LOOP AMPLITUDES

Till here we focussed on tree-level amplitudes.

In a ħ expansion, we know this usually means "considering only the classical approximation"

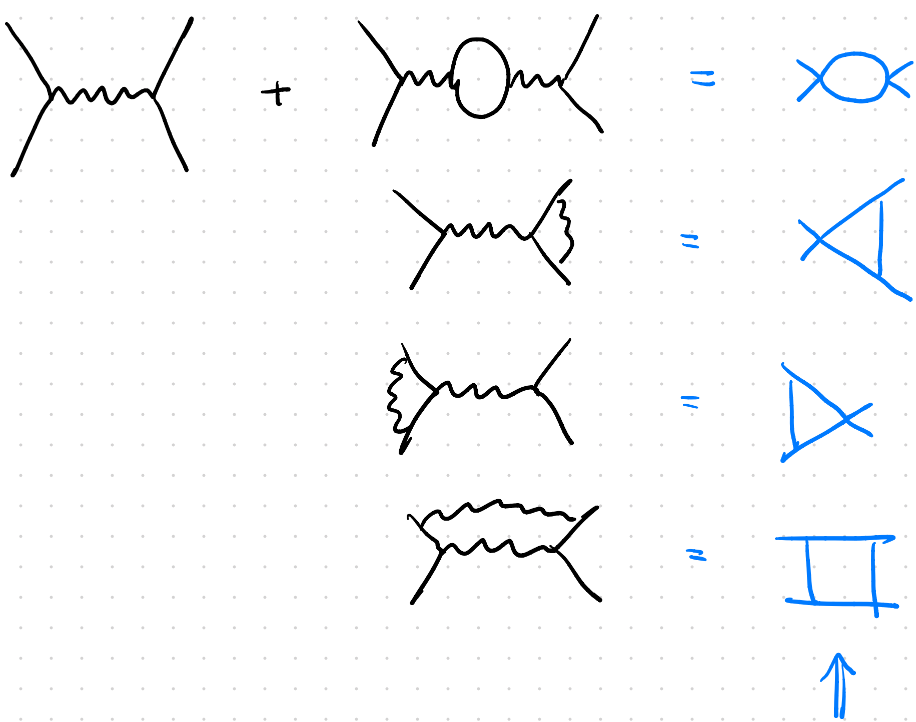
[subtleties with classical contributions coming from loop diagrams]

In next few lectures, we'll move to consider loop corrections.

We will focus mainly on 1-loop corrections to scattering Amplitudes.

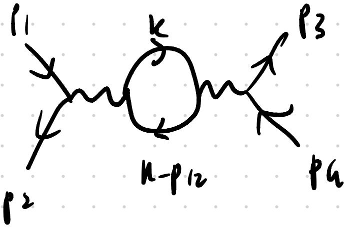
The starting point is how to handle LOOP INTEGRALS, which are the backbone of loop SCATTERING AMPLITUDES

Inside Feynman Diagrams at loop level, are "hidden" loop integrals, which bring new type of complexity, compared to tree-level



In this business, we deal a lot with Scalar Feynman Integrals

Let's see an example of this:



$$= \bar{u}_2 \gamma^\mu u_1 \frac{i}{p_{12}^2} \times$$

$$\times \int \frac{d^D k}{i\pi^{D/2}} \frac{\mathrm{Tr}\left[\gamma_\mu \not{k} \gamma_\nu (\not{k} - \not{p}_{12})\right]}{(k^2 - m^2 + i\varepsilon)\left((k-p_{12})^2 - m^2 + i\varepsilon\right)}$$

in view of what comes next work in generic "D"

