Scattering Amplitudes in Quantum Field Theory WS 2021

Prof. Lorenzo Tancredi

https://www.groups.ph.tum.de/ttpmath/teaching/ws-2021/

Sheet 02: Spinor helicity formalism (29/10/2021)

In this exercise sheet, we establish familiarity with the spinor-helicity formalism and derive several relations that are useful to simplify results when computing scattering amplitudes.

Exercise 1 - Massless spin-1/2 fermions

Part a

As a first step, we derive explicit representations for spinors $u_L(p), u_R(p)$ in the Weyl representation. Recall, that in the Weyl representation

$$\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}, \quad \gamma_5 = \begin{bmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{bmatrix}, \tag{1}$$

where $\sigma_\mu = (\mathbb{1}, \boldsymbol{\sigma})$ and $\bar{\sigma}_\mu = (\mathbb{1}, -\boldsymbol{\sigma})$.
Start from the massless Dirac equation in momentum space

$$p_\mu \gamma^\mu U(p) = 0\,, \quad U = N_p \begin{pmatrix} u_L(p) \\ u_R(p) \end{pmatrix}, \tag{2}$$

which can be written as

$$u_R(p) = \frac{\boldsymbol{\sigma p}}{|\boldsymbol{p}|} u_R(p) \tag{3}$$

$$u_L(p) = -\frac{\boldsymbol{\sigma p}}{|\boldsymbol{p}|} u_L(p)\,. \tag{4}$$

Show that the quantity

$$u_R(p) \equiv \mathrm{i}\sigma_2 \big(u_L(p)\big)^*\,, \tag{5}$$

solves Eq.(3), assuming that $u_L(p)$ solves Eq.(4).
Focusing on the left-handed spinor, we write Eq.(4) as

$$\boldsymbol{n\sigma} \begin{pmatrix} a \\ b \end{pmatrix} = - \begin{pmatrix} a \\ b \end{pmatrix}, \tag{6}$$

where we used

$$u_L = \begin{pmatrix} a \\ b \end{pmatrix}, \quad \boldsymbol{n} = \frac{\boldsymbol{p}}{|\boldsymbol{p}|} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}. \tag{7}$$

Solve this equation, together with the completeness relation[1]

$$\sum_{\lambda\in\{L,R\}} U_\lambda(p)\otimes \bar{U}_\lambda(p) = p_\mu \gamma^\mu\,, \tag{8}$$

to show that

$$u_L(p) = \begin{pmatrix} -\sin\frac{\theta}{2}e^{-\mathrm{i}\frac{\phi}{2}} \\ \cos\frac{\theta}{2}e^{\mathrm{i}\frac{\phi}{2}} \end{pmatrix}\,, \qquad N_p = \sqrt{2p_0}\,. \tag{9}$$

Discuss how the spinor transforms under the little group associated to its momentum p^μ.

Part b

For the remainder of this exercise, we identify

$$U_L = \sqrt{2p_0}\begin{pmatrix} u_a \\ 0 \end{pmatrix} \equiv |p]\ ,\ \ U_R = \sqrt{2p_0}\begin{pmatrix} 0 \\ u^{\dot a} \end{pmatrix} \equiv |p\rangle\ , \tag{10}$$

$$\bar{U}_L = \sqrt{2p_0}\begin{pmatrix} 0 & u_{\dot a} \end{pmatrix} \equiv \langle p|\ ,\ \ \bar{U}_R = \sqrt{2p_0}\begin{pmatrix} u^a & 0 \end{pmatrix} \equiv [p|\ . \tag{11}$$

1. Derive the identities

$$\langle p|\,\gamma^\mu\,|p] = [p|\,\gamma^\mu\,|p\rangle = 2p^\mu\,, \tag{12}$$

   and

$$\left(\,[p|\,\gamma^\mu\,|q\rangle\,\right)^* = \langle p|\,\gamma^\mu\,|q]\ . \tag{13}$$

2. Prove the relation

$$\sigma_2\sigma^\mu\sigma_2 = (\bar\sigma^\mu)^T \tag{14}$$

3. Use Eq.(14) to prove, that for an odd number of gamma matrices, the following equation holds

$$\langle p|\prod_{i=1}^{2n+1}\gamma^{\mu_i}\,|q] = [q|\prod_{i=2n+1}^{1}\gamma^{\mu_i}\,|p\rangle\ . \tag{15}$$

   In particular, Eq.(15) implies that $\langle p|\,\gamma^\mu\,|q] = [q|\,\gamma^\mu\,|p\rangle$.

