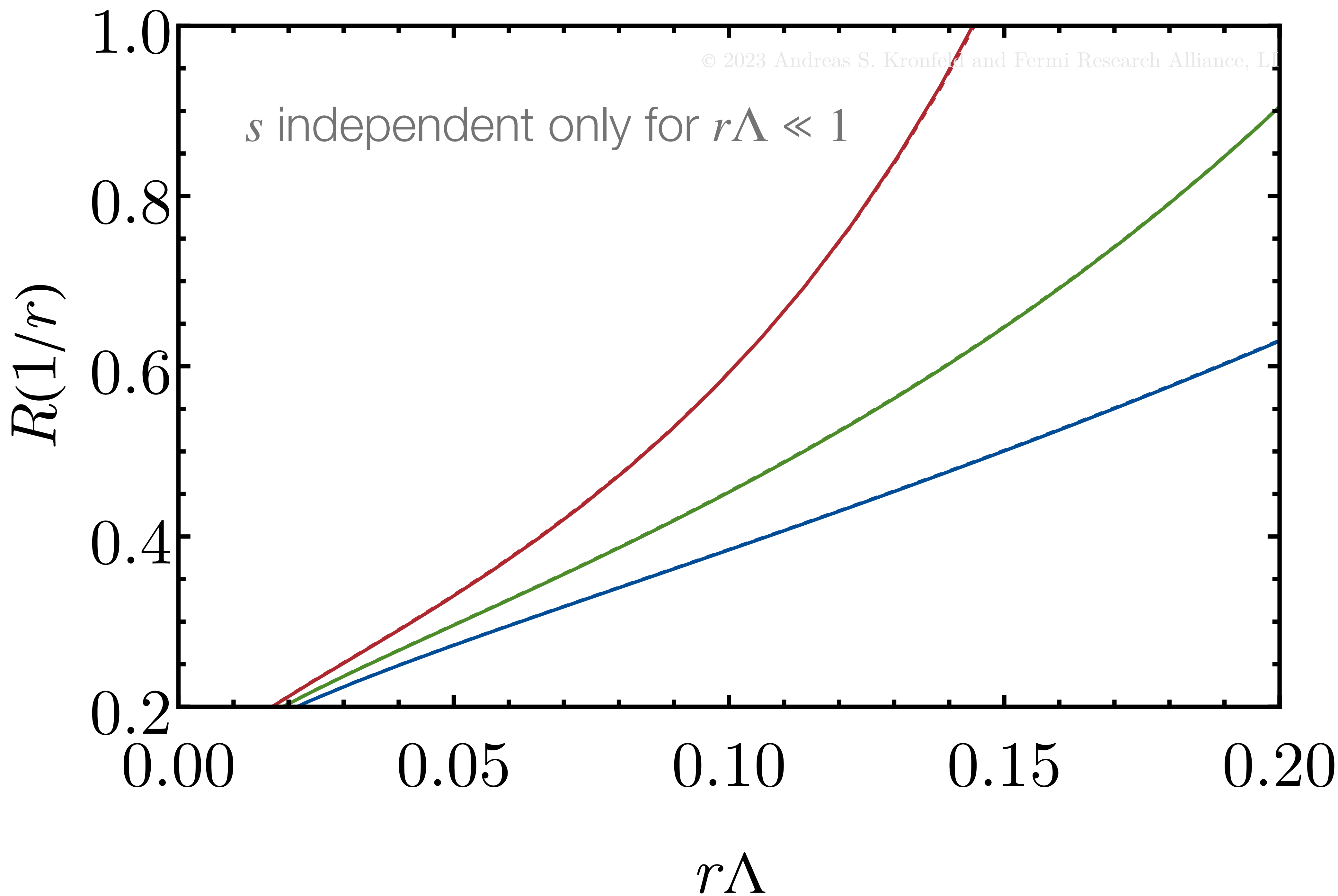
$\mu = s/r$



$s = ½$
$s = 1$
$s = 2$

Horrible Series ($p = 1$)

