8 - BCFW Recursion

WS 2021

TUM

ON-SHELL RECURSIONS RELATIONS

With Proof of uniqueness of YM theories for interaction of spin 1, we have seen a first example of how general requirements, as those of Little Group covariance + Locality, are enough to fix three- and four-point scattering amplitudes.

⟹ we'll see now that this can be generalised for ALL tree-level scattering amplitudes, such that they can be computed only resorting to <u>ON-SHELL</u>, <u>GAUGE INVARIANT DATA</u>, without even having to use Feynman Diagrams!

Again, this is based on assumption of <u>LOCALITY</u>

· Tree-level on-shell amplitude A_n n-point scattering

we assume all particles to be massless!

$$p_i^2 = 0 \qquad i = 1, \dots, n \qquad \text{and} \qquad \sum_{i=1}^{n} p_i = 0$$

A_n function of p_i and $\underbrace{\varepsilon_j \, ; \, u, \, \bar{u}}_{\text{external states}}$

$\Rightarrow$ helicity Amplitude will be function of spinor products
$|i\rangle$ $|i]$ $\langle i|$ $[i|$ only

Let us imagine to perform the following shift:

$$p_i \to \hat{p}_i^\mu = p_i^\mu + z\, r_i^\mu$$

some of the r_i^μ may be zero!

- $z \in \mathbb{C}$
- $\sum_{i=1}^{n} r_i = 0$
- $r_i \cdot r_j = 0 \quad \forall\, i, j$
- $p_i \cdot r_i = 0 \quad \forall\, i$

note that

- $\sum_{i=1}^{n} p_i = 0 \Longrightarrow \sum_{i=1}^{n} \hat{p}_i = 0$

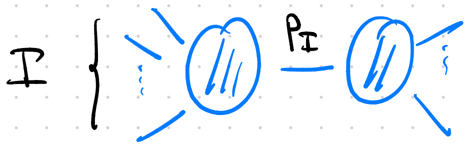
mom conservation

- $\hat{p}_i^2 = (p_i + z r_i)^2 = p_i^2 + z^2 r_i^2 + 2z\, p_i \cdot r_i = 0$

shifted momenta are ON-SHELL

- take a subset of I momenta with $2 \leq \# I \leq n-2$

and define $P_I^\mu = \sum_{i \in I} p_i^\mu$ (total momentum)



after shift: $\hat{P}_I^\mu = \sum_{i\in I} p_i^\mu + z \sum_{i \in I} r_i^\mu = P_I^\mu + z R_I^\mu$

where $R_I^\mu = \sum_{i \in I} r_i^\mu$ then

$$\hat{P}_I^2 = P_I^2 + 2z\, P_I \cdot R_I \quad \Longleftarrow \text{Linear in } z!$$

$$\hat{P}_I^2 = P_I^2 + 2z\, P_I \cdot R_I$$

we are interested in studying $\hat{P}_I^2 = 0$ (LOCAL FACTORISATION!)

then we can write:

$$\hat{P}_I^2 = -\frac{P_I^2}{z_I}(z - z_I) \quad \text{with} \quad z_I = -\frac{P_I^2}{2P_I \cdot R_I}$$

$$\hat{P}_I^2 = 0 \implies \underline{z = z_I}$$

So we have a new set of ON-SHELL (massless!) momenta $\hat{p}_i$, which fulfil momentum conservation. We want to study shifted amplitude as function of z.

@ tree-level we have seen that Scattering Amplitudes can only have POLES, no branch cuts (logs, roots etc)

$A_n(z)$ must be a RATIONAL FUNCTION of z

$A_n = A_n(z=0)$ original AMPLITUDE

$A_n(z)$ can have poles for $z = z_j$

* these have to be SIMPLE POLES $\sim \frac{1}{(z-z_j)^1}$

$\Rightarrow$ All poles are of the type $\frac{1}{\hat{P}_I^2}$ (from propag!)