FOR NOW $D \in \mathbb{N}$ integer

$$\times \frac{i}{p_{12}^2} \bar{u}_3 \gamma^\nu u_4$$

$$= \bar{u}_2 \gamma^\mu u_1 \frac{i}{p_{12}} \Pi^{(1\ell)}_{\mu\nu} \frac{i}{p_{12}} \bar{u}_3 \gamma^\nu u_4$$

pol Tensor @ 1 loop

Inside here hidden integrals of the type

$$\int \frac{d^D k}{i\pi^{D/2}} \frac{\left\{ k^\mu k^\nu ; \; k^\mu p_{12}^\nu ; \; k^\nu p_{12}^\mu ; \; p_{12}^\mu p_{12}^\nu \right\}}{(k^2 - m^2 + i\varepsilon)\left((k-p_{12})^2 - m^2 + i\varepsilon\right)} \qquad p_{12} = p_1 + p_2$$

tensor Integrals

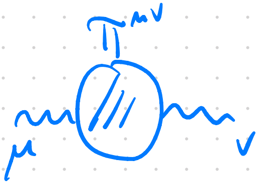
First step towards their calculation is so called tensor-reduction

1. Notice that $\Pi^{\mu\nu}(\underbrace{p_{12}^2}_{q})$ can depend only on $q = (p_1 + p_2)$

2. Lorentz Covariance implies that

$$\Pi^{\mu\nu}(q^2) = F_1 \, q^\mu q^\nu + F_2 \, g^{\mu\nu}$$

Scalar FORM FACTORS

3. 

must be gauge invariant

this means word Identity must be satisfied

$$q^\mu \Pi_{\mu\nu} = q^\nu \Pi_{\mu\nu} = 0$$

$$\Rightarrow F_1 \, q^2 q^\nu + F_2 \, q^\nu = 0$$

$$\Rightarrow \quad F_1 = - \frac{F_2}{q^2} = - \frac{F}{q^2}$$

$$\Pi^{\mu\nu} = F \underbrace{\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right)}$$

numerator of transverse photon propagator!

there is only one <u>form factor</u> $\Rightarrow$ Combination

of SCALAR integrals.

Explicitly, we can define a projector:

$$P_{\mu\nu} = c\left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}\right)$$ such that

$$P_{\mu\nu}\Pi^{\mu\nu} = F \Rightarrow cF\left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}\right)\left(\frac{q^\mu q^\nu}{q^2} - g^{\mu\nu}\right)$$

$$= cF\left[1 - 1 - 1 + D\right] \equiv F$$

$$\Rightarrow c = \frac{1}{D-1}$$ so that

$$P_{\mu\nu} = \frac{1}{D-1}\left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}\right)$$ is the projector to apply on Feynman Diags.

Apply $P_{\mu\nu}$ on integrals of the type seen before :

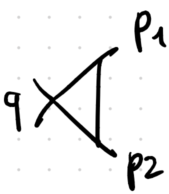
$$\int \frac{d^Dk}{i\pi^{D/2}} \frac{\{k^\mu k^\nu;\ k^\mu q^\nu;\ k^\nu q^\mu;\ q^\mu q^\nu\}}{(k^2-m^2+i\varepsilon)\,((k-q)^2-m^2+i\varepsilon)}$$

get "SCALAR INTEGRALS" :

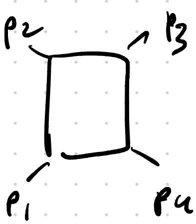
$$\int \frac{d^Dk}{i\pi^{D/2}} \frac{\{k\cdot k,\ k\cdot q;\ q\cdot q\}^n}{(k^2-m^2+i\varepsilon)\,((k-q)^2-m^2+i\varepsilon)}$$

raised to various powers

similarly for 3-point and 4-point

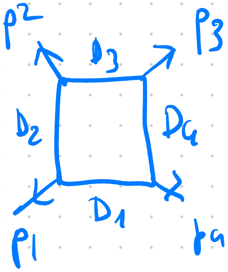


$$\sim \int \frac{d^Dk}{i\pi^{D/2}} \frac{\{k\cdot k,\ k\cdot p_1,\ p_i\cdot p_j\}^n}{(k-m_1^2+i\varepsilon)((k-p_1)^2-m_2^2+i\varepsilon)\,((k-p_{12})^2-m_3^2+i\varepsilon)} \qquad i = 1,2$$



$$\sim \int \frac{d^Dk}{i\pi^{D/2}} \frac{\{k\cdot k;\ k\cdot p_i,\ p_i\cdot p_j\}}{(k-m_1^2)((k-p_1)^2-m_2^2)\,((k-p_{12})^2-m_3^2)\,((k-p_{123})^2-m_4^2)} \qquad i,j=1,2,3$$