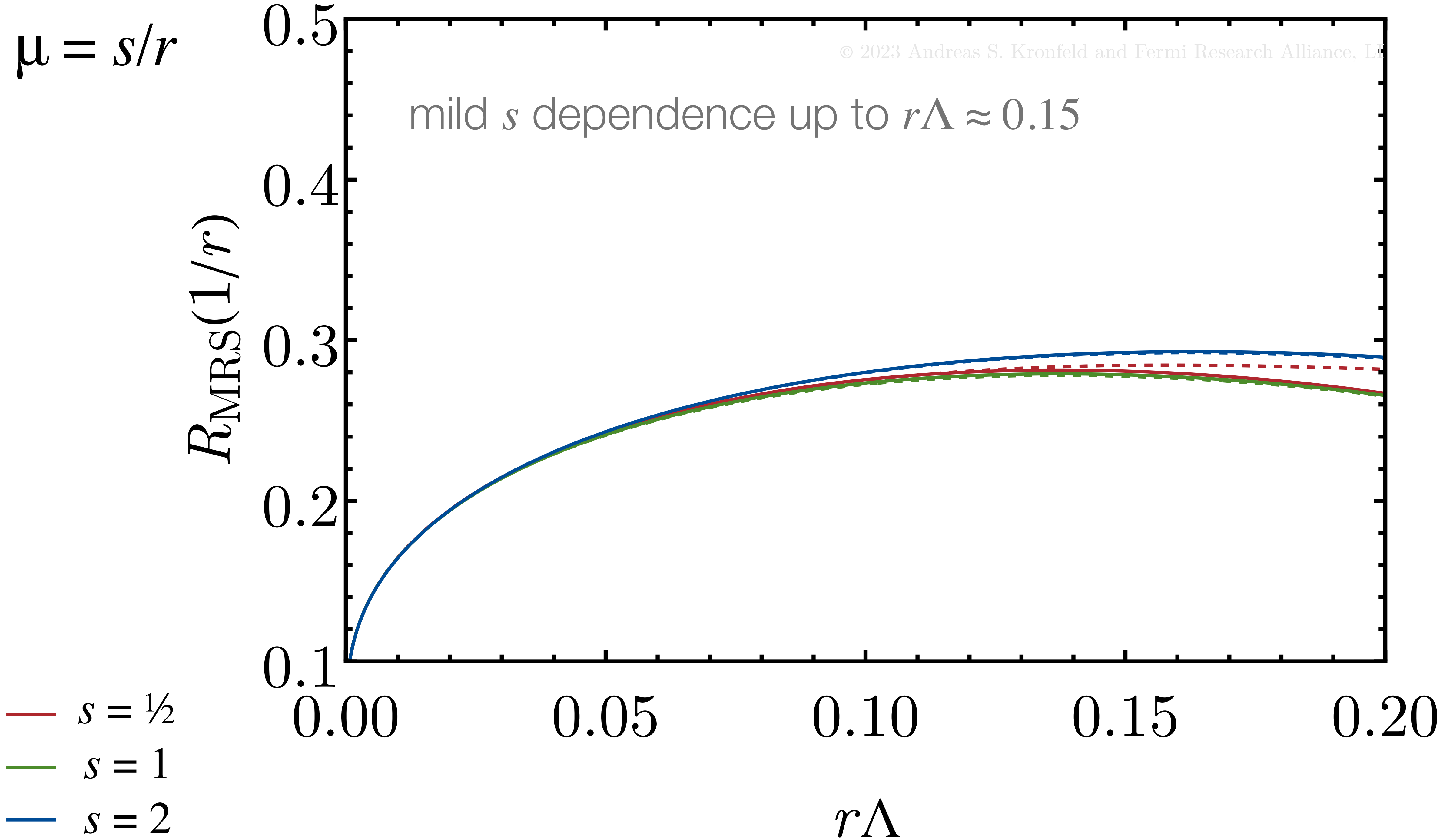
$\mu = s/r$



$s = ½$
$s = 1$
$s = 2$

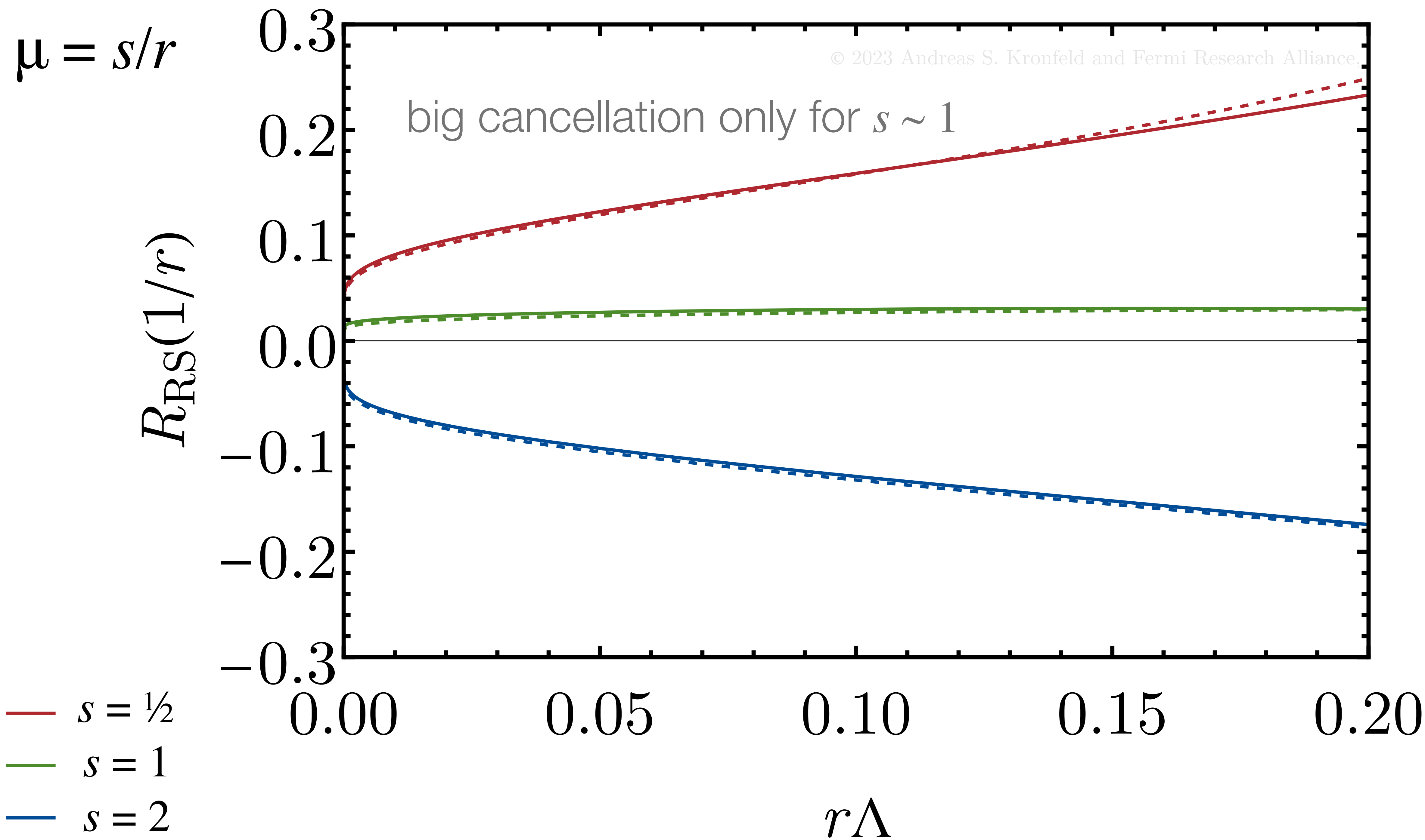
MRS Series

$\mu = s/r$