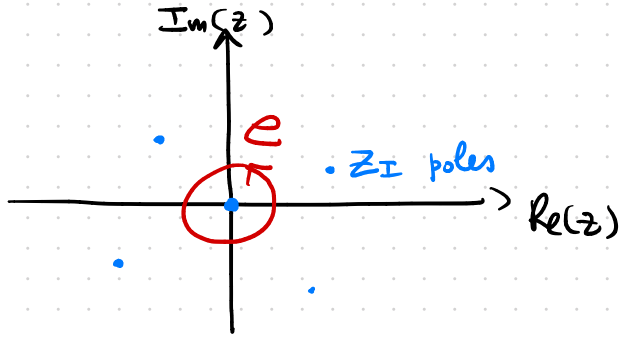
At tree-level there cannot be more equal propagators if all external momenta are generic

$$* \quad \frac{1}{\hat{P}_I^2} = - \frac{z_I}{P_I^2} \frac{1}{(z - z_I)} \, ; \qquad z_I = - \frac{P_I^2}{2 P_I \cdot R_I} \neq 0$$

poles are away from origin of $z - \mathbb{C}$ plane!

this is AGAIN all consequence of LOCALITY !

Now: Consider now $\frac{A_n(z)}{z} \leftarrow$ single pole for consid. above



$$f(z) = \frac{A_n(z)}{z}$$

CAUCHY THEOREM :

$$\oint_C \frac{A_n(z)}{z}\, dz = A_n(0) = A_n$$

but DEFORMING contour to infinity we get
(GLOBAL RESIDUE THEOREM)

$$\underbrace{\operatorname*{Res}_{z=0}\left(\frac{A_n(z)}{z}\right)}_{A_n} + \sum_{z_I} \operatorname*{Res}_{z=z_I}\left(\frac{A_n(z)}{z}\right) = \underbrace{\operatorname*{Res}_{z=\infty} \frac{A_n(z)}{z}}_{B_n}$$

"Boundary" term

$$\Rightarrow \quad A_n = -\sum_{z_I} \operatorname*{Res}_{z=z_I}\left(\frac{A_n(z)}{z}\right) + B_n$$

$$\underset{z\to\infty}{\text{Res}} \frac{A_n(z)}{z} \Rightarrow z \to \frac{1}{y} \qquad \underbrace{\oint dz \frac{A_n(z)}{z}}_{z\to\infty} = \underbrace{\oint \frac{dy}{y} A_n(y)}_{y\to 0}$$

so $B_n = A_n(y \to 0) = A_n(z \to \infty)$

↑ First coefficient of Laurent exp close to ∞

$$A_n = -\sum_{z_I} \underset{z=z_I}{\text{Res}} \left(\frac{A_n(z)}{z}\right) + B_n$$

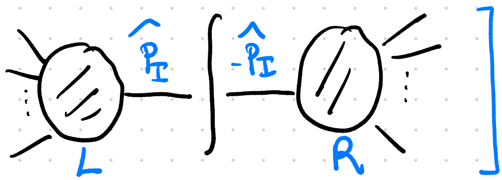
⇑

Locality says that @ $z = z_I$, $\hat{P}_I^2 = 0$,

Amplitude must FACTORISE on that POLE

z must flow here!

$A_n(z \sim z_I) \longrightarrow$



IMPLIES A SUM OVER ALL ONSHELL PARTICLE STATES THAT CAN BE EXCHANGED
⇒ helicity sum etc

$$= \hat{A}_L(z_I) \frac{1}{\hat{P}_I^2} \hat{A}_R(z_I)$$

$$\Rightarrow -\frac{z_I}{z - z_I} \hat{A}_L(z_I) \frac{1}{P_I^2} \hat{A}_R(z_I)$$

⇑ not shifted

so that finally

$$\underset{z=z_I}{\mathrm{Res}} \frac{A_n(z)}{z} = - \hat{A}_L(z_I) \frac{1}{P_I^2} A_R(z_I)$$

this intermediate particle is now ON-SHELL

these are shifted ON-SHELL amplitudes!

