Scattering Amplitudes in Quantum Field Theory WS 2021
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https://www.groups.ph.tum.de/ttpmath/teaching/ws-2021/
Sheet 01: The Lorentz group and its representations (22/10/2021)

Exercise 1 - The $(\frac{1}{2}, \frac{1}{2})$ representation of the Lorentz group

In this exercise, we will show how to obtain a vector representation of SO(1, 3) as a direct product of spin-$\frac{1}{2}$ representation of SU(2). We start from the fact that the Lorentz algebra so(1, 3)

$$\begin{aligned} [J_i, J_j] &= \mathrm{i}\,\epsilon_{ijk} J_k\,, \\ [J_i, K_j] &= \mathrm{i}\,\epsilon_{ijk} K_k\,, \\ [K_i, K_j] &= -\mathrm{i}\,\epsilon_{ijk} J_k\,, \end{aligned} \tag{1}$$

which is equivalent to su(2) ⊗ su(2). Through the identification $N_i^{\pm} = \frac{1}{2}\left(J_i \pm \mathrm{i}K_i\right)$, we obtain

$$\begin{aligned} [N_i^{\pm}, N_j^{\pm}] &= \mathrm{i}\,\epsilon_{ijk} N_k^{\pm}\,, \\ [N_i^{+}, N_j^{-}] &= 0\,, \end{aligned} \tag{2}$$

where N_i^+ (N_i^-) are the generators of left-handed (right-handed) SU(2) transformations.
To find the vector representation $(\frac{1}{2}, \frac{1}{2})$, we construct a state with two indices, $\nu^{\dot{a}b}$, which has a dotted, left-handed index $\dot{a}$ that transforms under N^+ and a undotted, right-handed index b that transforms under N^-. Since generators $N^{\pm}$ are Hermitian, we limit the discussion to the case where the object $\nu^{\dot{a}b}$ is a Hermitian 2×2 complex matrix.

Part a

Derive $\mathrm{SU_L}(2)$ and $\mathrm{SU_R}(2)$ transformations on the dotted and undotted indices of $\nu^{\dot{a}b}$. You should find that

$$\nu^{\dot{a}b} \to (\nu')^{\dot{c}d} = \left[e^{\frac{\mathrm{i}}{2}\boldsymbol{\theta}\boldsymbol{\sigma} - \frac{1}{2}\boldsymbol{\phi}\boldsymbol{\sigma}}\right]^{\dot{c}}_{\ \dot{a}} \nu^{\dot{a}b} \left[e^{\frac{\mathrm{i}}{2}\boldsymbol{\theta}\boldsymbol{\sigma} + \frac{1}{2}\boldsymbol{\phi}\boldsymbol{\sigma}}\right]^{d}_{\ b}\,. \tag{3}$$

<u>Hint</u>: start from the Lorentz transformation

$$L = e^{\mathrm{i}\boldsymbol{\theta}\boldsymbol{J} + \mathrm{i}\boldsymbol{\phi}\boldsymbol{K}}\,, \tag{4}$$

where $\boldsymbol{\theta}$ and $\boldsymbol{\phi}$ are rotation and boost parameters, respectively. Use the definition of $N^{\pm}$ together with the fact that the dotted (undotted) index should transform under a left-handed (right-handed) SU(2) transformations.

Part b

Next, we write

$$\nu^{\dot{a}b} = \nu^{\mu}\left(\sigma_{\mu}\right)^{\dot{a}b}\,, \tag{5}$$

where $\sigma_0 = \mathbb{1}\,, \sigma_i = \sigma^{(i)}$, are the identity and the Pauli matrices, respectively. Argue, that $\nu^{\dot{a}b}$ in Eq.(5) is indeed a Hermitian, but otherwise completely general 2×2 complex matrix as required.

Part c

Expand the r.h.s. of this Eq.(3) to first order in the parameters $\boldsymbol{\theta}, \boldsymbol{\phi}$ and write the result in the form of Eq.(5). Then, show that the new coefficients $(\nu')^\mu$ are related to ν^μ via a Lorentz transformation.