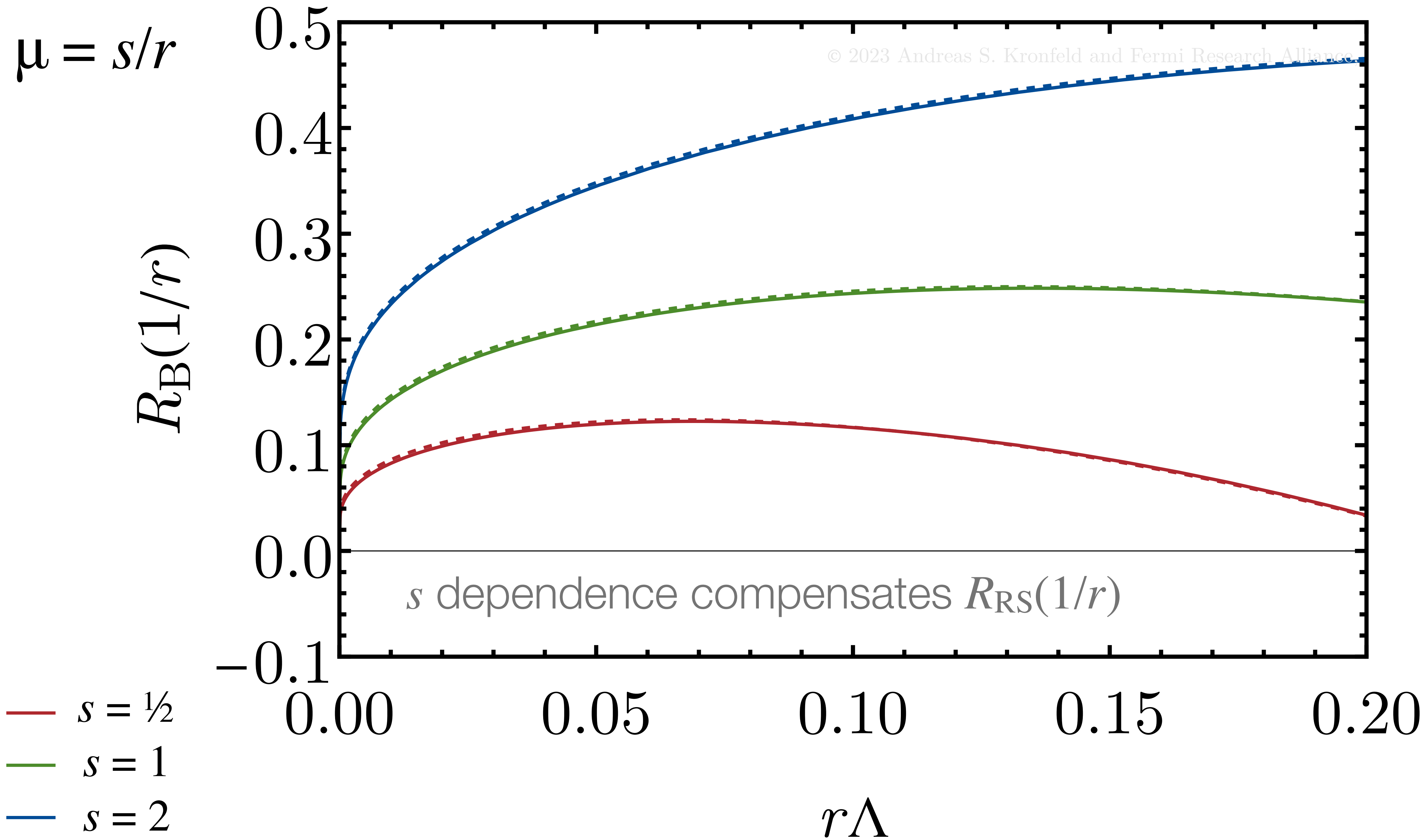
Renormalon Subtracted Series

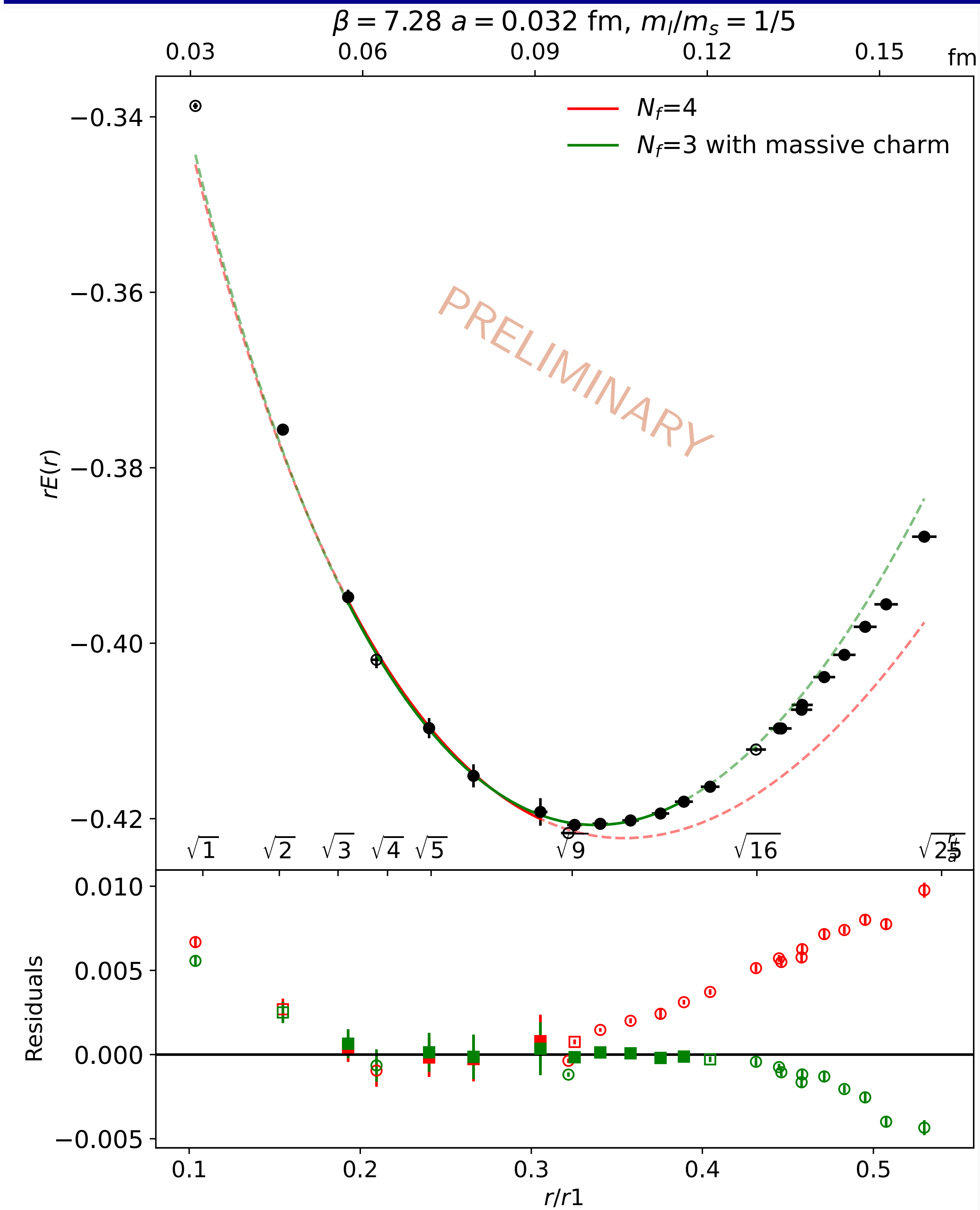
$\mu = s/r$



Borel Sum (the series convergent in $1/\alpha_s$)