Note that @ 1 loop all scalar products can be rewritten in terms of denominators !



$$= \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{\underbrace{k^2}_{D_1}\,\underbrace{(k-p_1)^2}_{D_2}\,\underbrace{(k-p_{12})^2}_{D_3}\,\underbrace{(k-p_{123})^2}_{D_4}}$$

massless <u>BOX</u>

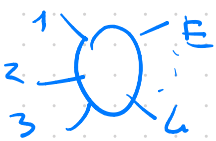
$$\left\{ \begin{matrix} k\cdot k \\ k\cdot p_1 \\ k\cdot p_2 \\ k\cdot p_3 \end{matrix} \right\}$$ independent scalar products $\Rightarrow$

$$k\cdot k = D_1$$

$$k\cdot p_1 = \frac{D_2 - D_1 - p^2}{2}$$

etc

Introduce a general notation for Feynman Ints



$$\int \frac{d^D k}{i\pi^{D/2}} \frac{1}{D_1^{n_1} D_2^{n_2} \cdots D_E^{n_E}} = I(n_1, n_2, \ldots n_E)$$

external legs !

Let's pause to have a look at these integrals:

Simplest integral <u>TADPOLE</u>  $= \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{(k^2-m^2+i\varepsilon)}$

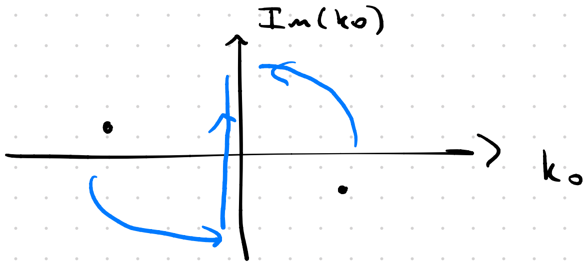
Comes for example from

$$\int \frac{d^D k}{i\pi^{D/2}} \left\{ \frac{k^2}{(k^2-m^2)((k-q)^2-m^2)} = \underbrace{\frac{1}{(k-q)^2-m^2}}_{k\to k+q} + \frac{m^2}{(\ )(\ )} \right\}$$

ISSUES

1. Divergence for $k^2=m^2$, regulated by $+i\varepsilon$

$\Rightarrow$ Wick Rotation



Consider Euclidean integrals

$$\int \frac{d^D k}{\cancel{i}\pi^{D/2}} \frac{1}{k^2+m^2}$$

2. more subtle problem, for physical number of dimensions $D=4$ most of these ints. are not well defined (DIVERGENT)

$$\int \frac{d^D k}{\pi^{D/2}} \frac{1}{k^2+m^2} \sim \int \frac{d^D k}{\pi^{D/2}} \frac{1}{k^2} \qquad k^2 \gg m^2 \text{ UV region}$$

$$\sim \frac{\Omega(D)}{\pi^{D/2}} \int^{\infty} dk \, \frac{k^{D-1}}{k^2} \qquad \text{diverges if } D \geq 2$$

this is called an UV divergences

We know from RENORMALISATION THEORY that if theory is renormalisable (as QED or QCD or SM)

then these divergences can be reabsorbed in FINITE number of physical quantities

We need a REGULARISATION proceedure in order to extract these divergences and properly "Dispose" of them ⟼

Noticing that tadpole converges for $D<2$, we could imagine to perform computation in general D and then collect divergences as POLES in $(D-4)$. This proceedure is conceptually delicate, let's see why.

