Scattering Amplitudes in Quantum Field Theory WS 2021

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https://www.groups.ph.tum.de/ttpmath/teaching/ws-2021/

Sheet 03: Spinor helicity formalism for massive quarks (5/11/2021)

In this exercise sheet, we will show how the spinor helicity formalism can be applied to massive quarks.

Exercise 1 - Spinor-Helicity with Massive quarks

Consider a top quark with momentum p_t such that $p_t^2 = m_t^2$. The top quark spinor $u_t(p_t)$ satisfies the Dirac equation

$$(\not{p}_t - m_t)u_t(p_t) = 0. \tag{1}$$

In order to use the spinor helicity formalism, we start by writing the top momentum as linear combination of two light-like momenta, i.e. $p_t = p_1 + p_\eta$, with $p_1^2 = 0, p_\eta^2 = 0, 2p_1 \cdot p_\eta = m_t^2, 2p_t \cdot p_\eta = m_t^2$.

Part a

A massive fermion can have two different polarisations which we will call $u_{t,\pm}(p_t)$. Argue that we can decompose the top quark spinor $u_t(p_t)$ for its two different polarisations in terms of two massless spinors as follows

$$\begin{aligned} u_{t,+}(p_t) &= \alpha\, |1\rangle + \beta\, |\eta] \\ u_{t,-}(p_t) &= \gamma\, |1] + \delta\, |\eta\rangle\,. \end{aligned} \tag{2}$$

Part b

Use the fact that the spinors in eq. (2) must satisfy the Dirac equation to find relations among $\alpha, \beta, \gamma, \delta$.

Part c

We can write the spinors in eq. (2) in the following compact form:

$$u_{t,+}(p_t) = N_+(\not{p}_t + m_t)\, |\eta]\,, \quad u_{t,-}(p_t) = N_-(\not{p}_t + m_t)\, |\eta\rangle \tag{3}$$

$$u_{t,+}(p_t) = \tilde{N}_+(\not{p}_t + m_t)\, |1]\,, \quad u_{t,-}(p_t) = \tilde{N}_-(\not{p}_t + m_t)\, |1\rangle\,. \tag{4}$$

Justify this remark.

Part d

Given that the sum over polarisations should give the correct density matrix,

$$\sum_\lambda u_{\lambda,t}(p)\bar{u}_{\lambda,t}(p) = (\not{p}_t + m_t) \tag{5}$$

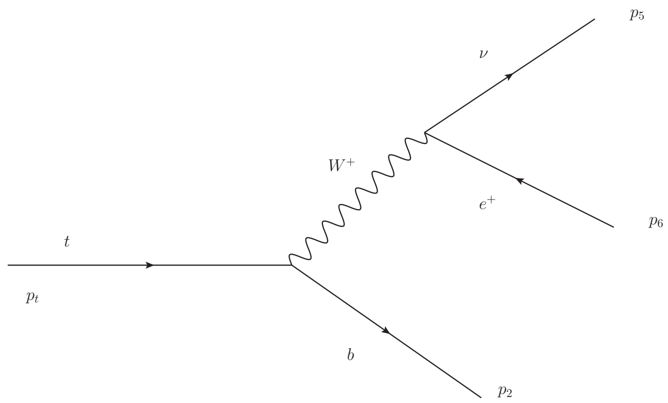
show that

$$u_{t,+}(p_t) = |1\rangle + \frac{m_t\, |\eta]}{[1\eta]}, \quad u_{t,-}(p_t) = |1] + \frac{m_t\, |\eta\rangle}{\langle 1\eta\rangle}. \tag{6}$$

Hint: Assume that the normalisation constants are independent of polarisations.

Exercise 2 - Top quark decay

Consider the semileptonic decay of the top quark $t \to bW(e^+\nu)$.



where we assume all fermions to be massless except the top quark.

Part a

Use the electroweak Feynman rules to write down an expression of the amplitude associated to this Feynman diagram. Which helicity configurations are allowed for the massless fermions?

Part b

Show that the amplitude can be written as

$$\mathcal{M}_\lambda = -ig_W^2 D_W \mathcal{A}\, \delta_{i_b,i_t} \quad \text{with} \quad \mathcal{A}_\lambda = \langle 5| \gamma^\mu |6]\, \langle 2| \gamma^\mu u_{t,\lambda}(p_t), \tag{7}$$

where g_W is the electroweak gauge coupling, $D_W = i/\big(s - M_W^2 + iM_W\Gamma_W\big)$, $\lambda = \pm$ labels the top quark polarisation and i_b, i_t are the color indices of the quarks.

Using eq.(6) show that

$$\mathcal{A}_+ = 2\,\langle 25\rangle\, \frac{[6\eta]}{[1\eta]} m_t, \quad \mathcal{A}_- = 2\,\langle 25\rangle\,[61]\,. \tag{8}$$

Part c

Calculate the sum over helicities of the squared amplitude $\sum_\lambda |\mathcal{A}_\lambda|^2$ and discuss your result.