Scattering Amplitudes in Quantum Field Theory WS 2021/2

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Sheet 13: Calculation of one-loop integrals
(Discussion: 04/02/2022)

In this exercise sheet we will get acquainted with the analytic calculation of one-loop integrals. To simplify the discussion, we will work in d-dimensional *euclidean* space, i. e.

$$g^{\mu\nu} = g_{\mu\nu} = \delta^{\mu}_{\nu} \equiv \mathrm{diag}(\underbrace{1, \dots, 1}_{d}) \,. \tag{1}$$

Some useful formulae are provided in section 5.

1 Generalised Feynman parameters

Using Swinger's trick

$$\frac{1}{D_j^{\nu_j}} = \frac{1}{\Gamma(\nu_j)} \int_0^{\infty} \mathrm{d}\alpha_j \; \alpha_j^{\nu_j - 1} \mathrm{e}^{-\alpha D_j} \tag{2}$$

for $D_j > 0, \; \Re(\nu_j) > 0$, prove that

$$\prod_{j=1}^{n} \frac{1}{D_j^{\nu_j}} = \Gamma(\nu) \prod_{j=1}^{n} \left[\int_0^{\infty} \mathrm{d}x_j \, \frac{x_j^{\nu_j - 1}}{\Gamma(\nu_j)} \right] \; \delta\Bigg(1 - \sum_{j=1}^{n} x_j\Bigg) \frac{1}{\Big(\sum_{j=1}^{n} x_j D_j\Big)^{\nu}} \tag{3}$$

with $\nu = \sum_{j=1}^{n} \nu_j$. *Hint: Use the fact that* $1 = \int_0^{\infty} \mathrm{d}t \; \delta\left(t - \sum_{j=1}^{n} \alpha_j\right)$.

2 Cheng - Wu theorem

Prove the Cheng - Wu theorem for the bubble integral. More specifically, convince yourselves that

$$\int \mathrm{d}x_1 \mathrm{d}x_2 \, \frac{\delta(1 - x_1 - x_2)}{(A x_1 + B x_2)^2} = \int \mathrm{d}x_1 \mathrm{d}x_2 \, \frac{\delta(1 - x_i)}{(A x_1 + B x_2)^2} \,, \qquad i = 1, 2 \,. \tag{4}$$

3 Feynman parameter representation

Consider the Feynman integral

$$I = \int \prod_{l=1}^{L} \frac{\mathrm{d}^d k_l}{\pi^{d/2}} \, \frac{1}{D_1 \, D_2 \, \dots D_n} . \tag{5}$$

In this exercise you should use (3) to prove the so-called *Feynman parametrization* formula

$$I = \Gamma\left(\nu - \frac{Ld}{2}\right) \prod_{j=1}^{n} \left[\int_0^{\infty} \mathrm{d}x_j \, \frac{x_j^{\nu_j - 1}}{\Gamma(\nu_j)} \right] \delta\Bigg(1 - \sum_{j=1}^{n} x_j\Bigg) \frac{U_G^{\nu - (L+1)d/2}}{F_G^{\nu - Ld/2}}, \tag{6}$$

where F_G and U_G are the two polynomials defined in class. To do so we can perform the following steps:

1. In the lectures we have seen that $Q = \sum_{j=1}^{n} x_j D_j$ is a quadratic form in the loop momenta k_l,

$$Q = k_i A_{ij}(x) k_j + 2B_i(x,p) k_i + C(x,p,m)$$

where x_j is the Feynman parameter introduced in (3). Perform a translation in the loop momenta so that the linear terms in k_l vanish.

2. Make a suitable rotation that brings the quadratic form in diagonal form.

3. Perform a change of variables that brings Q in the following form

$$Q = \frac{F_G}{U_G}\left[\sum_i K_i^2 + 1\right].$$

You should find that

$$U_G = \det A_{ij}\,, \qquad F_G = U_G\left[C(x,p,m) - B_i(x,p) A_{ij}^{-1} B_j(x,p)\right]\,.$$

4 Massless box

Consider the one-loop massless box

$$I = \int \frac{\mathrm{d}^d k}{i\pi^{d/2}} \frac{1}{k^2(k-p_1)^2(k-p_1-p_2)^2(k-p_1-p_2-p_3)^2} \tag{7}$$

with $p_i^2 = 0$ for $i = 1,2,3,4$ and $s = (p_1+p_2)^2$, $u = (p_2+p_3)^2$ are the standard (*minkowskian*) Mandelstam variables.

1. Using the formulas derived in the previous exercise, prove that

$$U_G = \sum_{i=1}^{4} x_i, \;\; F_G = S x_1 x_3 + U x_2 x_4$$

where $S = -s$ and $U = -u$ are *euclidean* Mandelstam invariants (i.e. after Wick Rotation). Assume from now on that $S, U > 0$.

2. Argue that for this set of values, the integral is real.

3. Using the Feynman parameter representation (6) calculate the one-loop massless box in $d = 4 - 2\epsilon$ dimensions in terms of the hypergeometric function ${}_2F_1(a,b;c;z)$.
Hint: use the change of variables $x_1 \to \eta_1 \xi_1,\; x_2 \to \eta_1(1-\xi_1),\; x_3 \to \eta_2 \xi_2,\; x_4 \to \eta_2(1-\xi_2)$.

4. **Bonus:** Go back to the Feynman parametrization for the box before integration. Use it to compute the first three coefficients of the Laurent expansion in ϵ for the Box. You should find

$$I = \frac{r_\Gamma S^{-2-\epsilon}}{x}\left(\frac{4}{\epsilon^2} - \frac{2\log x}{\epsilon} - \pi^2 + \mathcal{O}(\epsilon)\right)$$

with $x = U/S$ and $r_\Gamma = \frac{\Gamma(1-\epsilon)^2\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}$.

Check that you would obtain the same result by expanding the result that you found at point 3. *Hint: Note that you should be able to arrange expansions such that γ_E never shows up in the intermediate results.*

5 Useful formulae

$$\int_0^\infty \mathrm{d}t\; t^{\nu-1}\mathrm{e}^{-t} = \Gamma(\nu) \tag{8}$$

$$\int_0^\infty \mathrm{d}t\; \frac{t^{a-1}}{(1+t)^{a+b}} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{9}$$

$$\int_0^\infty \mathrm{d}t\; t^a(1+t)^b = \frac{\Gamma(1+a)\Gamma(-1-a-b)}{\Gamma(-b)} \tag{10}$$

$$\int_0^1 \mathrm{d}t\; \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a} = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\; {}_2F_1(a,b;c;z) \tag{11}$$

$$\Gamma(1+b\epsilon) = 1 - \gamma_E\, b\, \epsilon + \frac{1}{12}\left(6\gamma_E^2+\pi^2\right) b^2\, \epsilon^2 + O\left(\epsilon^3\right) \tag{12}$$

$${}_2F_1(1,1;1-\epsilon;x) = -\frac{1}{x-1} + \frac{\epsilon\log(1-x)}{x-1} + \epsilon^2\left(-\frac{\mathrm{Li}_2(x)}{x-1} - \frac{\log^2(1-x)}{2(x-1)}\right) + \mathcal{O}(\epsilon^3) \tag{13}$$

$${}_2F_1(1,-\epsilon;1-\epsilon;x) = 1 + \epsilon\log(1-x) - \epsilon^2\mathrm{Li}_2(x) + \mathcal{O}(\epsilon^3) \tag{14}$$

where $\gamma_E = 0.577216...$ is the Euler-Mascheroni constant.