$\mu = s/r$



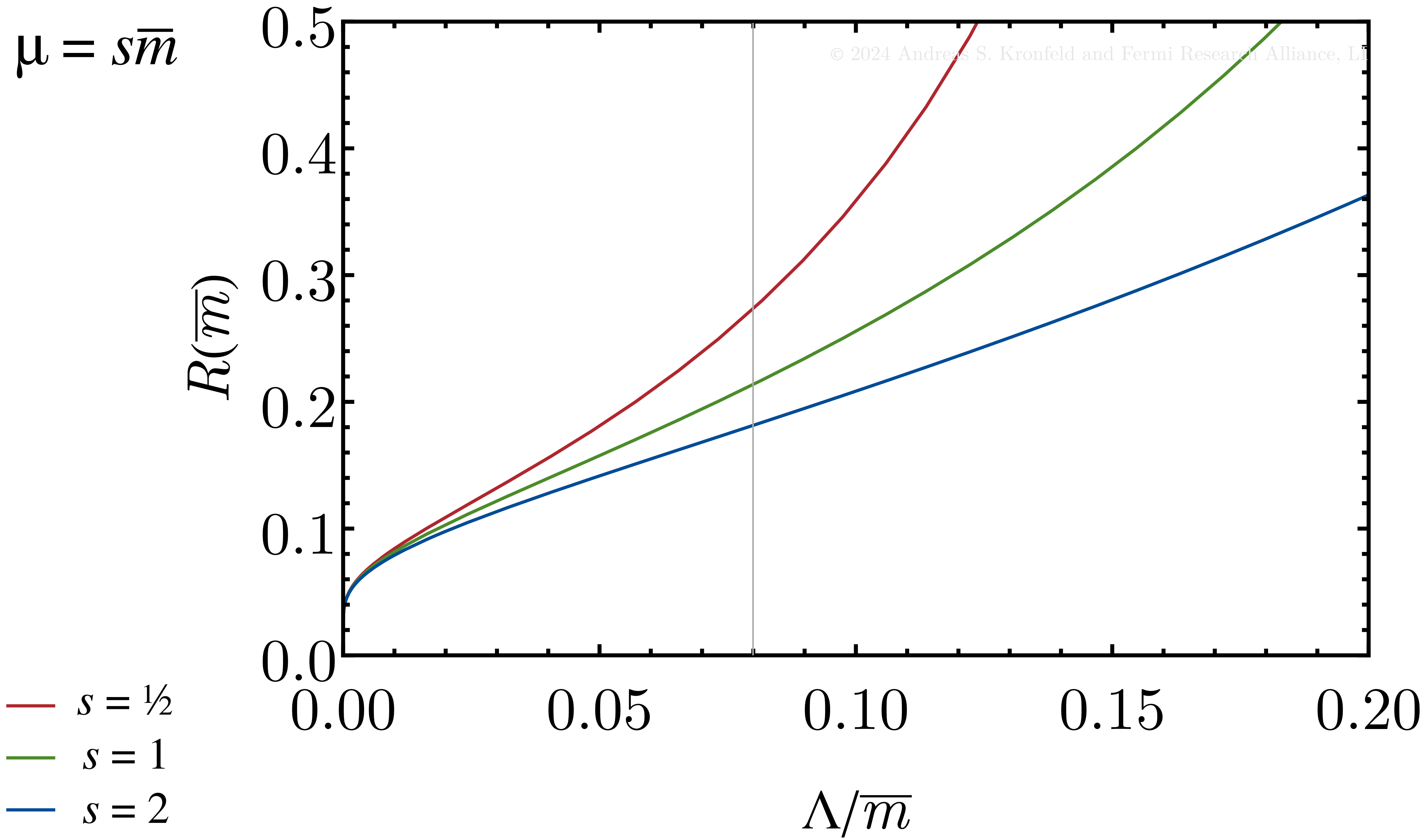


- Start fits from $r/a = \sqrt{3}$
- From TUMQCD2019
  PT works up to $\sim 0.13 fm$
- Charm effects noticeable already at $r > 0.1 fm$
- Charm effects:
  limit to 2-loop accuracy
- Drop on-axis points due to large discretization effects
- Model average (AIC) over valid fit ranges
- Correlated fits,
  blocked jackknife

← Example: Finest ensemble, 2-loops no us-resum., MRS

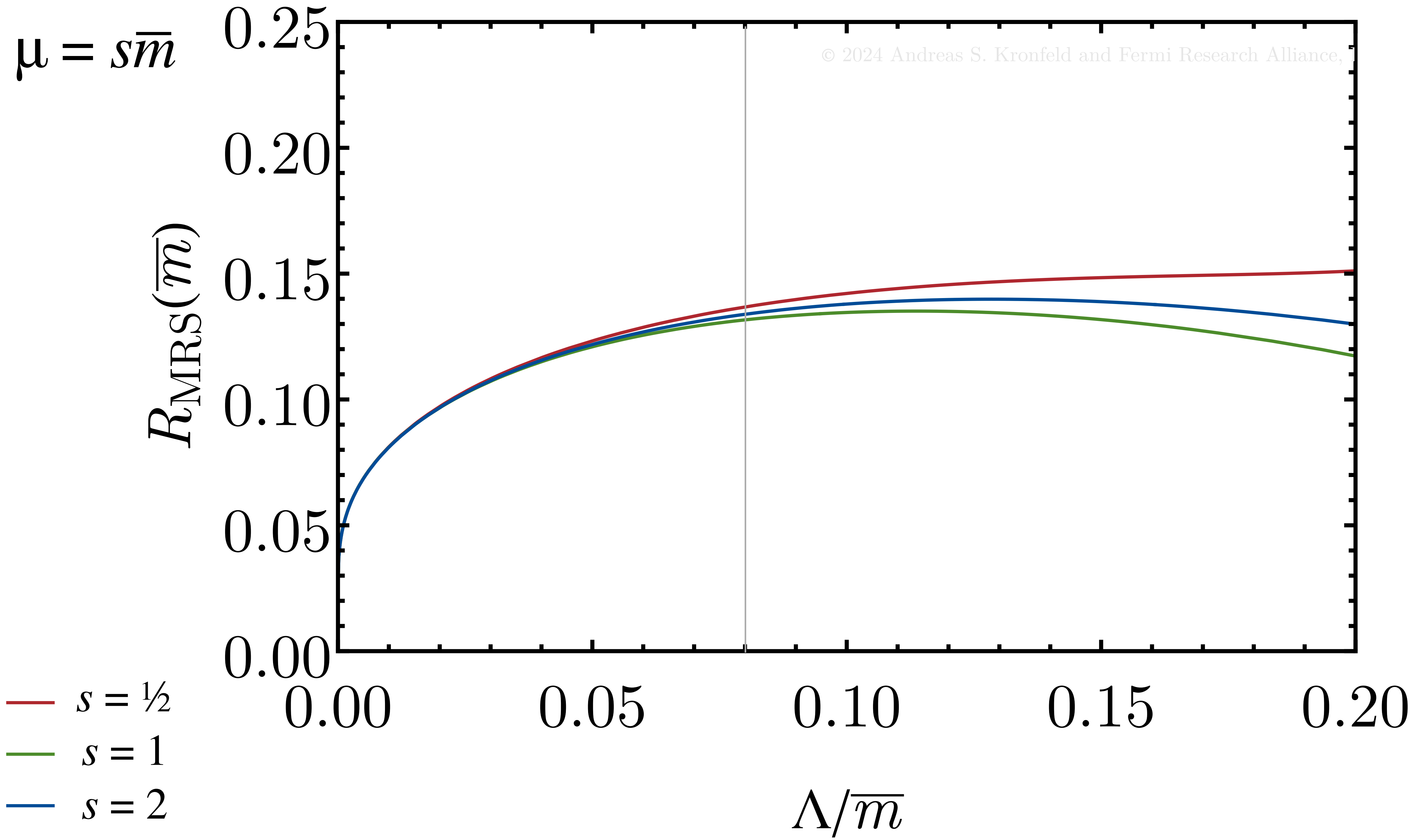
Pole Mass's Horrible Series ($p = 1$)

