Scattering Amplitudes in Quantum Field Theory SS 2022

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https://www.ph.nat.tum.de/ttpmath/teaching/ss-2022/

Sheet 02: Box coefficients of a five-point one-loop gluon amplitude

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Recap of integrand reduction using unitarity cuts

In the lectures we have seen that one-loop n-point amplitudes up to $\mathcal{O}(\epsilon^0)$ can be reduced to a linear combination of basis integrals, which involves scalar boxes, triangles, bubbles and tadpoles. The coefficients of these so-called "master integrals" can be computed from tree-level on-shell amplitudes that result after performing generalised "unitarity cuts" on the original amplitude.

As a non-trivial application, in this exercise we will consider the computation of the coefficients of the boxes that appear in the reduction of the one-loop five-gluon amplitude, $A_5^{\text{1-loop}}(1^+,\ 2^+,\ 3^-,\ 4^-,\ 5^-)$.

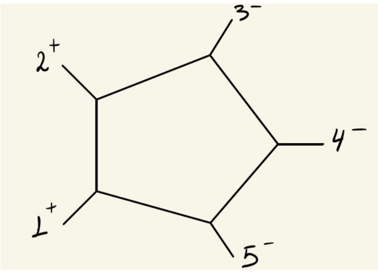

Figure 1: All momenta are considered incoming.

Box contributions to one-loop n-point amplitudes

Before working on the five-point amplitude, it is instructive to derive some general results concerning box integrals and how they contribute to a generic one-loop n-point amplitude.

1. Consider the decomposition of a tensor 4-point integral,

$$\int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{\prod_{j=1}^{r}(l\cdot u_j)}{D_0 D_1 D_2 D_3} = d \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{1}{D_0 D_1 D_2 D_3} + \sum_{n=1}^{4} d_n \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{(l\cdot n_4)^n}{D_0 D_1 D_2 D_3}$$
$$+ \text{ lower-point integrals.}$$

   where,

$$D_0 = l^2,\ D_1 = (l+q_1)^2,\ D_2 = (l+q_2)^2,\ D_3 = (l+q_3)^2 \tag{1}$$

   and q_i are the so-called region momenta. Argue that at the integrand level *in 4 space-time dimensions* (i.e. dropping all contributions proportional to $(l\cdot n_\epsilon)$ in the integrand), the above expression will have only the following contributions, modulo lower point integrals

$$\frac{d_{0123}(l)}{D_0 D_1 D_2 D_3} = \frac{d + \tilde{d}(l\cdot n_4)}{D_0 D_1 D_2 D_3},\ \text{with } \tilde{d} = d_1 \tag{2}$$

i.e., we do not need to keep any higher powers of $(l \cdot n_4)$. Note that, since we are working at *the integrand level*, we cannot neglect $(l \cdot n_4)$, since it would only drop after one performs the integration over the transverse space!

2. For a generic n-point one-loop amplitude, at *the integrand level* we have the following decomposition

$$A_n^{\text{one-loop}} = \sum_{0 \le i_1 < i_2 < i_3 < i_4 < n} \frac{d_{i_1 i_2 i_3 i_4}(l)}{D_{i_1} D_{i_2} D_{i_3} D_{i_4}} + \text{lower point contributions.} \tag{3}$$

We can focus on the contribution of a specific box and write the previous formula as,

$$A_n^{\text{one-loop}} = \frac{d_{0123}(l)}{D_0 D_1 D_2 D_3} + \text{other boxes} + \text{lower point contributions.} \tag{4}$$

In order to compute the coefficient d_{0123} we will use unitarity cuts. The idea is to perform a quadruple cut in both sides of (4) in such a way that we can isolate d_{0123}.

