Scattering Amplitudes in Quantum Field Theory WS 2021/2

Prof. Lorenzo Tancredi (TUM)

https://www.groups.ph.tum.de/ttpmath/teaching/ws-2021/

Double Sheet 08-09: Soft and collinear singularities
(Discussion: 14/01/2022)

In this exercise sheet we continue to study soft and collinear limits of tree-level amplitudes.

Spin correlations in collinear splittings

In the lecture we have derived the collinear splitting $q^* \to qg$. In this case, the helicity of the resulting quark is determined by the parent quark. We will now consider the two remaining collinear splittings, $g^* \to q\bar{q}$ and $g^* \to gg$.

Parameterization

Following the lecture, we parametrize splittings $a(p_{12}) \to a_1(p_1) + a_2(p_2)$ in Sudakov decomposition $p_{1,2} = x_{1,2}p + y_{1,2}\bar{p} \pm p_\perp$. Here, p and $\bar{p}$ denote light-like momenta, to which $p_\perp$ is orthogonal. Taking the collinear limit $\boldsymbol{p}_{1,2} \parallel \boldsymbol{p}$ corresponds to neglecting terms $y_i \sim p_\perp^2$, i.e. we are interested in $a(p) \to a_1(zp + p_\perp) + a_2((1-z)p - p_\perp) + \mathcal{O}(p_\perp^2)$.

Part a: $g^* \to q\bar{q}$

1. Consider the partial amplitude for the splitting of a gluon in a quark-antiquark pair

$$\mathrm{i}\mathcal{M}(q_1, \bar{q}_2, \dots) = gT^a_{ij}\bar{U}(p_2)\gamma_\mu U(p_1)\frac{-g^{\mu\nu}}{s_{12}}\widetilde{\mathcal{M}}_\nu(p_{12}, \dots)\,, \tag{1}$$

where $\widetilde{\mathcal{M}}_\nu$ is the amplitude that describes the emission of an off-shell gluon with momentum p_{12} and ν is a four-dimensional index where the external gluon is attached. Argue, that one can make the replacement

$$g^{\mu\nu} \to -\sum_\lambda \varepsilon^\mu_\lambda(p,\bar{p})\,[\varepsilon^\nu_\lambda(p,\bar{p})]^* + \mathcal{O}\left(p_\perp^2\right)\,, \tag{2}$$

in Eq.(1), where p and $\bar{p}$ are the two light-like momenta used for the Sudakov decomposition. Hence, the scattering amplitude can be written as

$$\mathrm{i}\mathcal{M}(q_1, \bar{q}_2, \dots) = \frac{g}{s_{12}}\sum_\lambda \bar{U}(p_2)\,\not{\varepsilon}_\lambda(p,\bar{p})\,U(p_1)\widetilde{\mathcal{M}}(p^{-\lambda}, \dots) + \mathcal{O}\left(p_\perp^2\right)\,. \tag{3}$$

2. Take the collinear limit in Eq.(3). You should find

$$\mathrm{i}\mathcal{M}(q_1^L, \bar{q}_2^L, \dots) = \sqrt{2}g\left\{-\frac{z}{[12]}\widetilde{\mathcal{M}}(p^-, \dots) + \frac{1-z}{\langle 12\rangle}\widetilde{\mathcal{M}}(p^+, \dots)\right\}, \tag{4}$$

$$\mathrm{i}\mathcal{M}(q_1^R, \bar{q}_2^L, \dots) = \sqrt{2}g\left\{-\frac{1-z}{[12]}\widetilde{\mathcal{M}}(p^-, \dots) + \frac{z}{\langle 12\rangle}\widetilde{\mathcal{M}}(p^+, \dots)\right\}. \tag{5}$$

3.

Part b: $g^* \to gg$

The second amplitude that we consider is the splitting of an off-shell gluon into two collinear gluons. It reads

$$\begin{aligned} \mathrm{i}\mathcal{M}(g_1, g_2, \dots) = g\varepsilon_{\mu_1}^{\lambda_1}\varepsilon_{\mu_2}^{\lambda_2}\Big[g^{\mu_1\mu_2}(p_2-p_1)^{\mu_3} + g^{\mu_2\mu_3}(-p_{12}-p_2)^{\mu_1} + g^{\mu_3\mu_1}(p_1+p_{12})^{\mu_2}\Big] \\ \times \frac{-g_{\mu_3\nu}}{s_{12}}\widetilde{\mathcal{M}}^{\nu}(p_{12}, \dots)\,. \end{aligned} \tag{6}$$

