Scattering Amplitudes in Quantum Field Theory
WS 2021/2

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https://www.groups.ph.tum.de/ttpmath/teaching/ws-2021/

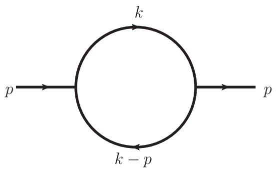
Sheet 14: The one-loop bubble with dispersive techniques
(Discussion: 11/02/2022)

In this exercise sheet we will get acquainted with dispersion techniques by studying the example of the one-loop bubble.

Exercise 1: The one-loop bubble in $d = 4$

In the following we compute the one-loop bubble close to $d = 4$ space-time dimensions from it's discontinuity by using the subtracted dispersion relation. The bubble integral in *Minkowski metric* is defined as

$$\mathrm{Bub}_{a_1,a_2}(p^2, m^2) =$$



$$= \int \frac{\mathrm{d^d}k}{\mathrm{i}\pi^{d/2}} \frac{1}{[k^2 - m^2]^{a_1} \, [(k-p)^2 - m^2]^{a_2}} \,. \tag{1}$$

Part a

Using Cutkosky rules,[1] we can write the discontinuity of the one-loop bubble as

$$\mathrm{Disc}_{p^2 > 4m^2}\left[\mathrm{Bub}^{(d)}_{1,1}\right] = \int \frac{\mathrm{d^d}k}{\mathrm{i}\pi^{d/2}} \big[-2\pi\mathrm{i}\, \delta\big(k^2 - m^2\big)\, \theta\big(k^0\big) \big] \times \big[-2\pi\mathrm{i}\, \delta\big((k-p)^2 - m^2\big)\, \theta\big(p^0 - k^0\big) \big] \,. \tag{2}$$

Compute the integral explicitly in the rest-frame $p^\mu = (M, \mathbf{0})$. You should obtain

$$\mathrm{Disc}_{p^2 > 4m^2}\left[\mathrm{Bub}^{(d)}_{1,1}(p^2)\right] = 2\mathrm{i}\, \frac{\pi^{3/2}}{\Gamma\big(\frac{d-1}{2}\big)} \frac{1}{\sqrt{p^2}} \left(\sqrt{\frac{p^2}{4} - m^2} \right)^{d-3} \,. \tag{3}$$

Part b

The bubble has a UV divergence at $d = 4$. In the language of dispersion relations, this manifests itself as the fact that the dispersive integrand

$$2\pi\mathrm{i} f(y) = \oint_{\mathcal{C}} \frac{f(z)}{z - y} \,\mathrm{d}z \,, \tag{4}$$

where f is the bubble as a function of p^2, does not vanish on the boundary $|z| \to \infty$ for $d = 4$.

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[1] The cut corresponds to the replacement $\Pi_F(q) = \left[q^2 - m^2 + i\sigma\right]^{-1} \to (-2\pi\mathrm{i})\, \delta\big(q^2 - m^2\big)\, \theta(E_q)$, where E_q is the energy that flows across the cut.

To circumvent this caveat, one can use the subtracted dispersion relation

$$f(y) = f(y_0) + \frac{y - y_0}{\pi} \int_{z_0}^{\infty} \frac{\mathrm{d}z}{(z-y)(z-y_0)} \, \Im\{f(z)\} \,, \tag{5}$$

where boundary terms have been dropped assuming $f(z)/z^2 \to 0$ as $|z| \to \infty$.

As we have seen in the previous exercise, the Bubble has a branch cut starting at $z_0 = 4m^2$. Use Eq. (5) to compute the bubble for *euclidean kinematics*, i.e. at $y = -p^2 = p_E^2$. Follow these steps:

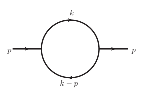
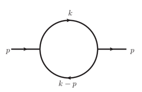
1. Chose as subtraction point $y_0 = 0$. Show that the one-loop bubble for $p_E^2 = 0$ reduces to a Tadpole integral with a square propagator; Write an analytic result for this tadpole and show that it has a UV poles close to $d = 4$;

2. Consider the second term in Eq. (5) and obtain the imaginary part of f from the bubble's discontinuity computed in the previous exercise. Explain, why the integral is convergent at $d = 4$ in the *subtracted* version, i.e. after *including* the factor $1/(z - y_0)$. Explain, why the subtraction was *not* required in $d = 2$;

3. Compute the second term in Eq. (5), neglecting terms $\mathcal{O}(d - 4)$.

Part c

Find the analytic continuation of the previous result and show that the imaginary part coincides with Eq. (3) using

$\mathrm{Disc}_{p^2>4m^2}\Big[$  $\Big] = 2\mathrm{i}\,\Im\Big[$  $\Big]\,.$ (6)

Exercise 2: The one-loop bubble's divergences

In the following, we consider the bubble in *euclidean* space. In particular, we have[2]

$$\mathrm{Bub}_{a_1,a_2}(p_E^2, m^2) = \int \frac{\mathrm{d}^{\mathrm{d}} k_E}{\pi^{d/2}} \frac{1}{[k_E^2 + m^2]^{a_1} \, [(k_E - p_E)^2 + m^2]^{a_2}} \,. \tag{7}$$

Note that

$$\mathrm{Bub}_{a_1,a_2}(p_E^2, m^2) = (-1)^{a_1+a_2} \mathrm{Bub}^M_{a_1,a_2}(-p^2, m^2) \,, \tag{8}$$

where the bubble in Minkowski space reads

$$\mathrm{Bub}^M_{a_1,a_2}(p^2, m^2) = \int \frac{\mathrm{d}^{\mathrm{d}} k}{\mathrm{i}\pi^{d/2}} \frac{1}{[k^2 - m^2]^{a_1} \, [(k-p)^2 - m^2]^{a_2}} \,. \tag{9}$$

[2]C.f. sheet 12, exercise 2a.

Part a

The bubble in $d-2$ dimensions can be related to the one in d dimensions by the following *dimensional recurrence* identity

$$\mathrm{Bub}_{1,1}^{(d-2)} = 2\,\mathrm{Bub}_{1,2}^{(d)}\,. \tag{10}$$

To prove the above formula, you can follow these steps:

1. Derive the Feynman parameter representation[3] of $\mathrm{Bub}_{1,1}^{(d)}$;
2. In this representation, show that the shift $d \to d-2$ corresponds to taking the derivative w.r.t m^2;
3. Identify $\partial_{m^2}\mathrm{Bub}_{1,1}^{(d)} = 2\mathrm{Bub}_{1,2}^{(d)}$ to verify Eq. (10).

Part b

Use the known IBP reduction[4] for $\mathrm{Bub}_{1,2}^{(d)}$ and express the bubble in d dimensions through the bubble in $d-2$ dimensions and the tadpole in d dimensions. You should find

$$\mathrm{Bub}_{1,1}^{(d)} = -\frac{(2-d)}{2(3-d)m^2}\,\mathrm{Bub}_{0,1}^{(d)} + \frac{(4m^2+p_E^2)}{2(3-d)}\,\mathrm{Bub}_{1,1}^{(d-2)}\,. \tag{11}$$

Discuss what Eq.(10) implies for the bubble's UV divergence close to $d=4$ and how this result is related to the first exercise.

[3] C.f. sheet 13, exercise 3.
[4] C.f. sheet 12, exercise 2a.