Scattering Amplitudes in Quantum Field Theory WS 2021/2

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https://www.groups.ph.tum.de/ttpmath/teaching/ws-2021/

Double Sheet 10 - 11: BCFW recursion relations

(Discussion: 21/01/2022)

In this exercise sheet we discuss several aspects and applications of the BCFW recursion relations.

1 Parke-Taylor formula for non-adjacent helicities

In the lectures you have used the BCFW recursion to prove the Parke-Taylor formula for the MHV n-gluon amplitudes for the case where the two equal helicity gluons are adjacent. Using a similar line of reasoning as done in the lectures, prove the Parke-Taylor formula for the MHV n-gluon amplitudes in the case where the two equal helicity gluons are non-adjacent, ie

$$A_n(1^+, ..i^-, ..., j^-, ...n^+) = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle...\langle n1\rangle} . \tag{1}$$

2 Split helicity NMHV six-gluon amplitude

In this exercise we will use the BCFW recursion to compute the split-helicity NMHV amplitude $A_6[1^-2^-3^-4^+5^+6^+]$. In what follows we shall denote as $[i, j\rangle$-shift the following:

$$|\hat{i}] = |i] + z\,|j] , \quad |\hat{j}] = |j] , \quad |\hat{i}\rangle = |i\rangle , \quad |\hat{j}\rangle = |j\rangle - z\,|i\rangle . \tag{2}$$

Part a

Choose the $[1, 2\rangle$-shift and discuss which terms contribute to the BCFW recursion for this amplitude.

Part b

Use the above diagrams to write the $A_6[1^-2^-3^-4^+5^+6^+]$ tree amplitude in the following representation:

$$A_6[1^-2^-3^-4^+5^+6^+] = \frac{\langle 3|\,1+2\,|6]^3}{P_{126}^2\,[21]\,[16]\,\langle 34\rangle\,\langle 45\rangle\,\langle 5|\,1+6\,|2]} + \frac{\langle 1|\,5+6\,|4]^3}{P_{156}^2\,[23]\,[34]\,\langle 56\rangle\,\langle 61\rangle\,\langle 5|\,1+6\,|2]} \tag{3}$$

We use the abbreviation $P_{ijk} = p_i + p_j + p_k$.

Part c

Discuss the analytic structure of eq. (3), by answering the following questions

1. Which expressions in the denominators do *not* represent physically allowed poles? Identify the non-physical pole and prove that its residue is equal to zero.
2. Focus on the physical poles and compare with the MHV n-gluon amplitudes. What type of new intermediate states can produce poles in NMHV amplitudes compared to MHV ones? Can you justify why these new poles can never appear in the MHV amplitudes?

Part d

Let us consider now another NMHV 6-gluon amplitude, the alternating helicity amplitude $A_6[1^+2^-3^+4^-5^+6^-]$. Show that the BCFW recursion relations based on the $[2,3\rangle$-shift gives the following representation for the 6-point alternating helicity gluon amplitude:

$$A_6[1^+2^-3^+4^-5^+6^-] = \{M_2\} + \{M_4\} + \{M_6\} \tag{4}$$

where

$$\{M_i\} = \frac{\langle i, i+2\rangle^4 [i+3, i-1]^4}{\tilde{P}_i^2 \langle i| \tilde{P}_i |i+3] \langle i+2| \tilde{P}_i |i-1] \langle i, i+1\rangle \langle i+1, i+2\rangle [i+3, i-2] [i-2, i-1]} \tag{5}$$

with $\tilde{P}_i = P_{i,i+1,i+2}$.

3 Scalar QED

Consider the Lagrangian for massless scalar QED:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - |D\phi|^2 - \frac{1}{4}\lambda|\phi|^4 \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - |\partial\phi|^2 + ieA^\mu\left[(\partial_\mu\phi^*)\phi - \phi^*\partial_\mu\phi\right] - e^2A^\mu A_\mu\phi^*\phi - \frac{1}{4}\lambda|\phi|^4.\end{aligned} \tag{6}$$

The Feynman rules give a scalar-scalar-photon 3-vertex $ie(p_2 - p_1)^\mu$ (both momenta outgoing), a 2-scalar 2-photon 4-vertex $-2ie^2\eta_{\mu\nu}$, and a 4-scalar vertex $-i\lambda$. For $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$.

Part a

Using (6), show that $A_4(\phi\phi^*\phi\phi^*)$ reads

$$A_4(\phi\phi^*\phi\phi^*) = -\lambda + \tilde{e}^2\left(1 + \frac{\langle 13\rangle^2 \langle 24\rangle^2}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 41\rangle}\right) \tag{7}$$

where $\tilde{e} = \sqrt{2}e$.

Part b

1. Compute $A_4(\phi\phi^*\phi\phi^*)$ using BCFW recursion, using a $[1,3\rangle$-shift.
2. Compare your result with (7). For which values of λ and $\tilde{e}$ do the two results converge? What is the meaning of this special limit?