· the subamplitudes $A_{L,R}$ involve fewer LEGS

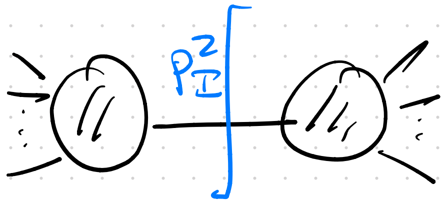
so if $B_n = 0$ (no residue at infinity) we can get A_n from ON-SHELL AMPLITUDES with fewer particles

$\Longrightarrow$ BASIS OF RECURSION RELATION

Sufficient but NOT necessary condition ("too strong") is that $A_n(z) \longrightarrow 0$ as $z \to \infty$!

In jargon, if $A_n(z) \to 0$ for $z \to \infty$ we say that we started from a <u>VALID SHIFT</u>

$$A_n = \sum_I \hat{A}_L(z_I)\,\frac{1}{P_I^2}\,A_R(z_I)$$

$$= \sum_I$$ 

sum over all possible factorisation channels

sum over all helicity-states etc is <u>IMPLICIT</u>.

Generic on-shell Recursion Relation $\Rightarrow$ build <u>higher point</u> scattering amplitudes from <u>lower point</u>, <u>gauge invariant</u>, <u>on-shell</u> building blocks!

<u>MOST FAMOUS ONE</u>:

BCFW = BRITO, CACHAZO, FENG, WITTEN

Uses one special type of <u>SHIFT</u>.

BCFW Recursion

In BCFW recursion we shift only two momenta say p_i, p_j and choose $z_{i,j}$ such that

$$|\hat{i}\rangle = |i\rangle \qquad |\hat{i}] \to |i] + z|j]$$

$$|\hat{j}] = |j] \qquad |\hat{j}\rangle \to |j\rangle - z|i\rangle$$

called $[i,j\rangle$ shift

this means

$$\cancel{p}_i = |i\rangle[i| + |i]\langle i| \to |i\rangle[i| + z|i\rangle[j| + |i]\langle i| + |j]\langle i|$$

$$= \cancel{p}_i + z\left(|i\rangle[j| + |j]\langle i|\right)$$

$$\cancel{p}_j = |j\rangle[j| + |j]\langle j| \to \cancel{p}_j - z\left(|i\rangle[j| + |j]\langle i|\right)$$

what is corresponding shift momentum ?

$$r_i^\mu = -r_j^\mu = q^\mu \quad \text{such that} \quad \cancel{q} = |i\rangle[j| + |j]\langle i|$$

$$\Longrightarrow \quad q^\mu = \frac{1}{2}[j|\gamma^\mu|i\rangle$$

INDEED:

- $\hat{p}_i + \hat{p}_j = p_i + p_j$ mom conservation is preserved

- $q \cdot p_i = q \cdot p_j = 0$ $\left\{ \begin{array}{l} p_i^\mu = \frac{[i\,\gamma^\mu\,i\rangle}{2} \qquad p_j = \frac{[j\,\gamma^\mu\,j\rangle}{2} \\ \text{or} \quad p_i \cdot q = \frac{1}{2}(\not{p}_i \not{q} + \not{q} \not{p}_i) \end{array} \right.$

- $q \cdot q = 0$ $\left\{ \begin{array}{l} [j\,\gamma^\mu\,i\rangle\,[j\,\gamma_\mu\,i\rangle \sim \langle i\,i\rangle [j\,j] = 0 \\ \text{or} \quad \not{q}\not{q} \text{ alternatively.} \end{array} \right.$

$\boxed{\text{IMPORTANT}}$: q^μ with these properties <u>CANNOT</u> exist as <u>REAL momentum</u> $\Rightarrow$ it must be complex!

$$p_i = (E, 0, 0, E) \qquad p_j = (E, 0, 0, -E)$$

then $q \cdot p_i = q \cdot p_j = 0 \Rightarrow q = (0, q_1, q_2, 0)$

$q \cdot q = q_1^2 + q_2^2 = 0$

$\Longrightarrow q^\mu = 0$ if $q_i \in \mathbb{R}$!