$\mu = s\overline{m}$



Pole Mass's MRS Series

$\mu = s\overline{m}$

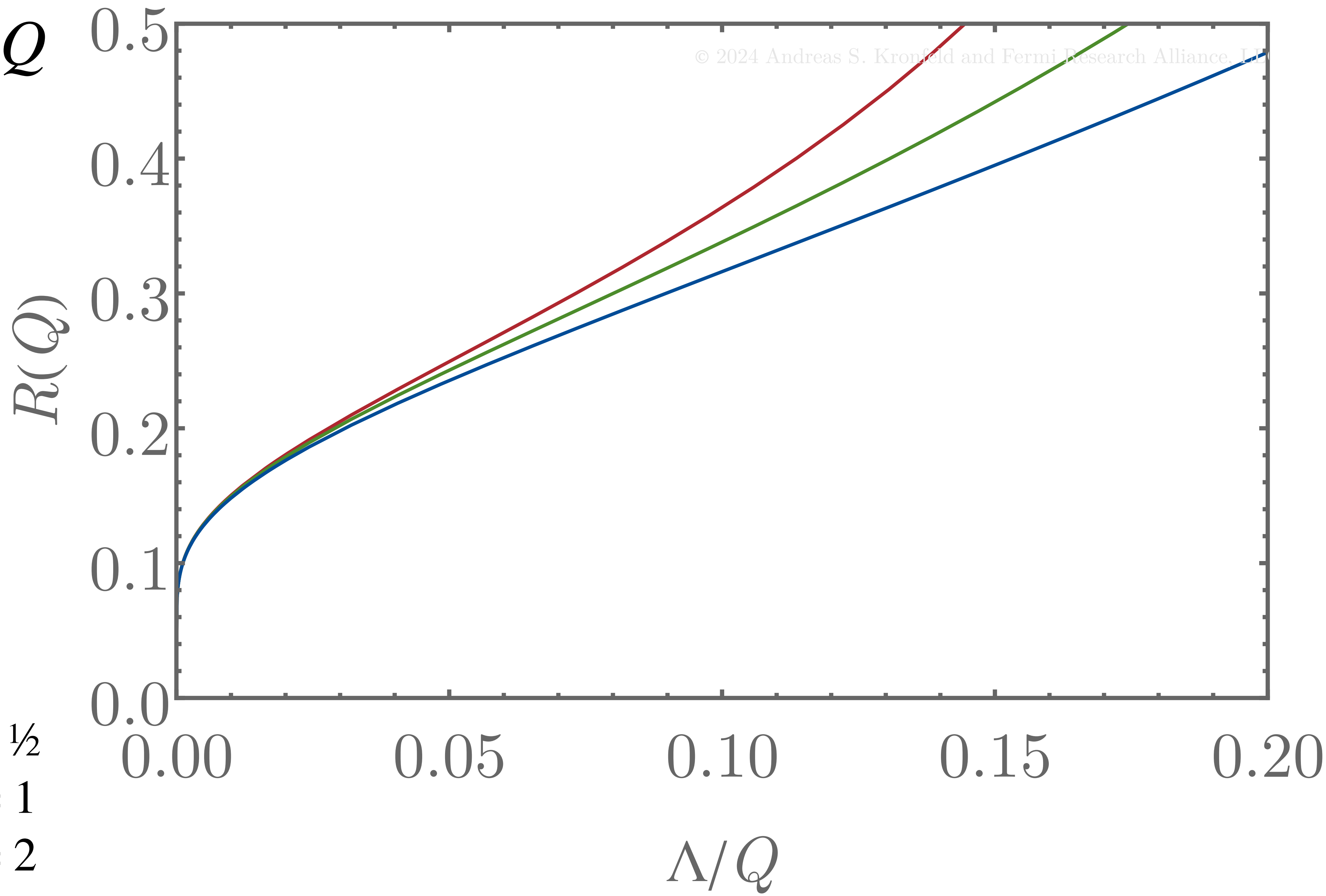


Fitting with Power Corrections

- The Λ on the horizontal axis is $\Lambda_{\overline{\text{MS}}}$—
  - fits to data will have this as free parameter, i.e., optimization will stretch/shrink the curves to fit.
- Let's go back to the plots and get a feel for adding small amounts of order $(\Lambda/q)^{2 \text{ or } 3 \text{ or } 4}$, $(\Lambda r)^9$, or Λr. ★
- Disentangling power-law and logarithmic dependence seems hard for $\tilde{R}(q)$ and $R(1/r)$, but not for $F(1/r)$ and $R_{\text{MRS}}(1/r)$.

Bjorken Sum Rule's Horrible Series (p = 2)