4. Use Eq.(14) again, to prove the equivalent identities

$$\langle p|\prod_{i=1}^{2n}\gamma^{\mu_i}\,|q\rangle = -\,\langle q|\prod_{i=2n}^{1}\gamma^{\mu_i}\,|p\rangle\ , \tag{16}$$

$$[p|\prod_{i=1}^{2n}\gamma^{\mu_i}\,|q] = -\,[q|\prod_{i=2n}^{1}\gamma^{\mu_i}\,|p]\ , \tag{17}$$

   for an even number of gamma matrices.

[1] We use $\otimes$ to denote the outer product, the Dirac conjugate reads $\bar{U} = U^\dagger\gamma^0$.

5. Argue, why the following expressions evaluate to zero

$$0 = \langle p| \prod_{i=1}^{2n} \gamma^{\mu_i} |q] = \langle p| \prod_{i=1}^{2n+1} \gamma^{\mu_i} |q\rangle = [p| \prod_{i=1}^{2n+1} \gamma^{\mu_i} |q] \, . \tag{18}$$

6. Prove that

$$(\sigma^\mu)_{a\dot{a}} (\sigma_\mu)_{b\dot{b}} = 2 \left(\mathrm{i}\sigma_2\right)_{ab} \left(\mathrm{i}\sigma_2\right)_{\dot{a}\dot{b}} = 2\varepsilon_{ab}\varepsilon_{\dot{a}\dot{b}} \, , \tag{19}$$

and use this result to derive the Fierz identity

$$\langle p| \gamma^\mu |q] \langle k| \gamma_\mu |l] = 2 \langle pk\rangle [lq] \, . \tag{20}$$

7. Derive the Schouten identities

$$\begin{aligned} \langle ij\rangle \langle kl\rangle + \langle ik\rangle \langle lj\rangle + \langle il\rangle \langle jk\rangle = 0 \\ [ij] [kl] + [ik] [lj] + [il] [jk] = 0 \end{aligned} \tag{21}$$

Hint: Write $\langle ij\rangle \langle kl\rangle + \langle ik\rangle \langle lj\rangle + \langle il\rangle \langle jk\rangle = \langle iq\rangle$ and argue that $|q\rangle = 0$.

Exercise 2 - Massless spin-one bosons

In this exercise, we use the representation

$$\varepsilon_+^\mu(p,r) = -\frac{[r| \gamma^\mu |p\rangle}{\sqrt{2}\,[rp]} \, , \quad \varepsilon_-^\mu(p,r) = \frac{\langle r| \gamma^\mu |p]}{\sqrt{2}\,\langle rp\rangle} \, , \tag{22}$$

for the two polarisation vectors of a massless spin-one particle with momentum p^μ.[2]

Part a

Show that this representation satisfies transversality

$$p_\mu \, \varepsilon_\pm^\mu(p,r) = r_\mu \, \varepsilon_\pm^\mu(p,r) = 0 \tag{23}$$

and normalisation

$$\left[\varepsilon_\pm^\mu(p,r)\right]^* \varepsilon_{\pm,\mu}(p,r) = -1 \, . \tag{24}$$

Part b

Prove the relation

$$\sum_{\lambda\in\{+,-\}} \left[\varepsilon_\lambda^\mu(p,r)\right]^* \varepsilon_\lambda^\nu(p,r) = -g^{\mu\nu} + \frac{p^\mu r^\nu + p^\nu r^\mu}{p\cdot r} \, . \tag{25}$$

[2]Recall, that we are working in the axial gauge $r^2 = r_\mu A^\mu = 0$.

Part c

Prove that for two vector bosons with the same reference vector r,

$$\varepsilon^{\mu}_{\pm}(p,r)\,\varepsilon_{\pm,\mu}(q,r) = 0\,. \tag{26}$$

Part d

In the case where the reference vector of one vector boson is the momentum of an other, prove that

$$\varepsilon^{\mu}_{+}(p,q)\,\varepsilon_{-,\mu}(q,r) = \varepsilon^{\mu}_{+}(p,r)\,\varepsilon_{-,\mu}(q,p) = 0\,. \tag{27}$$

Part e (Bonus)

Derive polarisation vectors in the two-dimensional spinor representation, i.e. compute

$$\varepsilon^{\dot{a}a}_{\pm} = \varepsilon^{\mu}_{\pm}\,(\sigma_{\mu})^{\dot{a}a}\;. \tag{28}$$