With the replacement in Eq. (2), the amplitude in Eq.(6) can be written as

$$\mathrm{i}\mathcal{M}(g_1, g_2, \dots) = g\sum_{\lambda}\mathrm{Split}_{\lambda}(g_1^{\lambda_1}, g_2^{\lambda_2})\widetilde{\mathcal{M}}(p^{-\lambda}, \dots) + \mathcal{O}(p_\perp^2) \tag{7}$$

where

$$\mathrm{Split}_{\lambda}(g_1^{\lambda_1}, g_2^{\lambda_2}) = \frac{\varepsilon_{\mu_1}^{\lambda_1}(p_1, r_1)\,\varepsilon_{\mu_2}^{\lambda_2}(p_2, r_2)\,\varepsilon_{\mu_3}^{\lambda}(p, \bar{p})}{s_{12}}\Big[g^{\mu_1\mu_2}\,(p_2-p_1)^{\mu_3} - 2g^{\mu_2\mu_3}p_2^{\mu_1} + 2g^{\mu_3\mu_1}p_1^{\mu_2}\Big]\,. \tag{8}$$

1. Discuss, why in the collinear limit only $r_1 = r_2 = \bar{p}$ is a sensible choice of reference vectors.
2. Compute the collinear limit of $\mathrm{Split}_{\lambda}(g_1^{\lambda_1}, g_2^{\lambda_2})$ in Eq. (8) for all independent helicity configurations. You should find

$$\mathrm{Split}_{+}(g_1^{+}, g_2^{+}) = 0\,, \quad \mathrm{Split}_{-}(g_1^{+}, g_2^{+}) = -\frac{\sqrt{2}}{[12]}\frac{1}{\sqrt{z}\sqrt{1-z}}\,, \tag{9}$$

and

$$\mathrm{Split}_{+}(g_1^{+}, g_2^{-}) = -\frac{\sqrt{2}}{[12]}\frac{(1-z)^2}{\sqrt{z}\sqrt{1-z}}\,, \quad \mathrm{Split}_{-}(g_1^{+}, g_2^{-}) = -\frac{\sqrt{2}}{\langle 12\rangle}\frac{z^2}{\sqrt{z}\sqrt{1-z}}\,. \tag{10}$$

Infrared singularities in the $e^+e^- \to 3\,j$ cross-section

In this exercise, we study the soft and collinear behavior of the process $e^+e^-(p) \to q\bar{q}g$. In particular, we study the singular behavior of the amplitude and observe the appearance of logarithmic singularities in the real-emission cross section.

Computation of the amplitude

Consider the momentum-conserving all-incoming amplitude $e_1^+ + e_2^- + \bar{q}_3 + q_4 + g_5 \to 0$ where $p_{12345} = 0$. Write it as[1]

$$\mathrm{i}\mathcal{M}_{e^+e^-\bar{q}qg} = \mathrm{i}g(\sqrt{2}e)^2 Q_q T^{a_5}_{i_4\bar{i}_3}\mathcal{A}(1_{e^+}^{\lambda_1}, 2_{e^-}^{\lambda_2}, 3_{\bar{q}}^{\lambda_3}, 4_q^{\lambda_4}, 5_g^{\lambda_5}) \tag{11}$$

[1]Recall, that we use the normalisation $\mathrm{Tr}(T^aT^b) = \delta^{ab}$.

1. Draw the Feynman diagrams that contribute to Eq. (11) and convince yourself that

$$\mathcal{A}(1^{\lambda_1},\ldots,5^{\lambda_5}) = \frac{1}{\sqrt{2}}\overline{U}(p_1)\gamma^\mu U(p_2)\,\frac{-g_{\mu\nu}}{s_{12}}\,\overline{U}(p_3)\left\{\frac{\not\epsilon_5 \not p_{35}\gamma^\nu}{s_{35}} + \frac{\gamma^\nu \not p_{45}\not\epsilon_5}{s_{45}}\right\}U(p_4)\,. \tag{12}$$

Using parity and helicity conservation along the massless electron and quark line, we need $2^3/2 = 4$ helicity configurations

$$\begin{aligned} &\mathcal{A}(1^L_{e^+},2^L_{e^-},3^L_{\bar q},4^L_q,5^+_g)\,, \quad &&\mathcal{A}(1^L_{e^+},2^L_{e^-},3^L_{\bar q},4^L_q,5^-_g)\,,\\ &\mathcal{A}(1^L_{e^+},2^L_{e^-},3^R_{\bar q},4^R_q,5^+_g)\,, \quad &&\mathcal{A}(1^L_{e^+},2^L_{e^-},3^R_{\bar q},4^R_q,5^-_g)\,. \end{aligned} \tag{13}$$

2. Confirm that

$$\mathcal{A}(1^L_{e^+},2^L_{e^-},3^L_{\bar q},4^L_q,5^+_g) = \frac{[24]^2}{[12]\,[35]\,[54]}\,, \tag{14}$$

by explicit computation. Hint: It is convenient to choose $r_5 = p_4$, such that the second term in Eq. (12) vanishes.