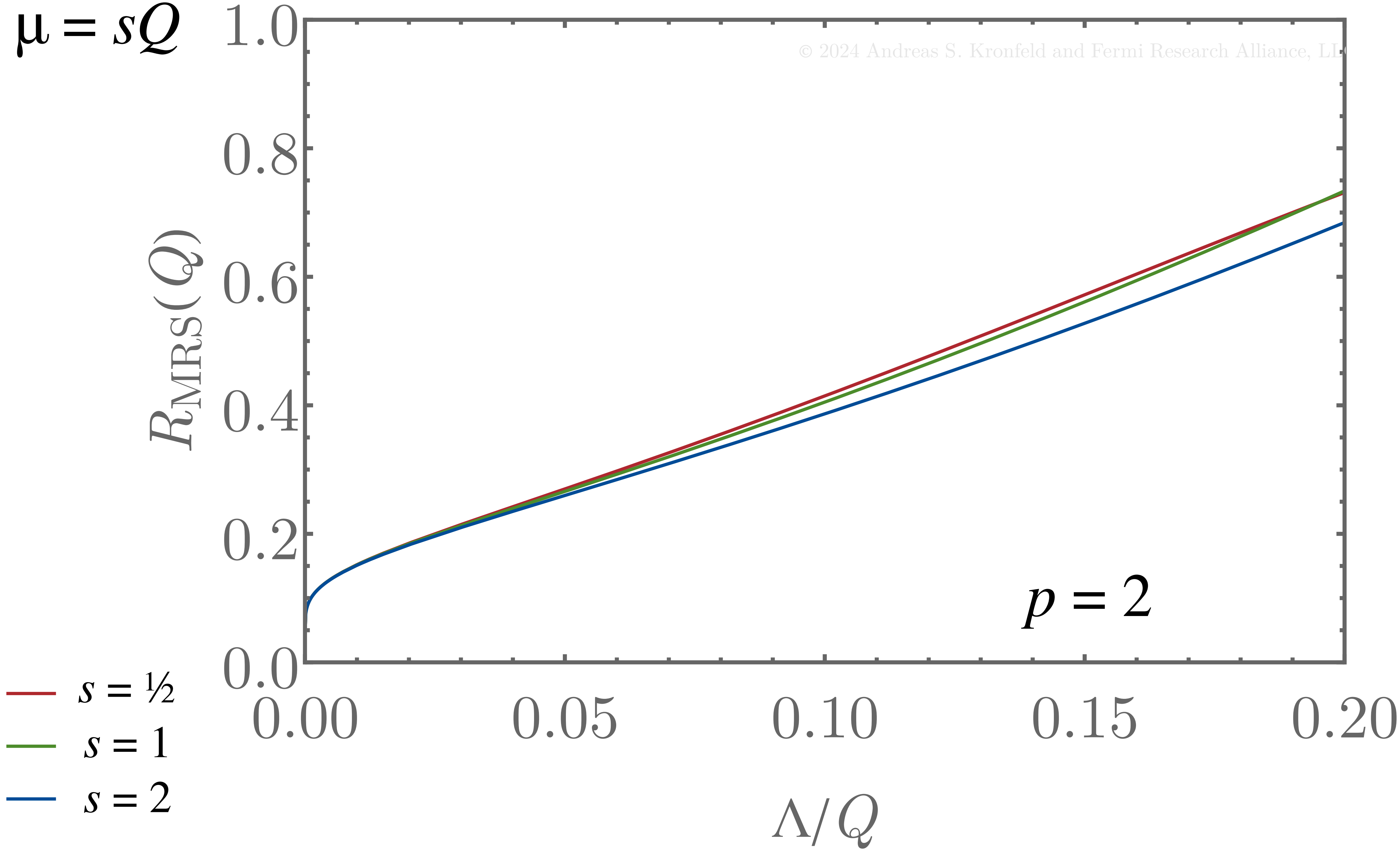
$\mu = sQ$



s = ½
s = 1
s = 2

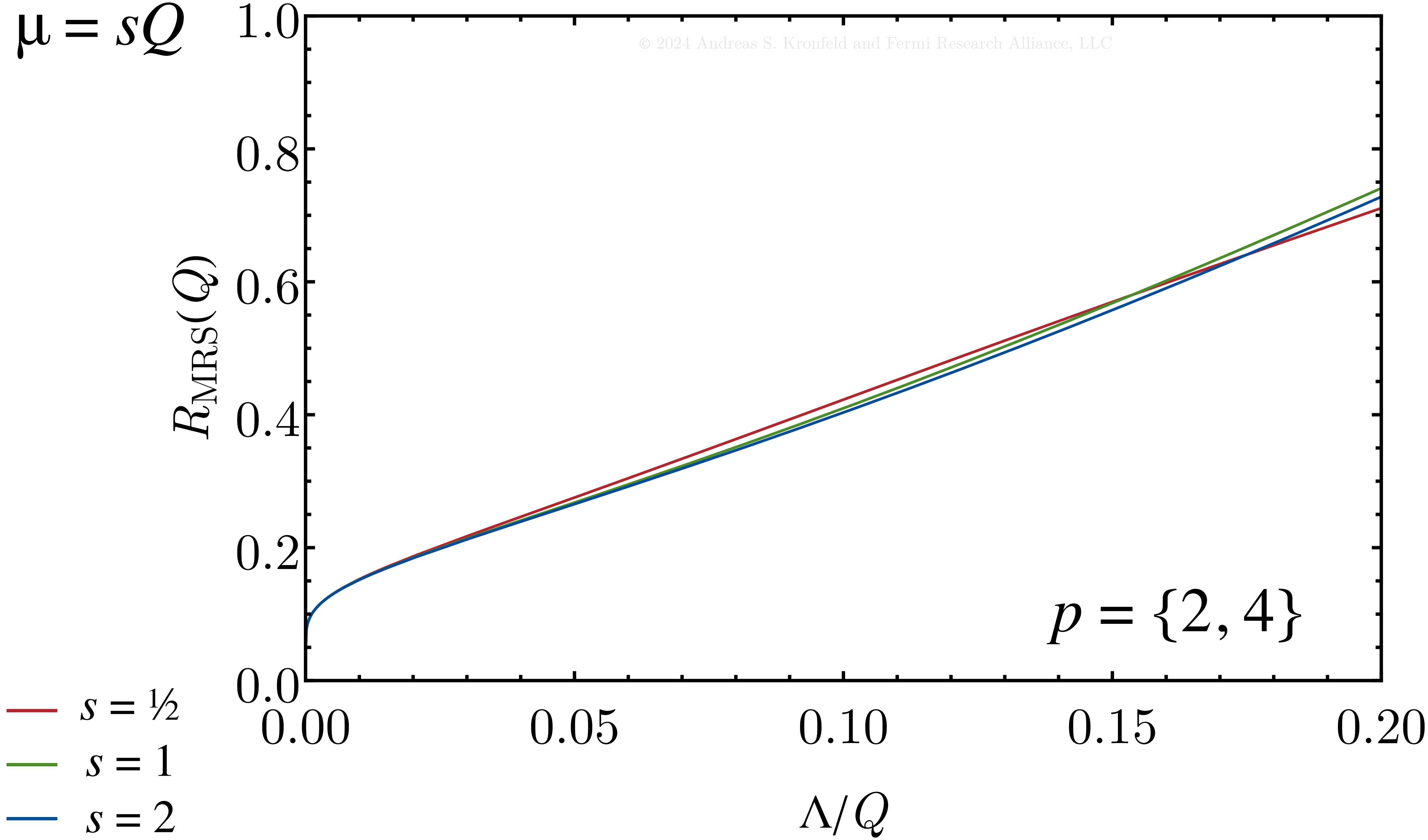
Bjorken Sum Rule’s MRS Series

$\mu = sQ$



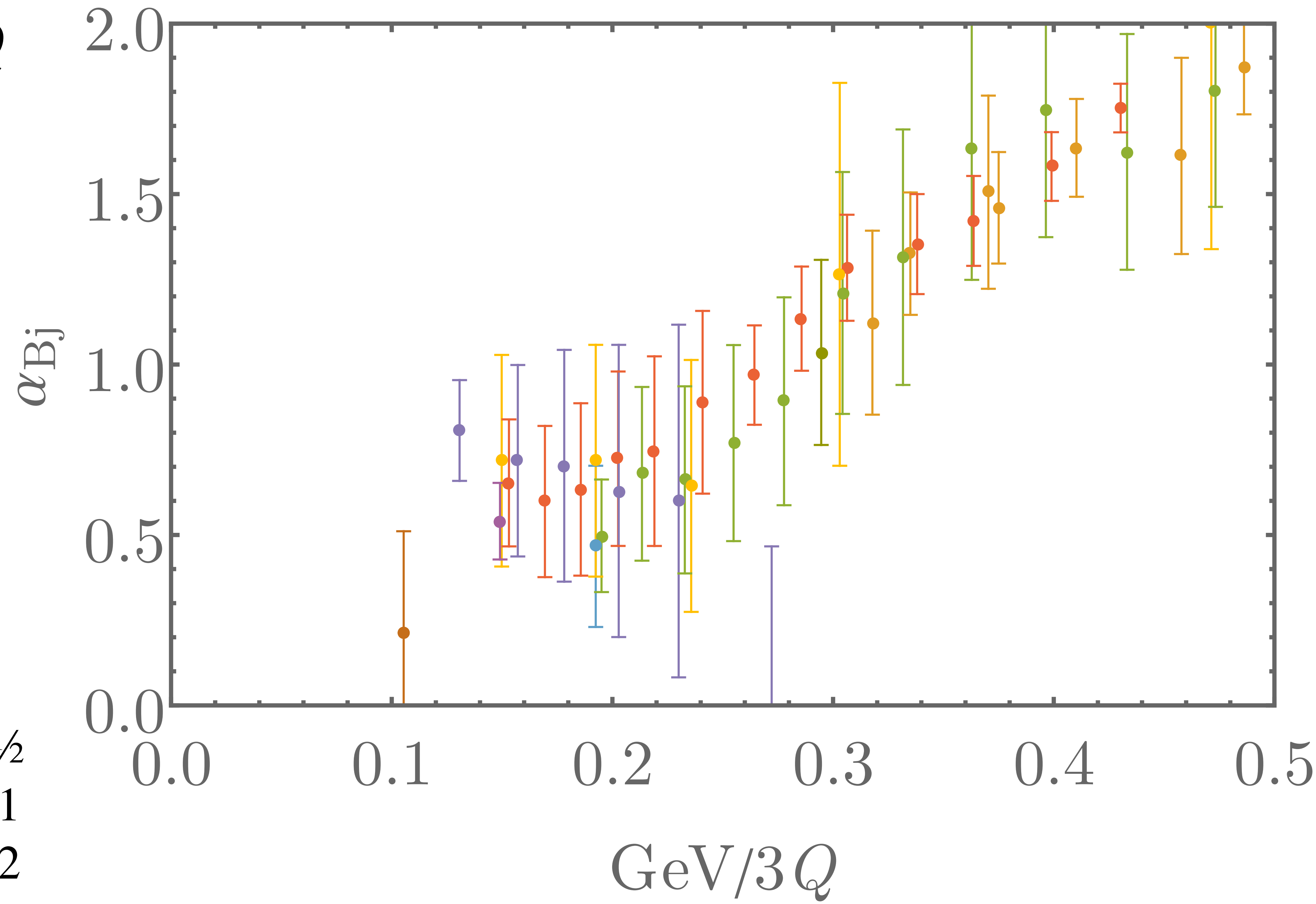
Bjorken Sum Rule’s MRS Series

$\mu = sQ$



Bjorken Sum Rule Experimental Data

$\mu = Q$



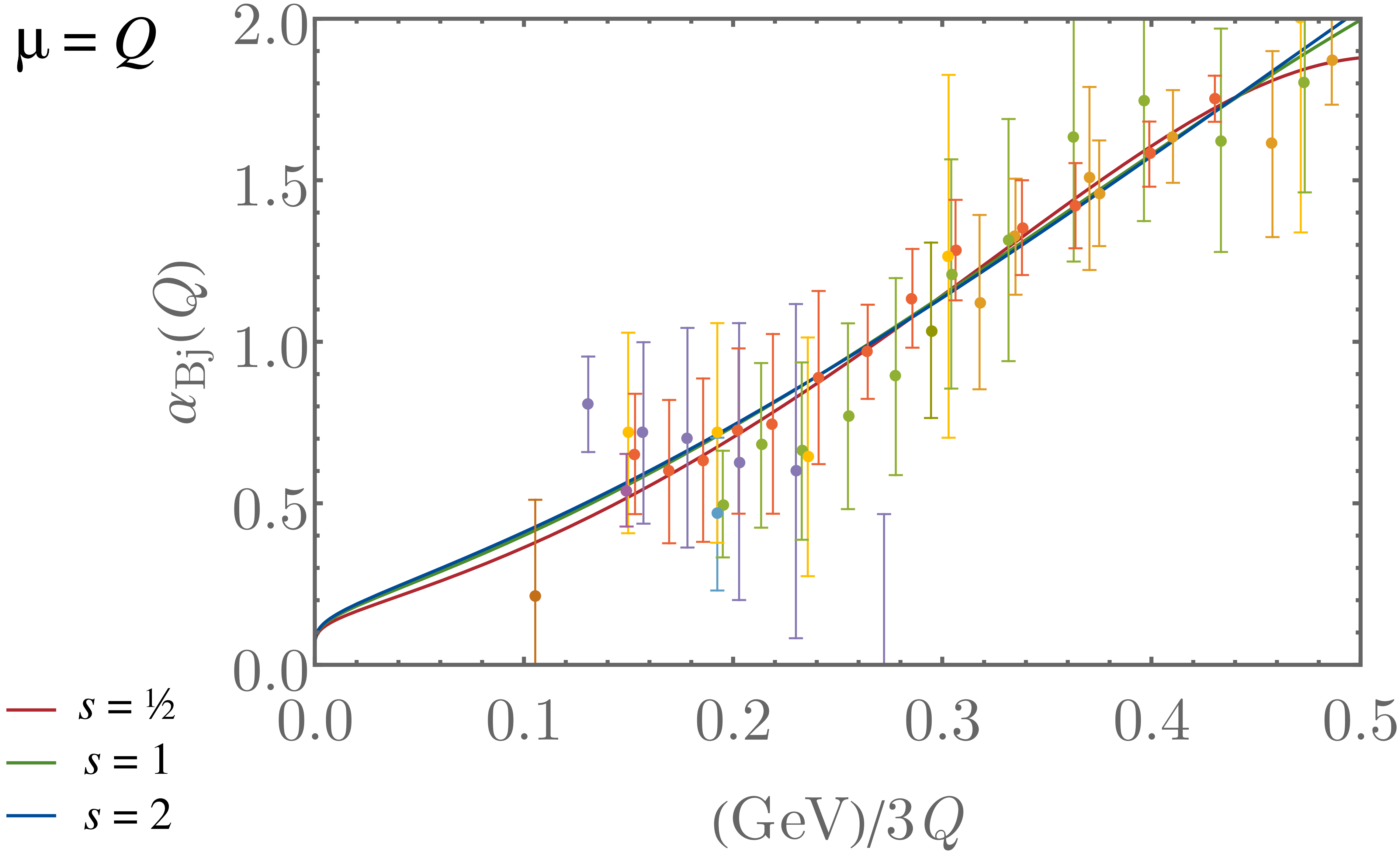
$s = ½$

$s = 1$

$s = 2$

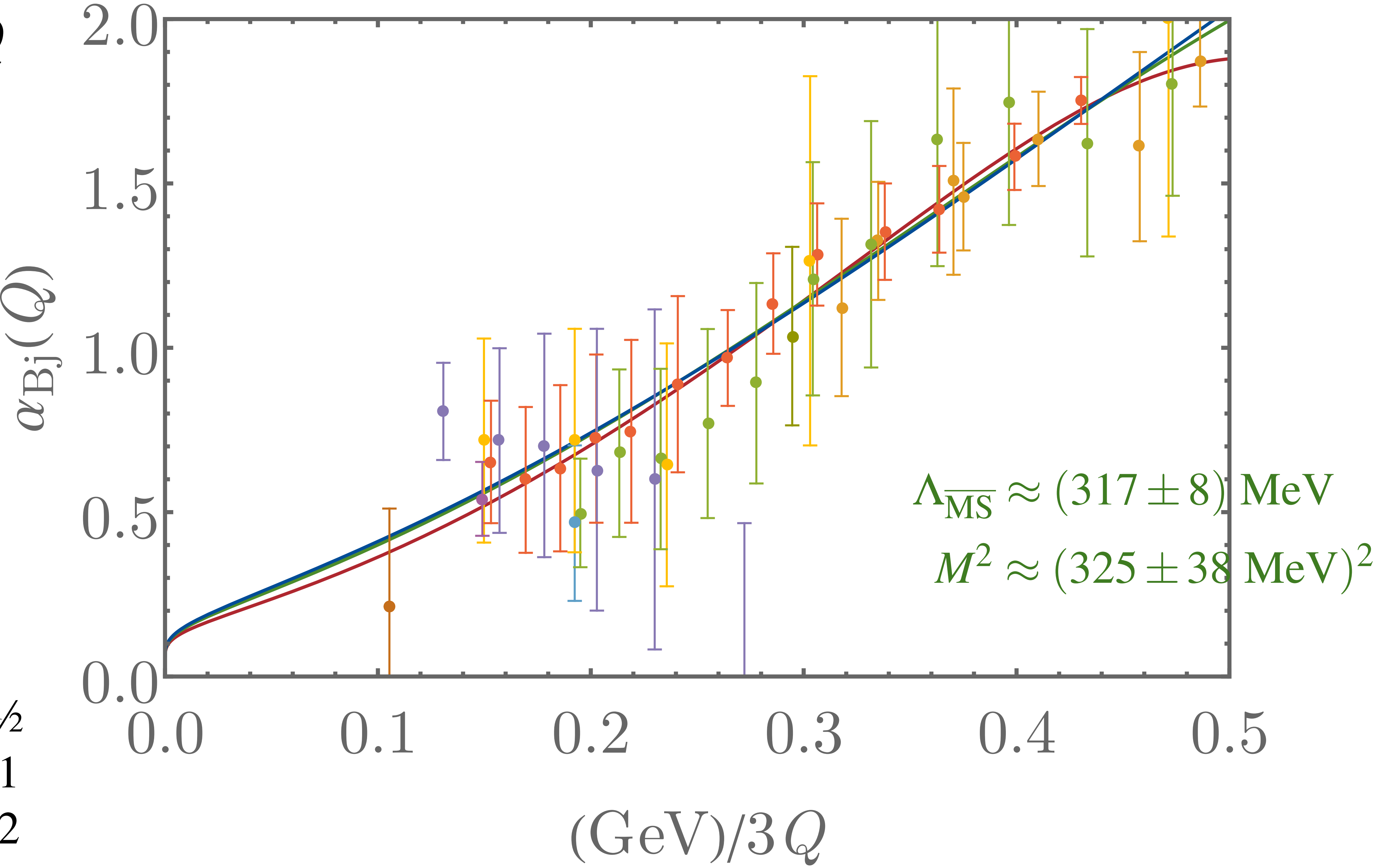
Bjorken Sum Rule Two-Parameter Fits

μ = Q



Bjorken Sum Rule Two-Parameter Fits

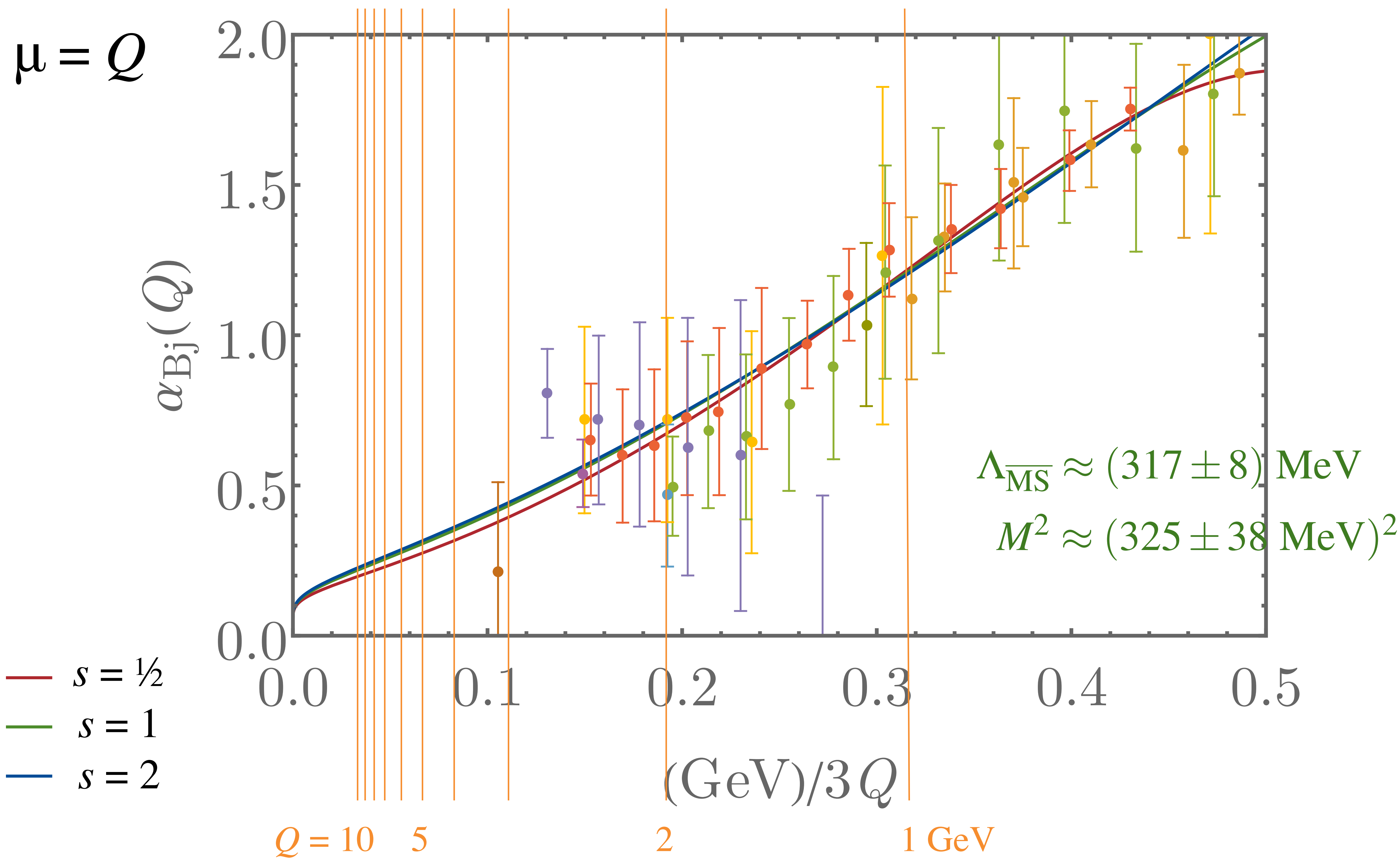
μ = Q



s = ½
s = 1
s = 2

Bjorken Sum Rule Two-Parameter Fits