$$\bigcirc = \int \frac{d^Dk}{\pi^{D/2}} \frac{1}{(k^2+m^2)} = \frac{\Omega(D)}{\pi^{D/2}} \int_0^\infty \frac{dk\, k^{D-1}}{(k^2+m^2)}$$

$$= \underbrace{\frac{2\pi^{D/2}}{\Gamma(\frac{D}{2})}}_{} \frac{1}{\pi^{D/2}} (m^2)^{\frac{D-2}{2}} \int_0^\infty \frac{dx\, x^{D-1}}{(x^2+1)}$$

$\left(\frac{D}{2}-1\right)!$ if $\frac{D}{2}-1$ integer ! $\Gamma(\frac{D}{2})$ gives analytic cont. for all D

$\mathcal{U}$

GAMMA FUNCTION

generalisation of

FACTORIAL

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt$$

well def if $x > 0$ real num!

$$n! = \Gamma(n+1) = \int_0^\infty e^{-t}\, t^{n}\, dt \quad \text{for } n \in \mathbb{N}$$

In fact

$$\Gamma(1+x) = \int_0^\infty \underset{g'}{e^{-t}}\, \underset{f}{t^{x}}\, dt = \underbrace{-e^{-t} t^x \Big|_0^\infty}_{\substack{= \\ 0 \\ \text{if } x>0}} + \underbrace{x \int_0^\infty e^{-t}\, t^{x-1}\, dt}_{x\,\Gamma(x)}$$

$$\Gamma(1+x) = x\,\Gamma(x)$$

and since $\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1$

$$\Gamma(2) = 1 \cdot \Gamma(1) = 1$$

$$\Gamma(3) = \Gamma(1+2) = 2 \cdot \Gamma(2) = 2$$ etc .... Factorial

IMPORTANT

$$\Gamma(x) = \frac{\Gamma(1+x)}{x}$$ can be used to analytically continue $\Gamma(z)$ to whole complex plane! (and first to $x<0$!)

poles at $x = 0, -1, -2, -3$ etc

but well defined for example for $x = -\frac{1}{2}$

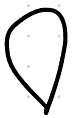
$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{-1/2} = -2\,\Gamma\left(\frac{1}{2}\right) = -2\int_0^\infty dt\, e^{-t}\, t^{-\frac{1}{2}}$$

$t^{\frac{1}{2}} = u \qquad du = +\frac{1}{2} t^{-1/2} dt \qquad dt = 2u\, du$

$$\int_0^\infty 2\cancel{u}\, du\; e^{-u^2} \frac{1}{\cancel{u}} = 2\int_0^\infty du\, e^{-u^2} = 2\,\frac{\sqrt{\pi}}{2} = \sqrt{\pi}$$

so we have $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$ (example of analytic continuation)

Let's continue with the <u>TADPOLE</u>.



$$= \frac{2\pi^{D/2}}{\Gamma\left(\frac{D}{2}\right)} \frac{1}{\pi^{D/2}} (m^2)^{\frac{D-2}{2}} \int_0^\infty \frac{dx\ x^{D-1}}{(x^2+1)}$$

$x^2 = t \Rightarrow$

$$= \frac{\cancel{2}\,(m^2)^{\frac{D-2}{2}}}{\Gamma\left(\frac{D}{2}\right)} \cancel{\frac{1}{2}} \int_0^\infty dt\ \frac{t^{\frac{D}{2}-\frac{1}{2}}}{t^{\frac{1}{2}}(1+t)}$$

$$= \frac{(m^2)^{\frac{D-2}{2}}}{\Gamma\left(\frac{D}{2}\right)} \underbrace{\int_0^\infty dt\ \frac{t^{\frac{D}{2}-1}}{(1+t)}}$$

this is a well known function $\Rightarrow$ <u>BETA FUNCTION</u>

Ma

Indeed

$$\int_0^\infty dt\, \frac{t^{x-1}}{(1+t)^{x+y}} = B(x,y) \left\{ = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \right\}$$