3. Using general formulas derived in the lecture, confirm that the amplitude $\mathcal{A}(1^L_{e^+},2^L_{e^-},3^L_{\bar q},4^L_q,5^+_g)$ in Eq. (14) has the appropriate behavior in the soft ($p_5 \to 0$) and the collinear ($\boldsymbol{p}_5 \parallel \boldsymbol{p}_3$ and $\boldsymbol{p}_5 \parallel \boldsymbol{p}_4$) limits.

Computation of the amplitude squared

1. The amplitude $\mathcal{A}$ is invariant under charge and parity conjugation,

$$C_f\mathcal{A}(q_f^{\lambda_f},\bar q_{\bar f}^{\lambda_{\bar f}},p_g^{\lambda_g},\ldots) = \mathcal{A}(q_{\bar f}^{-\lambda_f},\bar q_f^{-\lambda_{\bar f}},p_g^{\lambda_g},\ldots)\,, \tag{15}$$

$$P\mathcal{A}(q_f^{\lambda_f},\bar q_{\bar f}^{\lambda_{\bar f}},p_g^{\lambda_g},\ldots) = \mathcal{A}(q_f^{-\lambda_f},\bar q_{\bar f}^{-\lambda_{\bar f}},p_g^{-\lambda_g},\ldots)\,, \tag{16}$$

where C_f is the charge-conjugation operator and P the parity one. Exploit this invariance, together with re-labelling and the result in Eq. (14), to convince yourself that

$$\begin{aligned} \mathcal{A}(1^L_{e^+},2^L_{e^-},3^L_{\bar q},4^L_q,5^-_g) &= \frac{\langle 13\rangle^2}{\langle 21\rangle\,\langle 45\rangle\,\langle 53\rangle}\,,\\ \mathcal{A}(1^L_{e^+},2^L_{e^-},3^R_{\bar q},4^R_q,5^+_g) &= \frac{[23]^2}{[12]\,[45]\,[53]}\,,\\ \mathcal{A}(1^L_{e^+},2^L_{e^-},3^R_{\bar q},4^R_q,5^-_g) &= \frac{\langle 14\rangle^2}{\langle 21\rangle\,\langle 35\rangle\,\langle 54\rangle}\,. \end{aligned} \tag{17}$$

2. Derive the following amplitude squared and summed over polarizations and color

$$|\mathcal{M}_{e^+e^-\bar q q g}|^2 = 48\,g^2e^4Q_q^2C_F\left[\frac{s_{13}^2+s_{23}^2+s_{14}^2+s_{24}^2}{s_{12}s_{35}s_{45}}\right]\,. \tag{18}$$

Computation of the real-emission cross-section (Bonus)

Finally, we turn to the real-emission cross section for the process $e^+(q_1)e^-(q_2) \to q(p_1)\bar{q}(p_2)g(k)$. We will work in the center-of-mass system $Q = q_1 + q_2 = (\sqrt{s}, \mathbf{0})$.

1. Show that the spin-averaged matrix element squared reads

$$\overline{|M|^2} = 6\, g^2 e^4 Q_q^2 C_F \,\frac{(p_1\cdot q_1)^2 + (p_1\cdot q_2)^2 + (p_2\cdot q_1)^2 + (p_2\cdot q_2)^2}{(q_1\cdot q_2)\ (p_1\cdot k)\ (p_2\cdot k)}\,. \tag{19}$$

2. Explain why the integration of the real-emission cross section

$$\sigma^{q\bar{q}g} \equiv \mathcal{N} \int \frac{\mathrm{d}^3\boldsymbol{p}_1}{2E_1}\,\frac{\mathrm{d}^3\boldsymbol{p}_2}{2E_2}\,\frac{\mathrm{d}^3\boldsymbol{k}}{2E_k}\,(2\pi)^4\delta^4(Q - p_1 - p_2 - k)\,\frac{(p_1\cdot q_1)^2 + (p_1\cdot q_2)^2 + (p_2\cdot q_1)^2 + (p_2\cdot q_2)^2}{(q_1\cdot q_2)\ (p_1\cdot k)\ (p_2\cdot k)}\,, \tag{20}$$

   can be parameterized in terms of two energies and three angles.

3. Parameterize the quark energies as $x_{1,2} = 2E_{1,2}/\sqrt{s} = 2\,(Q\cdot p_{1,2})\,/s$ and find the domain of integration for $x_{1,2}$. Use momentum conservation to show that $2\,(p_{1,2}\cdot k) = s(1 - x_{2,1})$.

4. With the parameterization above, derive

$$\sigma^{q\bar{q}g} \equiv \widetilde{\mathcal{N}} \int \mathrm{d}x_1\mathrm{d}x_2\,\frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}\,. \tag{21}$$

5. Describe the type and origin of the singularities in Eq. (21). Are they physical?