Part d

Compute the determinant

$$\det\left(\nu^{\dot{a}b}\right) \,, \tag{6}$$

and discuss the result.

Exercise 2 - A geometric picture of spin

The goal of this exercise is to provide a geometrical picture of what it means to "perform a rotation in spinor space".

As a starting point, take a sphere and think of it as a null cone of light rays emitted from the origin

$$-t^2 + x^2 + y^2 + z^2 = 0 \Rightarrow x^2 + y^2 + z^2 = t^2. \tag{7}$$

For $t = 1$ we get the unit sphere. In the usual spherical coordinates this looks like figure 1[1].

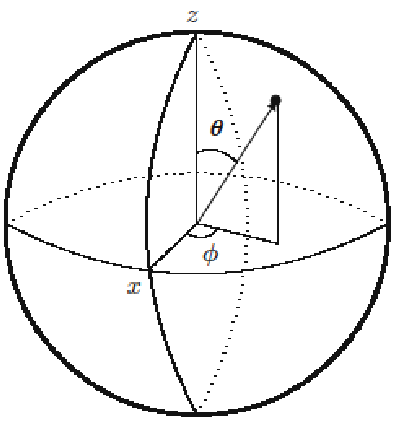

Figure 1: Unit sphere

We now superimpose the complex plane with the $z = 0$ plane, such that the x axis ($\theta = \pi/2$, $\phi = 0$) corresponds to the real axis and the y axis ($\theta = \pi/2$, $\phi = \pi/2$) with the imaginary one, see fig. 2.
Let us now pick any point on the unit sphere and connect it with the point at the north pole ($\theta = 0$), and then extend this line as shown in figure 3. This line will intersect the complex plane at some point $\tilde{z} = a + ib$, with $a, b \in \mathbb{R}$.[2] In polar coordinates the point $\tilde{z}$ is

[1] All figures are taken from *Symmetry and the Standard Model*, by Matthew Robinson, Springer.
[2] In the picture the point $\tilde{z}$ is depicted as z. To avoid confusion with the z axis we will denote it as $\tilde{z}$.

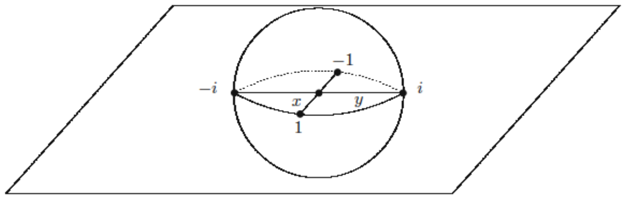

Figure 2: Unite sphere superimposed on the complex plane.

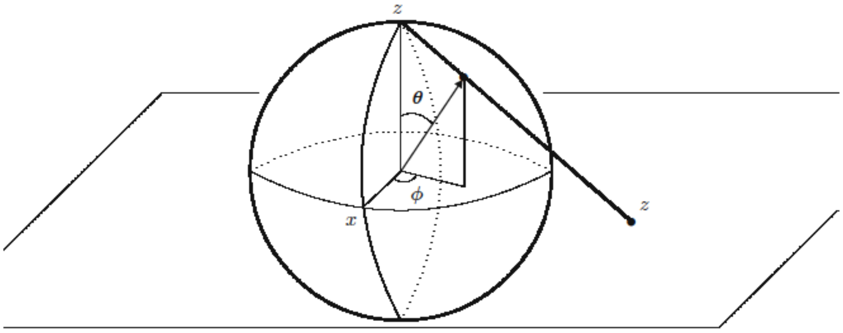

Figure 3: Stereographic projection

$$\tilde{z} = r\mathrm{e}^{i\phi} \tag{8}$$

Note that the angle ϕ is the same as in our spherical coordinates on the unit sphere and $r = r(\theta)$.

