Scattering Amplitudes in Quantum Field Theory WS 2021/2

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https://www.groups.ph.tum.de/ttpmath/teaching/ws-2021/

Sheet 06: Spinor-helicity formalism and color ordering (26/11/2021)

In this exercise, we compute the amplitude squared and summed over helicities for four-gluon scattering at tree level.

1 Trace-based decomposition

In the last exercise sheet, we derived the color-decomposed amplitude[1]

$$\mathcal{M}_{4g}(1,2,3,4) = \text{Tr}(1234)\mathcal{M}(1234) + \text{perm}(234)\,, \tag{1}$$

where

$$\begin{aligned} \mathcal{M}(1^+2^+3^-4^-) &= g^2 \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\,, \\ \mathcal{M}(1^+2^-3^+4^-) &= g^2 \frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\,, \\ \mathcal{M}(1^+2^-3^-4^+) &= g^2 \frac{\langle 14\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\,. \end{aligned} \tag{2}$$

Part a: Color algebra

In the lecture, we have discussed that we can extend $\text{SU}(N) \to \text{U}(N)$ for gluon amplitudes at tree-level, since they do not contain fermions. Hence, we can use $T^a_{ij}T^a_{kl} = \delta_{il}\delta_{kj}$ when manipulating strings of color generators.

1. First, show that the following identities hold

$$\begin{aligned} \text{Tr}(1abc)\text{Tr}(1def)^\dagger &= \text{Tr}(abcfed)\,, \\ \text{Tr}(AT^aBT^aC) &= \text{Tr}(AC)\text{Tr}(B)\,. \end{aligned} \tag{3}$$

2. Using Eq.(3), show that squared traces in $|\mathcal{M}_{4g}(1,2,3,4)|^2$ yield

$$\text{Tr}(1jkl)\text{Tr}(1jkl)^\dagger = N^4\,. \tag{4}$$

3. Using Eq.(3), show that interference terms of traces in $|M_{4g}(1,2,3,4)|^2$ yield

$$\text{Tr}(1jkl)\text{Tr}(1\sigma_{jkl})^\dagger = N^2\,, \tag{5}$$

   for the five permutations σ *without* the identity ($\sigma_{jkl} \neq jkl$).

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[1] We use the convention $\text{Tr}(T^{a_1}\ldots T^{a_n}) = \text{Tr}(1\ldots n)$.

4. Using the results for color traces in Eq.(4) and Eq.(5), we can write

$$\sum_{\text{col}} |M_{4g}(1,2,3,4)| = N^4 \sum_{i=1}^{6} |M_i|^2 + N^2 \sum_{\substack{i,j=1 \\ i \neq j}}^{6} M_i M_j^\dagger \,, \tag{6}$$

where $M_{1..6}$ denote the partial amplitudes. Finally, using the photon decoupling equations, cyclicity and reversal identities, convince yourself that

$$\sum_{\text{col}} |M_{4g}(1,2,3,4)| = 2N^2(N^2-1)\left\{ |\mathcal{M}(1234)|^2 + |\mathcal{M}(1243)|^2 + |\mathcal{M}(1324)|^2 \right\}. \tag{7}$$

Part b: Polarisation sum

1. Compute the sum over helicities for the first partial amplitude in Eq.(7) using the results in Eq.(2). You should find

$$\sum_{\text{pol}} |\mathcal{M}(1234)|^2 = 2g^4 \left[\frac{s_{12}^2}{s_{23}^2} + \frac{s_{13}^4}{s_{12}^2 s_{23}^2} + \frac{s_{23}^2}{s_{12}^2} \right]. \tag{8}$$

2. Use the result in Eq.(8) to compute the final result

$$\sum_{\text{pol,col}} |M_{4g}(1,2,3,4)| = 2g^4 N^2(N^2-1) \times \frac{(s_{12}^2 + s_{13}^2 + s_{14}^2)^3}{s_{12}^2 s_{13}^2 s_{14}^2} \,. \tag{9}$$