Scattering Amplitudes in Quantum Field Theory WS 2021/2

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https://www.groups.ph.tum.de/ttpmath/teaching/ws-2021/

Sheet 07: BCJ relations (03/12/2021)

In the lectures and in exercise sheet 5, we have seen that the four-gluon amplitude $\mathcal{M}gggg(1234)$ can be decomposed into color-ordered gauge-invariant *partial amplitudes* as follows

$$\mathcal{M}gggg(1234) = \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\mathcal{M}(1234) + \mathrm{perm}(234)\,, \tag{1}$$

and that the six color ordered amplitudes $\mathcal{M}(1ijk)$ are not all linearly independent. In particular we have shown that using *cyclicity*, *reversal* and the *photon decoupling identity*, there are only two independent color ordered amplitudes. We have also stated that in the case n-gluons scattering, this statement generalizes and one can prove that there are only $(n-2)!$ independent color-ordered amplitudes.

In this exercise we introduce one extra type of relations, called the BCJ relations, from Bern Carrasco and Johansson [1], which allows us to reduce the number of independent amplitudes to $(n-1)!$. We will do this in the special case of four gluon scattering.

1 New relations among color ordered amplitudes for *gggg* scattering

First of all, we would like to write the scattering amplitude for the scattering of four gluons in a different color basis, ie we write[1]

$$\mathcal{M}gggg(1234) = \frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u} \tag{2}$$

where n_i are numerators that in general depend on the kinematic variables and the polarizations, while the c_i are the three obvious tree-level color factors

$$c_s \equiv f^{a_1a_2b}f^{ba_3a_4}, \quad c_t \equiv f^{a_1a_3b}f^{ba_4a_2}, \quad c_u \equiv f^{a_1a_4b}f^{ba_2a_3}\,. \tag{3}$$

Part a

1. Write each c_i in terms of the three traces $\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})$ and those with ordering 1243, 1324. Check that they satisfy the Jacobi identity

$$c_s + c_t + c_u = 0. \tag{4}$$

2. Use your expressions to convert (2) to a basis with those three traces.

Part b

1. Compare your expressions for $\mathcal{M}gggg(1234)$ and read off the relationship between n_i and the color-ordered amplitudes. For example you should find

$$\mathcal{M}(1234) = -\frac{n_s}{s} + \frac{n_u}{u}. \tag{5}$$

Find the respective relations for $\mathcal{M}(1243)$, $\mathcal{M}(1324)$.

2. Show that it follows directly from these expressions that the color-ordered amplitudes satisfy the $n = 4$ photon decoupling relation.

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[1] Throughout the exercise we use the notation $s = s_{12},\ t = s_{13},\ u = s_{14}$.

Part c

Suppose that there is a choice of the n_i that satisfy the same relation as the color factor c_i,

$$n_s + n_t + n_u = 0. \tag{6}$$

Show that (6) implies

$$t\mathcal{M}(1324) - s\,\mathcal{M}(1234) = 0\,. \tag{7}$$

2 Color-Kinematics duality

In the exercise above, we have derived, without knowing it, an example of a BCJ relation. The BCJ proposal (otherwise known as the Color-Kinematics duality) states that one can always find a representation for the scattering amplitude such that the kinematics factors fulfil dual relations to those fulfilled by the color factors, ie

1. if $c_i = -c_j$ then $n_i = -n_j$
2. the kinematics factors n_i fulfil the "jacoby identity": $c_i + c_j + c_k = 0 \quad \leftrightarrow \quad n_i + n_j + n_k = 0\,.$

Part a

For the four gluon amplitude, choose $\mathcal{M}(1234)$ and $\mathcal{M}(1324)$ as the two independent color-ordered amplitudes. Using your results from exercise (1.b.1), impose the jacoby identity on the n_i and show that the following relation holds, by choosing the independent numerators $\hat{n}_1,\ \hat{n}_2$ among the $\{n_s,\ n_u,\ n_t\}$ appropriately:

$$\begin{pmatrix} \mathcal{M}(1234) \\ \mathcal{M}(1324) \end{pmatrix} = \begin{pmatrix} -\frac{1}{s} & \frac{1}{u} \\ -\frac{1}{t} & -\frac{1}{u} - \frac{1}{t} \end{pmatrix} \begin{pmatrix} \hat{n}_1 \\ \hat{n}_2 \end{pmatrix}. \tag{8}$$

Part b

Re-derive (7) through (8) by imposing the Color-Kinematics duality.

References

[1] Z. Bern, J. J. M. Carrasco and H. Johansson, Phys. Rev. D **78** (2008), 085011 doi:10.1103/PhysRevD.78.085011, arXiv:0805.3993 1