$\mu = Q$



Two or More Power Corrections

Next Approximation

- If there is another power correction with $p_2 > p_1 = p$, then f_k will grow in a similar but slower fashion.

- Apply previous procedure with p_1; then repeat with p_2:

$$
\begin{aligned}
\boldsymbol{f}^{\{p_1,p_2\}} &\equiv \mathbf{Q}^{(p_2)} \cdot \mathbf{Q}^{(p_1)} \cdot \boldsymbol{r} \\
\Rightarrow \boldsymbol{r} &= \mathbf{Q}^{(p_1)^{-1}} \cdot \mathbf{Q}^{(p_2)^{-1}} \cdot \boldsymbol{f}^{\{p_1,p_2\}} \\
&= \left[\frac{p_2}{p_2 - p_1} \mathbf{Q}^{(p_1)^{-1}} + \frac{p_1}{p_1 - p_2} \mathbf{Q}^{(p_2)^{-1}} \right] \cdot \boldsymbol{f}^{\{p_1,p_2\}}
\end{aligned}
$$

- Extension to any sequence of higher powers by induction.

Summary

Summary

- MRS formulas for growth and normalization both follow from RGE and hold exactly at low orders.
- Generalized to any sequence of power corrections $\leftrightarrow$ dominant, subdominant, sub-subdominant, … growth.
- Scale dependence of total is mild: even though details of cancellation depend on $s = \frac{1}{2}, 1, 2$
- MRS shape not like leading power, when latter matters.
- Standard to sum logarithms; let's sum factorials too!

Thank you for your attention

Questions?