Scattering Amplitudes in Quantum Field Theory SS 2022

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Sheet 01: Photon mass in the Schwinger model
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Warm up on Schouten Identities in $D = 2$ dimensions

Let us define three D-dimensional vectors p^μ, q^μ, and l^μ, with $D \in \mathbb{N}$. Let us also define the Levi-Civita tensor in $D = 2$ as $\epsilon^{\mu\nu}$ and the one in $D = 3$ as $\epsilon^{\mu\nu\rho}$. We will also use the common notation $\epsilon^{\mu\nu\rho} p_\mu q_\nu l_\rho = \epsilon^{pql}$ etc.

Argue that the following three statements are equivalent

1. Given three vectors p, q, and l in $D = 3$ dimensions that are confined onto a two-dimensional subspace, then
$$\epsilon^{pql} = 0\,. \tag{1}$$
2. If p, q, and l are defined in $D = 2$ dimensions, the following identity holds
$$\epsilon^{pq} l^\mu - \epsilon^{lq} p^\mu - \epsilon^{pl} q^\mu = 0\,. \tag{2}$$
3. Finally, in $D = 2$ dimensions, given one vector l^μ
$$\epsilon^{\mu\nu} l^\rho - \epsilon^{\rho\nu} l^\mu - \epsilon^{\mu\rho} l^\nu = 0\,. \tag{3}$$

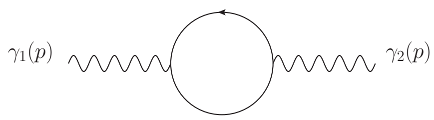
All three statements are referred to as Schouten identities in $D = 2$ space-time dimensions.

Photon vacuum polarization

In this exercise, we study the photon vacuum polarization in 1+1 dimensional QED, the so-called Schwinger model [1].

Amplitude

Let us consider QED with a massless electron in $D = 2$ space-time dimensions. Use Feynman diagrams to show that the one-loop correction to the photon vacuum polarization reads

$$\Pi_{12}^{\gamma}(p) = \Pi^{\mu\nu} \varepsilon_{1,\mu} \varepsilon_{2,\nu} =$$


$$= -e^2 \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{\mathrm{Tr}\left(\not{\varepsilon}_1 \not{l} \not{\varepsilon}_2 (\not{l} + \not{p})\right)}{l^2 (l+p)^2}\,, \tag{4}$$

where ε_i denote two-dimensional polarization vectors of the virtual photon with momentum p ($p^2 \neq 0$) and we use dimensional regularisation, $D = 2 - 2\epsilon$.

Preliminary considerations

1. Argue that the integral in eq. (4) can be written as

$$\Pi^{\gamma}_{12}(p) = -\,e^2\,\mathrm{Tr}\left(\not\varepsilon_1\gamma_\alpha\not\varepsilon_2\gamma_\beta\right)\int\frac{\mathrm{d}^D l}{(2\pi)^D}\,\frac{l^\alpha(l+p)^\beta}{l^2(l+p)^2} \tag{5}$$

$$= -\,e^2\,\mathrm{Tr}\left(\not\varepsilon_1\gamma_\alpha\not\varepsilon_2\gamma_\beta\right)\sum_{i=1}^{2}T_i^{\alpha\beta}\int\frac{\mathrm{d}^D l}{(2\pi)^D}\,\frac{A_i\,l^2+B_i\,(l+p)^2+C_i\,p^2}{l^2(l+p)^2}\,, \tag{6}$$

where the exact form of A_i, B_i and C_i is immaterial, except that they do not depend on the loop momentum, and we defined $T_1^{\alpha\beta}=g^{\alpha\beta}$ and $T_2^{\alpha\beta}=p^\alpha p^\beta/p^2$.

2. Explain which numerator in eq. (6) gives a non-zero contribution.

3. Argue that you could infer the value of C_i by taking a (double) residue in the loop momentum directly in eq. (5).

Integrand reduction in "van Neerven - Vermaseren" basis

For two-point functions in two dimensions, the physical and transverse spaces in the "van Neerven - Vermaseren" basis are both one-dimensional; they are spanned by unit vectors $v_1^\mu = p^\mu/\sqrt{p^2}$ and $n_\perp^\mu$, respectively. Furthermore, we denote the unit-vector spanning the remaining, "-2ϵ"-dimensional space by n_ϵ^μ.[1] Then, by construction, $v_1\cdot n_\perp = v_1\cdot n_\epsilon = n_\perp\cdot n_\epsilon = 0$.

