Scattering Amplitudes in Quantum Field Theory SS 2022

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https://www.ph.nat.tum.de/ttpmath/teaching/ss-2022/

Sheet 05: Differential equations and iterated integrals
Discussion: 15/07/2022

Simplifying differential equations

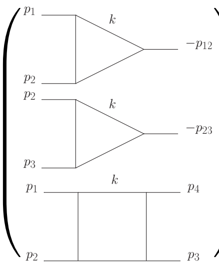
In this exercise we study the massless one-loop box topology[1]

$$\text{box}_{a_1,a_2,a_3,a_4} = \int \frac{\mathrm{d}^{\mathrm{d}}k}{[k^2]^{a_1}\,[(k+p_1)^2]^{a_2}\,[(k+p_{12})^2]^{a_3}\,[(k+p_{123})^2]^{a_4}}\,, \tag{1}$$

as a function of s_{12}, s_{23}, and $s_{31} = -s_{12} - s_{23}$ in $d = 4 - 2\epsilon$ dimensions. Note that we define as usual $s_{ij} = (p_i + p_3)$ and all momenta are considered incoming.

From IBP reduction, we find that this topology has three master integrals; for this exercise we choose

$$\boldsymbol{I} = \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} \text{box}_{1,1,1,0} \\ \text{box}^{x(123)}_{1,1,1,0} \\ \text{box}_{1,1,1,1} \end{pmatrix} = \left(\begin{array}{c} \text{[diagrams]} \end{array} \right). \tag{2}$$



Note that we have defined the crossing $x(123)$ that maps the momenta $\{p_1 \to p_2, p_2 \to p_3, p_3 \to p_1\}$. We derive the corresponding system of differential equations and obtain

$$\frac{\partial}{\partial S}\boldsymbol{I} = \hat{M}_S \boldsymbol{I}\,, \quad S = s_{12}, s_{23}\,, \tag{3}$$

where

$$M_{s_{12}} = \begin{pmatrix} -\frac{1+\epsilon}{s_{12}} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{2\epsilon}{s_{12}s_{23}} - \frac{2\epsilon}{s_{23}(s_{12}+s_{23})} & \frac{2\epsilon}{s_{23}(s_{12}+s_{23})} - \frac{2\epsilon}{s_{12}s_{23}} & \frac{\epsilon}{s_{12}+s_{23}} - \frac{1+\epsilon}{s_{12}} \end{pmatrix}, \tag{4}$$

$$M_{s_{23}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1+\epsilon}{s_{23}} & 0 \\ \frac{2\epsilon}{s_{12}(s_{12}+s_{23})} - \frac{2\epsilon}{s_{12}s_{23}} & \frac{2\epsilon}{s_{12}s_{23}} - \frac{2\epsilon}{s_{12}(s_{12}+s_{23})} & \frac{\epsilon}{s_{12}+s_{23}} - \frac{1+\epsilon}{s_{23}} \end{pmatrix}. \tag{5}$$

To find a basis, in which the differential equation admits an ϵ-homogeneous form, do the following:

1. Compute the maximal cut of each integral, $I^{\text{mc}}_{1,2,3}$ in $d = 4$ dimensions, for example using the Baikov representation.[2]

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[1] For further reference, see e.g. [1].
[2] See also ref. [2].

2. Rescale each master integral by its maximial cut, i.e. consider a new basis

$$\boldsymbol{J} = \text{diag}\{I_1^{\text{mc}}, I_2^{\text{mc}}, I_3^{\text{mc}}\}\boldsymbol{I}\,. \tag{6}$$

Derive the DEQ in the new basis $\boldsymbol{J}$. You should find, that it admits an ϵ-homogeneous form. Hint: For $\partial_x \boldsymbol{I} = A\boldsymbol{I}$ and $I = T\boldsymbol{J}$, the transformed DEQ $\partial_x \boldsymbol{J} = \tilde{A}\boldsymbol{J}$ is given by $\tilde{A} = T^{-1}AT - T^{-1}\partial_x T$.