Part 1

1. Show explicitly that $r = \cot\left(\frac{\theta}{2}\right)$. This provides the so-called *stereographic projection*, which maps every point (θ, ϕ) on the unit sphere in space, to the point $\tilde{z} = \mathrm{e}^{i\phi}\cot\left(\frac{\theta}{2}\right)$ on the complex plan.

2. Rewrite $\tilde{z}$ as a ratio of two other complex numbers α, γ, by imposing $\alpha \in \mathbb{R}$ and $\alpha \geq 0$ (why can we assume this and still be completely general?). Write their explicit expressions and show that the complex point $\tilde{z}$ uniquely specifies two complex points.

Part 2

1. With your newly defined α, β, consider the map

$$(\theta, \phi) \to \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} cos(\theta/2) \\ \mathrm{e}^{-i\phi}\sin(\theta/2) \end{pmatrix} \tag{9}$$

and argue that

$$\begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = \psi_R^*$$

should be interpreted as a right-handed spinor, which we write as

$$\psi_R = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad \beta = \gamma^* .$$

Hint: In order to justify the ψ_R is a right-handed spinor, study what the positive and negative z directions, i.e. $(\theta, \phi) = \{(0,0), (\pi, 0)\}$, correspond to in the spinor space that we have just defined with this map.

2. Discuss the relation between the "spinor space" defined above and the usual representation in quantum mechanics

$$\psi_R = \cos(\theta/2) \,|\uparrow>_z + e^{i\phi} \sin(\theta/2) |\downarrow>_z \tag{10}$$

Part 3

Using the latest definitions for $\alpha \& \beta$, verify the following relations:

1. $t = \alpha\alpha^* + \beta\beta^*$
2. $x = \alpha\beta^* + \beta\alpha^*$
3. $y = i(\alpha\beta^* - \beta\alpha^*)$
4. $z = \alpha\alpha^* - \beta\beta^*$

With these relations, consider the matrix $A = \begin{pmatrix} \alpha\alpha^* & \alpha^*\beta \\ \alpha\beta^* & \beta\beta^* \end{pmatrix}$. Can you work out a relation between this matrix and the Pauli matrices? Discuss your result.

Part 4

To study the chirality of ψ_R proceed as follows:

1. Act on ψ_R with the right-handed spinor rotation of angle γ around the x axis, $R_x(\gamma)$, and calculate the corresponding spacetime transformation (t', x', y', z'). Discuss the result.
2. Act with the left-handed spinor boost transformation on ψ_R and show that it corresponds to the wrong spacetime boost transformation. What does that tell you about ψ_R?

Part 5

Now we have a right-handed spinor. To obtain a left-handed one, we can perform a parity trasformation

$$(x, y, z) \to (-x, -y, -z) \Rightarrow (\theta, \phi) \to (\theta', \phi') = (\pi - \theta, \phi + \pi) \tag{11}$$

1. Apply the above transformations to $\tilde{z}(\theta, \phi) \to \tilde{z}'(\theta', \phi')$. Show that $\tilde{z}' = -1/\tilde{z}^*$.

2. Check your results against the following definition of a left-handed spinor:

$$\psi_L = \begin{pmatrix} e^{-i\phi} \sin(\theta/2) \\ -\cos(\theta/2) \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = e^{-i\phi} \sin(\theta/2) \, | \uparrow>_z - \cos(\theta/2) \, | \downarrow>_z \tag{12}$$

3. Using these new definitions of $\alpha \,\&\, \beta$ for left-handed spinors, show that the relations of **Part 3** hold under a parity transformation.

4. Apply the left handed spinor transformation on ψ_L. Compare your results with **Part 4**.

Part 6

By now you should be convinced that we have a geometric understanding of what a right-handed and a left-handed spinor is. To conclude, starting from the fact that $\tilde{z} \to -1/\tilde{z}^*$ and $\tilde{z} = \alpha/\beta$ show that

$$-\frac{\beta^*}{\alpha^*} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}^* = i\sigma^2 \begin{pmatrix} \alpha \\ \beta \end{pmatrix}^* \tag{13}$$

Explain why a full rotation in space results in an overall minus sign in spinor space.