1. Decompose the loop momentum as $l^\mu = \alpha_1 v_1^\mu + \alpha_\perp n_\perp^\mu + \alpha_\epsilon n_\epsilon^\mu \equiv \alpha_1 v_1^\mu + l_\perp^\mu + l_\epsilon^\mu$. Recall that the off-shell massless two-point function receives contributions only from bubble integrals,[2] and that this contribution can be found from localizing the loop momentum in the *numerator* on the "double cut" $0 = l^2 = (l+p)^2$. What does the double cut condition imply for α_1 and $l_\perp^2 + l_\epsilon^2$.

2. Use $\mathrm{Tr}\left(\gamma^\alpha\gamma^\beta\gamma^\rho\gamma^\sigma\right) = 4\left[g^{\alpha\beta}g^{\rho\sigma} - g^{\alpha\rho}g^{\beta\sigma} + g^{\alpha\sigma}g^{\beta\rho}\right]$ and the previously obtained decomposition for l^μ to simplify the numerator in eq. (4). Argue, that $\varepsilon_{1,2}\cdot l_\epsilon = 0$ and that one can disregard terms in the integrand that are linear in $l_\perp$ or l_ϵ. The final result reads

$$\Pi^{\gamma}_{12}(p) = -4e^2\int\frac{\mathrm{d}^D l}{(2\pi)^D}\,\frac{1}{l^2(l+p)^2}\left\{2(l_\perp\cdot\varepsilon_1)(l_\perp\cdot\varepsilon_2) + \frac{p^2}{2}\left[\varepsilon_1\cdot\varepsilon_2 - \frac{(p\cdot\varepsilon_1)(p\cdot\varepsilon_2)}{p^2}\right]\right\}. \tag{7}$$

Angular integration

Consider the integral in eq. (7).

1. Argue that the denominator does not depend on the direction $l_\perp$ but only on $l_\perp^2$.

[1] By unit-vector, we imply that $n^2 = n_\perp^2 = n_\epsilon^2 = 1$.

[2] We have seen in the preliminary considerations, that only terms $\sim C_i$ in eq. (6) contribute.

2. With this information, derive the following formula

$$\int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{l_\perp^\alpha l_\perp^\beta}{l^2(l+p)^2} = \left[g_{(2)}^{\alpha\beta} - \frac{p^\alpha p^\beta}{p^2} \right] \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{l_\perp^2}{l^2(l+p)^2} \,, \tag{8}$$

where $g_{(2)}^{\alpha\beta}$ is the metric tensor of the 2-dimensional sub-space.
Hint: Consider a general, symmetric solution $A\, g_{(2)}^{\alpha\beta} + B\, p^\alpha p^\beta / p^2 + C\, g_{(\epsilon)}^{\alpha\beta}$ and derive A, B, and C by appropriate contractions.

3. Use eq. (8) together with the cut condition $l_\perp^2 = -p^2/4 - l_\epsilon^2$ and show that

$$\Pi_{12}^\gamma(p) = 4e^2 \left[\varepsilon_1 \cdot \varepsilon_2 - \frac{(p\cdot\varepsilon_1)(p\cdot\varepsilon_2)}{p^2} \right] \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{l_\epsilon^2}{l^2(l+p)^2} \,. \tag{9}$$

Integration of rational term

The result in eq. (9) shows that the only nonzero contribution to $\Pi_{12}^\gamma(p)$ stems from ultraviolet regularisation. Such contributions are known as "rational terms" in generalized unitarity. Show that

$$\int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{l_\epsilon^2}{l^2(l+p)^2} = \frac{\mathrm{i}}{4\pi} + \mathcal{O}(D-2) \,. \tag{10}$$

With this result, we finally arrive at

$$\Pi_{12}^\gamma(p) = \mathrm{i}\, \frac{e^2}{\pi} \left[\varepsilon_1 \cdot \varepsilon_2 - \frac{(p\cdot\varepsilon_1)(p\cdot\varepsilon_2)}{p^2} \right] \,. \tag{11}$$

Photon mass

Argue why the result in eq. (11) implies a photon mass $m_\gamma^2 = \frac{e^2}{\pi}$ in the Schwinger model, i.e. in $1+1$ dimensional QED.

References

[1] J. S. Schwinger, Phys. Rev. **128** (1962), 2425-2429