Bonus

You have seen in class that a canonical basis can be "guessed" from constructing a basis of master integrals with integrands that are dLogs. In the following, we apply this method focusing on $I_{1,3}$. To construct the canonical basis, follow these steps:

1. Express the box integral I_3 in Baikov representation in $d = 4$ and compute the discontinuity in s_{12} (Note that this corresponds to deforming the integration contour and take the residue in the Baikov variables $x_{1,3}$ that correspond to the first and third propagator)

2. Insert a numerator ansatz $c_0 + c_4 x_4$ into the Baikov representation. You should find

$$\sim \int \frac{\mathrm{d}x_2\,\mathrm{d}x_4}{x_2 x_4} \frac{[c_0 + c_4 x_4]}{\sqrt{s_{12}\left(s_{12}\left(4s_{23}^2 - 4s_{23}(x_2+x_4) + (x_2-x_4)^2\right) - 4s_{23}x_2x_4\right)}}\,, \tag{7}$$

where we omitted constant prefactors.

3. Show that the integral in eq. (7) can be written as an iterated integral over dLogs. Determine, which $c_{0,4}$ cancel rational prefactors and compare your result to the first part of the exercise. Hint: Useful primitives read

$$\int \frac{\mathrm{d}x}{x\sqrt{(x-a)(x-b)}} = -\frac{1}{\sqrt{ab}}\log\left(\frac{\sqrt{b(x-a)} - \sqrt{a(x-b)}}{\sqrt{b(x-a)} + \sqrt{a(x-b)}}\right)\,, \tag{8}$$

$$\int \frac{\mathrm{d}x}{\sqrt{(x-a)(x-b)}} = -\log\left(\frac{\sqrt{x-a} - \sqrt{x-b}}{\sqrt{x-a} + \sqrt{x-b}}\right)\,. \tag{9}$$

4. Finally, explain why the chosen numerator ansatz is sufficient and argue that a dLog form for $\text{Disc}_{s_{12}}\{I_3\}$ implies a dLog form for I_3 using dispersion relation.

Chen iterated integrals

In this exercise we prove three useful properties of Chen iterated integrals [3], which are defined as[3]

$$\mathcal{I}_{1,\ldots,n} = \int_\gamma \omega_1 \ldots \omega_n = \int_{0\le t_1 \le \cdots \le t_n} \mathrm{d}t_n f_n(t_n) \ldots \mathrm{d}t_1 f_1(t_1) \tag{10}$$

$$= \int_0^1 \mathrm{d}t_n f_n(t_n) \int_0^{t_n} \mathrm{d}t_{n-1} f_{n-1}(t_{n-1}) \ldots \int_0^{t_2} \mathrm{d}t_1 f_1(t_1)\,, \tag{11}$$

[3] For recent literature, see e.g. ref. [4, 5].

where ω_i are one-folds on a manifold $\mathcal{M}$, $\gamma : [0,1] \to \mathcal{M}$ is the path of integration and we defined the pullback $\gamma^* \omega_i = \mathrm{d}t_i f_i(t_i)$.
For convenience, we define

$$\mathcal{I}_{1,\dots,n}(x) = \int_0^x \mathrm{d}t_n f_n(t_n) \int_0^{t_n} \mathrm{d}t_{n-1} f_{n-1}(t_{n-1}) \dots \int_0^{t_2} \mathrm{d}t_1 f_1(t_1) \,. \tag{12}$$

Hence,

$$\mathcal{I}_{1,\dots,n} = \mathcal{I}_{1,\dots,n}(1) = \int_0^1 \mathrm{d}t_n f_n(t_n) \, \mathcal{I}_{1,\dots,n-1} \,, \tag{13}$$

where the recursion starts at $\mathcal{I} \equiv 1$.