(a) For the right-hand-side of (4), working on the box with coefficient d_{0123}, use the Van Neerven-Vermaseren decomposition for the loop momentum, using its three region momenta q_1, q_2, q_3 and any extra momentum n_4, to show that this *freezes* all components of the loop momentum l^μ to the two solutions

$$\bar{l}^\mu_\pm = -\frac{1}{2}\sum_{i=1}^{3} q_i^2 v^\mu \pm \frac{1}{2}\sqrt{-\left(q_1^2 v_1^\mu + q_2^2 v_2^\mu + q_3^2 v_3^\mu\right)^2} n_4^\mu. \tag{5}$$

(b) On the left-hand-side of (4), the quadruple cut can be written as the product of four tree-level amplitudes, i.e. $A_1^{\text{tree}}(\bar{l}_\pm) A_2^{\text{tree}}(\bar{l}_\pm) A_3^{\text{tree}}(\bar{l}_\pm) A_4^{\text{tree}}(\bar{l}_\pm)$. This will be equal to d_{0123}, which you have shown in previous steps to be $d_{0234} = d + \tilde{d}(l \cdot n_4)$. Show that the scalar box coefficient d can be written as,

$$d = \frac{D_+ + D_-}{2} \tag{6}$$

with

$$D_\pm = A_1^{\text{tree}}(\bar{l}_\pm) A_2^{\text{tree}}(\bar{l}_\pm) A_3^{\text{tree}}(\bar{l}_\pm) A_4^{\text{tree}}(\bar{l}_\pm). \tag{7}$$

What is the corresponding formula for $\tilde{d}$? Do we need to compute it and if yes, for what?

Box coefficients of $A_5^{\textbf{1-loop}}(1^+,\ 2^+,\ 3^-,\ 4^-,\ 5^-)$

1. Starting from figure 1, convince yourselves that the contributing boxes are

$$I(s_{12}),\ I(s_{23}),\ I(s_{34}),\ I(s_{45}),\ I(s_{51}),$$

where $I(s_{ij})$ represents the box - integral with $2p_i \cdot p_j$ as the massive leg.

2. We write for the amplitude at *the integral level*

$$A_5^{\text{1-loop}}(1^+,\ 2^+,\ 3^-,\ 4^-,\ 5^-) = d_{12}I(s_{12}) + d_{23}I(s_{23}) + d_{34}I(s_{34}) \\ + d_{45}I(s_{45}) + d_{51}I(s_{51}) + \text{lower point integrals} \tag{8}$$

and we focus on the computation of the coefficient of $I(s_{12})$, which we denote as d_{12} following the notation in eq. (2). The $I(s_{12})$ scalar integral is defined as

$$I(s_{12}) = \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{1}{l^2(l+p_{12})^2(l+p_{123})^2(l+p_{1234})^2} = \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{1}{l^2(l+q_2)^2(l+q_3)^2(l+q_4)^2} \tag{9}$$

with p_i denoting the inflow momenta, q_i the region-momenta and $p_{ijk} = p_i + p_j + p_k$. The diagram associated with (9) is shown in figure 2.

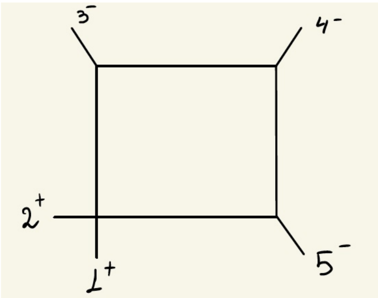

Figure 2: All momenta are considered incoming.

Now we'll use eq. (2) and the general decomposition for the amplitude (8), evaluated on the quadruple cut eq. (5), to fix the coefficients $d_{12}, \tilde{d}_{12}$, working only at the *integrand level.*

(a) Consider the helicity amplitudes that result from the quadruple cut of $A_5^{\text{1-loop}}(1^+,\ 2^+,\ 3^-,\ 4^-,\ 5^-)$ in such a way as to isolate the contribution of $I(s_{12})$. What choices are allowed for the helicities of the resulting tree amplitudes?

(b) After fixing the helicities, enforce the cut relations on the propagators and fix the loop momentum. For convenience you can define $l_i = l + q_i$, for $i = 1 \ldots 5$, with $q_5 = 0$ and solve the cut relations for an appropriate choice of l_i instead of l.

(c) Having fixed the loop momentum in the previous step, use (7) to compute $D_\pm$ in terms of specific helicity amplitudes and finally use (6) to obtain the box coefficient d_{12}.