Scattering Amplitudes in Quantum Field Theory
WS 2021/2

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https://www.groups.ph.tum.de/ttpmath/teaching/ws-2021/

Sheet 12: One-loop warm-up
(Discussion: 28/01/2022)

In this exercise sheet we will get acquainted with some standard one-loop integrals and related tricks and properties. To simplify the discussion, we will work in d-dimensional *euclidean* space, i. e.

$$g^{\mu\nu} = g_{\mu\nu} = \delta^{\mu}_{\nu} \equiv \mathrm{diag}(\underbrace{1,\ldots,1}_{d}) \,. \tag{1}$$

Exercise 1: Dimensional regularisation

In class, you have seen that

$$\mathrm{Tad}(m^2) = \int \frac{\mathrm{d^d}k}{\pi^{d/2}} \frac{1}{k^2+m^2} = (m^2)^{d/2-1}\,\Gamma\left(\frac{2-d}{2}\right) \,. \tag{2}$$

1. By direct computation, going directly in spherical coordinates, show that

$$\int \frac{\mathrm{d^d}k}{\pi^{d/2}} \frac{1}{k^2\,(k^2+m^2)} = -\frac{\mathrm{Tad}(m^2)}{m^2} \,. \tag{3}$$

2. Compute the same integral by splitting it into two integrals with the partial fraction decomposition

$$\frac{1}{k^2\,(k^2+m^2)} = \frac{1}{m^2}\left(\frac{1}{k^2} - \frac{1}{k^2+m^2}\right) \,. \tag{4}$$

   What do Eqs.(3,4) imply for the integral

$$\int \frac{\mathrm{d^d}k}{\pi^{d/2}} \frac{1}{k^2} \,. \tag{5}$$

   Discuss why you are allowed to perform the calculation in this way in dimensional regularisation.

Exercise 2: Integration-by-parts identities

Recall that for a scalar one-loop integral which depends on E external momenta $\{p\} = \{p_1,\ldots,p_E\}$[1]

$$I = \int \frac{\mathrm{d^d}k}{\pi^{d/2}}\, g(k,\{p\}) \,, \tag{6}$$

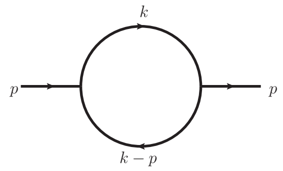
dimensional regularisation makes it possible to write the following integration-by-parts identities (IBPs)

$$0 = \int \frac{\mathrm{d^d}k}{\pi^{d/2}} \frac{\partial}{\partial k^\mu}\left[q^\mu\, g(k,\{p\})\right] \,, \quad q \in \{k, p_1, \ldots, p_E\} \,. \tag{7}$$

[1] The discussion of course extends to higher loop orders.

Part a - Bubble

Consider bubble integrals of the form

$$\mathrm{Bub}_{a_1,a_2}(p^2,m^2) = \quad$$ 

$$= \int \frac{\mathrm{d}^d k}{\pi^{d/2}} \frac{1}{[k^2+m^2]^{a_1}\,[(k-p)^2+m^2]^{a_2}}\,, \tag{8}$$

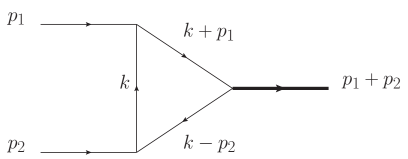
with $a_1, a_2 \in \mathbb{Z}$.

1. Show that $\mathrm{Bub}_{a_1,a_2} = \mathrm{Bub}_{a_2,a_1}$.
2. Derive both IBP relations for Bub_{a_1,a_2} with $q \in \{k,p\}$, for generic values of a_1 and a_2.
3. Consider the four IBPs obtained for $\{a_1 = 0,\, a_2 = 1\}$ and $\{a_1 = 1,\, a_2 = 1\}$ and use them to show

$$\mathrm{Bub}_{1,2} = \frac{(3-d)}{(4m^2+p^2)}\,\mathrm{Bub}_{1,1} + \frac{(2-d)}{2m^2(4m^2+p^2)}\,\mathrm{Bub}_{0,1}\,. \tag{9}$$

Part b - Massless triangle

Consider massless triangle integrals of the form

$$I_{a_1,a_2,a_3}(s) = \quad$$ 

$$= \int \frac{\mathrm{d}^d k}{\pi^{d/2}} \frac{1}{[k^2]^{a_1}\,[(k+p_1)^2]^{a_2}\,[(k-p_2)^2]^{a_3}}\,, \tag{10}$$

where $p_{1,2}^2 = 0$ and $p_{12}^2 = (p_1+p_2)^2 = s$.

1. Show the following properties of the integral in Eq. (10)
   - $I_{a_1,a_2,0} = I_{a_1,0,a_2} = 0$,
   - $I_{a_1,a_2,a_3} = I_{a_1,a_3,a_2}$,
   - $I_{0,a_2,a_3} = \mathrm{Bub}_{a_2,a_3}(s,0)$.
2. Using Eq. (7), derive the three IBP relations for I_{a_1,a_2,a_3} with $q \in \{k, p_1, p_2\}$.
3. Specialise the three identities to the *seed integral* $I_{1,1,1}$, i.e. put $a_1 = a_2 = a_3 = 1$. Show that one of the IBP relations is not linearly independent. Furthermore, show that this particular triangle can be reduced to a bubble as follows

$$I_{1,1,1} = \frac{-2(d-3)}{(d-4)\,s}\,\mathrm{Bub}_{1,1}(s,0)\,. \tag{11}$$