Shuffle relation

The so-called shuffle relation reads

$$\mathcal{I}_{1,\dots,r} \times \mathcal{I}_{r+1,\dots,r+s} = \sum_{\sigma \in \Sigma} \mathcal{I}^{\gamma}_{\sigma(1),\dots,\sigma(r+s)} \,, \tag{14}$$

where $\Sigma = \{\sigma \in S_{r+s} | \sigma(1) < \dots < \sigma(r) \wedge \sigma(r+1 < \dots < \sigma(s)\}$ denotes the set of all so-called shuffles. That means, that Σ is the set of all permutations of indices $1 \dots r+s$, which preserve the individual order $1, \dots, r$ and $r+1, \dots, r+s$.[4]
Prove eq. (14) as follows

1. Show that

$$\begin{aligned} \mathcal{I}_{1,\dots,r} \times \mathcal{I}_{r+1,\dots,r+s} = & \int_0^1 \mathrm{d}t_r f_r(t_r) \, \mathcal{I}_{1,\dots,r-1}(t_r) \, \mathcal{I}_{r+1,\dots,r+s}(t_r) \\ & + \int_0^1 \mathrm{d}t_{r+s} f_{r+s}(t_{r+s}) \, \mathcal{I}_{1,\dots,r}(t_{r+s}) \, \mathcal{I}_{r+1,\dots,r+s-1}(t_{r+s}) \,. \end{aligned} \tag{15}$$

   Hint: Use the fact that

$$\int_0^1 \mathrm{d}x \, \mathrm{d}y \, f(x,y) = \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y \, f(x,y) + \int_0^1 \mathrm{d}y \int_0^y \mathrm{d}x \, f(x,y) \,. \tag{16}$$

2. Argue, that eq. (14) follows from eq. (15) by recursion.

Concatenation

Prove that

$$\int_{\gamma_1 \circ \gamma_2} \omega_1 \dots \omega_n = \sum_{i=1}^{n} \int_{\gamma_1} \omega_1 \dots \omega_i \int_{\gamma_2} \omega_{i+1} \dots \omega_n \,, \tag{17}$$

[4] One can think of a shuffle as a way to slide two decks of cards into each other.

where γ_2 is a path that starts at the end of γ_1, i.e. $\gamma_1(1) = \gamma_2(0)$.
Hint: With $\gamma_{1,2}^* \omega_i = \mathrm{d}t_i f^{(1),(2)}(t_i)$ we can write

$$(\gamma_1 \circ \gamma_2)^* \omega_i = \mathrm{d}t_i \left[f_i^{(1)}(2t_i)\, \theta(1/2 - t_i) + f_i^{(2)}(2t_i - 1)\, \theta(t_i - 1/2) \right] . \tag{18}$$

To prove eq. (17), use eq. (18) on its l.h.s, expand the resulting formula, and disregard all vanishing contributions.

Reversal

Prove that

$$\int_{\gamma^{-1}} \omega_1 \dots \omega_n = (-1)^n \int_\gamma \omega_n \dots \omega_1 \,, \tag{19}$$

where γ^{-1} is the reversed path γ.
Hint: Use that for $\gamma^* \omega_i = \mathrm{d}t_i f_i(t_i)$, we have $(\gamma^{-1})^* \omega_i = \mathrm{d}t_i f_i(1 - t_i)$.

Bonus

Let us focus on a special case of iterated integrals, called multiple polylogarithms, where the integration kernels have simple poles, $f_i(t_i) = 1/t_i - a_i$. We write

$$G_{a_n,\dots,a_1}(x) = \int_0^x \frac{\mathrm{d}t_n}{t_n - a_n}\, G_{a_{n-1},\dots,a_1}(t_n) \,. \tag{20}$$

To accommodate trailing zeros, we define

$$G_{\underbrace{0,\dots,0}_{n}}(x) = \frac{1}{n!} \ln^n(x) \,. \tag{21}$$

Use the shuffle identity to show that

$$\int_0^x \frac{\mathrm{d}t}{t-1} \ln(t) = \mathrm{Li}_2(x) + \ln(x) \ln(x-1) \,. \tag{22}$$

References

[1] J. M. Henn, J. Phys. A **48** (2015), 153001 doi:10.1088/1751-8113/48/15/153001 [arXiv:1412.2296 [hep-ph]].

[2] H. Frellesvig and C. G. Papadopoulos, JHEP **04** (2017), 083 doi:10.1007/JHEP04(2017)083 [arXiv:1701.07356 [hep-ph]].

[3] K. T. Chen, Bull. Am. Math. Soc. **83** (1977), 831-879 doi:10.1090/S0002-9904-1977-14320-6

[4] S. Weinzierl, [arXiv:2201.03593 [hep-th]].

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