Scattering Amplitudes in Quantum Field Theory WS 2021/2

Prof. Lorenzo Tancredi (TUM)

https://www.groups.ph.tum.de/ttpmath/teaching/ws-2021/

Sheet 05: Spinor-helicity formalism and color ordering (19/11/2021)

In this exercise sheet, we compute the four-gluon scattering amplitude at tree level.

Amplitude $gggg$

In this exercises, to follow standard conventions, we choose all momenta to be outgoing, i.e. we consider

$$g_{\lambda_1,a_1}(p_1) \;+\; g_{\lambda_2,a_2}(p_2) \;+\; g_{\lambda_3,a_3}(p_3) \;+\; g_{\lambda_4,a_4}(p_4) \to 0\,, \tag{1}$$

where λ_i and a_i denote helicity and color in the adjoint representation of gluon i. Momentum conservation implies $p_{1234} = 0$.

Part a

Draw all diagrams that contribute to the four-gluon amplitude $\mathcal{M}_{gggg}(p_1, p_2, p_3, p_4)$.

Part b

1. Show that cyclicity of the color trace implies that the amplitude can be decomposed into color-ordered gauge-invariant *partial amplitudes* as follows[1]

$$\mathcal{M}_{gggg}(p_1, p_2, p_3, p_4) = \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\mathcal{M}(1234) + \mathrm{perm}(234)\,. \tag{2}$$

2. As we have seen in class, partial amplitudes are gauge invariant and invariant under *cyclic permutation* of arguments. Furhtermore, they obey *reversal identities* and the *photon decoupling equation*. Show that taking these identities into account, there are only two independent color ordered partial amplitudes, which we can choose as $\mathcal{M}(1234)$ and $\mathcal{M}(1243)$. Express all other color ordered partial amplitudes through $\mathcal{M}(1234)$ and $\mathcal{M}(1243)$.

Part c

Let us pick one of the two color ordered amplitudes, say $\mathcal{M}(1234)$, and fix the helicities of the external gluons $\mathcal{M}(1^{\lambda_1}2^{\lambda_2}3^{\lambda_3}4^{\lambda_4})$. How many helicity configurations are there in total?

1. We have seen in class that all helicity amplitudes with equal helicities and with only one different helicity are zero. Justify that this implies that there are only two helicity configurations that we need to compute, which we can choose as $\mathcal{M}(1^+2^+3^-4^-)$, $\mathcal{M}(1^+2^-3^+4^-)$, i.e. the different helicities can be adjacent or not.

2. Using the identities among color ordered amplitudes derived above, show that

$$\mathcal{M}(1^+2^-3^+4^-) = -\mathcal{M}(1^+3^+2^-4^-) - \mathcal{M}(1^+3^+4^-2^-)\,, \tag{3}$$

which implies in turn that the configuration with non adjacent equal helicities can be obtained from the adjacent one, by permutation of the external legs.

---

[1] According to the lecture, we rescale $T^a = \sqrt{2}t^a$, such that $\mathrm{Tr}(T^aT^b) = \delta^{ab}$ and $[T^a, T^b] = \mathrm{i}\sqrt{2}f^{abc}T^c$.

Part d

1. Using the color ordered feynman rules below, compute explicitly

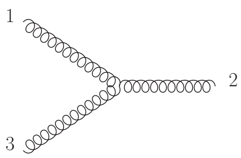
$$\mathcal{M}(1^+2^+3^-4^-) = g^2 \frac{\langle 12\rangle^4}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 41\rangle} \,. \tag{4}$$

2. Using eq. (3), show that

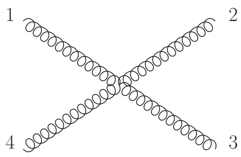
$$\mathcal{M}(1^+2^-3^+4^-) = g^2 \frac{\langle 13\rangle^4}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 41\rangle} \,. \tag{5}$$

Color-ordered Feynman rules

We use the following set of color-ordered Feynman rules



$$= \frac{\mathrm{i}g}{\sqrt{2}} \left[g^{\mu_1\mu_2}(p_1 - p_2)^{\mu_3} + \mathrm{cycl}(123) \right] , \tag{6}$$



$$= \frac{\mathrm{i}g^2}{2} \left[2g^{\mu_1\mu_3}g^{\mu_2\mu_4} - g^{\mu_1\mu_2}g^{\mu_3\mu_4} - g^{\mu_1\mu_4}g^{\mu_2\mu_3} \right] , \tag{7}$$

where all momenta are incoming.