SEE BELOW

$$\|$$

$$t = \frac{u}{1-u} \qquad dt = \frac{1}{(1-u)^2}\, du$$

$$\int_0^1 du\, \frac{1}{(1-u)^2} \left(\frac{u}{1-u}\right)^{x-1} \left(\frac{1}{1-u}\right)^{-x-y} =$$

$$= \int_0^1 du\; u^{x-1}\, (1-u)^{x+y+1-x-2} = \underline{\underline{\int_0^1 du\; u^{x-1}\, (1-u)^{y-1}}}$$

other integral representation.

it's easy to prove that

$$\Gamma(x)\Gamma(y) = \int_0^\infty dt \int_0^\infty du \; e^{-(t+u)} \; t^{x-1} \; u^{y-1}$$

$u = zw$

$t = (1-z)w$

$$J = \begin{pmatrix} w & z \\ -w & 1-z \end{pmatrix}$$

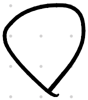
$$\det J = w(1-z) + wz = w$$

$$\Gamma(x)\Gamma(y) = \int_0^1 dz \int_0^\infty dw \; w \; e^{-w} \; (1-z)^{x-1} \; w^{x-1} \; z^{y-1} \; w^{y-1}$$

$$= \int_0^\infty dw \; e^{-w} \; w^{1+x-1+y-1} \int_0^1 dz \; z^{y-1} (1-z)^{x-1}$$

$$= \int_0^\infty dw \; e^{-w} \; w^{x+y-1} \; B(x,y) = \Gamma(x+y) \; B(x,y)$$

so finally

 $= \frac{(m^2)^{\frac{D-2}{2}}}{\Gamma\left(\frac{D}{2}\right)} \int_0^\infty dt \, \frac{t^{\frac{D}{2}-1}}{(1+t)}$

$B(x,y) = \int_0^\infty dt \, \frac{t^{x-1}}{(1+t)^{x+y}}$

$x = \frac{D}{2}$

$x+y = 1 \quad y = 1-\frac{D}{2}$

so

 $= \frac{(m^2)^{\frac{D-2}{2}}}{\cancel{\Gamma\left(\frac{D}{2}\right)}} \frac{\cancel{\Gamma\left(\frac{D}{2}\right)}\,\Gamma\left(1-\frac{D}{2}\right)}{\Gamma(1)} = (m^2)^{\frac{D-2}{2}} \, \Gamma\left(1-\frac{D}{2}\right)$

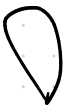
result we have obtained

integrating assuming $D < 2$

can now be extended everywhere

except poles

For example use $\Gamma(1+x) = x\,\Gamma(x)$



$$= (m^2)^{\frac{D-2}{2}}\,\Gamma\left(1-\frac{D}{2}\right) = (m^2)^{\frac{D-2}{2}}\,\frac{\Gamma\left(3-\frac{D}{2}\right)}{\left(\frac{2-D}{2}\right)\left(\frac{4-D}{2}\right)}$$

$$= 4\,(m^2)^{\frac{D-2}{2}}\,\Gamma\left(\frac{6-D}{2}\right)\frac{1}{(D-2)(D-4)}$$

see explicit poles for

$D=2,4$ etc...

what about $D=3$?

$$\int \frac{d^3k}{\pi^{3/2}}\,\frac{1}{k^2+m^2} \sim \int^{\Lambda} \frac{d^3k}{K^2} \sim \Lambda$$

linear divergence in UV

BUT

$$\bigcirc = (m^2)^{\frac{3}{2}-\frac{2}{2}}\,\Gamma\left(1-\frac{3}{2}\right) = (m^2)^{\frac{1}{2}}\,\Gamma\left(-\frac{1}{2}\right)$$

$$= -2\sqrt{\pi}\, m$$

FINITE & NEGATIVE

Similar to $\sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s)$ Riemann Zeta

$$\zeta(-1) = \sum_{n=1}^{\infty} n = -\frac{1}{12} \quad !$$

$\Rightarrow$ In Dimensional Regularization the INTEGRAL REPRESENTATION represents the final result ONLY if it converges. Otherwise it has NOTHING to do with it!