$\left.\right\}$ In fact $q^\mu = \frac{[j\,\gamma^\mu\,i\rangle}{2}$ has a <u>SPINOR PHASE</u>!

so recapping BCFW shift is $[i, j\rangle$

$$|i] \to |i] + z|j] \quad ; \quad |j\rangle = |j\rangle - z|i\rangle$$

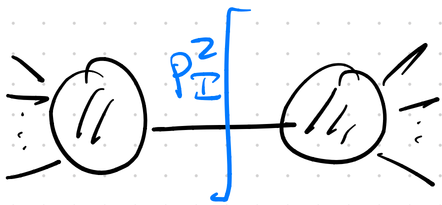
$$\begin{cases} p_i \to p_i + z \frac{1}{2} [j \gamma^\mu i\rangle \\ p_j \to p_j - z \frac{1}{2} [j \gamma^\mu i\rangle \end{cases}$$

General recursion: (assuming $A_n(z \to \infty) \to 0$)

$$A_n = \sum_I \hat{A}_L(z_I) \frac{1}{P_I^2} A_R(z_I)$$

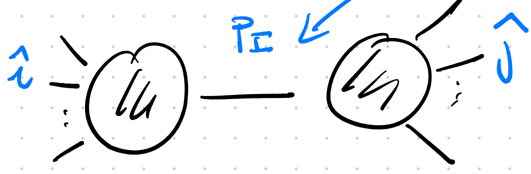
not shifted

$$= \sum_I$$



Becomes

$$= \sum_I$$



Sum over helicities implicit

if p_i, p_j on same side, P_I does NOT depend on z! there is no POLE

What happens to polarisation vectors under shift?

$$\varepsilon^\mu_{1+} \to -\frac{[r_i \gamma^\mu i\rangle}{\sqrt{2}\left([r_i i] + z[r_i j]\right)}$$

r_i = gauge momentum

new poles?

$\Rightarrow$ no! choose $r_i = p_j$ and $\hat{\varepsilon}^\mu_{1+} \equiv \varepsilon^\mu_{1+}$

so there is a gauge where no shift in pol vectors.

PARKE-TAYLOR FORMULA FOR N-GLUON SCATT.

We'll use now the BCFW recursion to prove PT formula for the MHV n-gluon amplitudes!

We will do this inductively, starting from

$n=3$

Adjacent helicities

$$\tilde{A}(1^+ 2^+ 3^- \dots n^-) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle \dots\dots \langle n 1\rangle}$$

- $n=3$ we derived amplitudes and found that LITTLE GROUP scaling alone imposes:

$$\mathcal{M}(1^+2^+3^-) = \left(c^{abc}\right) \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}$$

$$= \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \quad \checkmark \quad \text{true}$$

let's try now to build an inductive proof for n-gluons and two adjacent + helicities.

We will <u>cheat a bit</u> now: let's study PARKE TAYLOR and see how amplitudes behave under $[1,2\rangle$ shift

$$A(1^+2^+3^-\dots n^-) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle \dots \langle n1\rangle}$$

$$|\hat{1}] = |1] + z|2] \qquad |\hat{2}] = |2]$$

$$|\hat{1}\rangle = |1\rangle \qquad |\hat{2}\rangle = |2\rangle - z|1\rangle$$

$$\langle 12\rangle \to \langle 12\rangle - z\langle 11\rangle = \langle 12\rangle \quad \text{unshifted}$$

$$\langle 23\rangle \to \langle 23\rangle - z\langle 13\rangle \quad \text{shifted}$$

everything else <u>UNSHIFTED</u>.

$$A(1^+2^+3^-\dots n^-) \xrightarrow{z\to\infty} \frac{1}{z}$$

so $A(z) \to \infty$ no BOUNDARY Term!

this is checking! One can prove, completely in general, that if $[i,j\rangle$ ADJECENT then:

$$[i,j\rangle : \quad |++\rangle, \quad |+-\rangle, \quad |--\rangle, \quad |-+\rangle$$

$$A_n(z) : \quad \frac{1}{z} \qquad \frac{1}{z} \qquad \frac{1}{z} \qquad z^3$$

$\Uparrow$ only non allowed shift

arXiv: 0801.2385

Arkani-Hamed / Kaplan

for example $[n\, 1\rangle$ on our amplitude means

$$|1\rangle \to |\hat{1}\rangle = |1\rangle - z|n\rangle$$

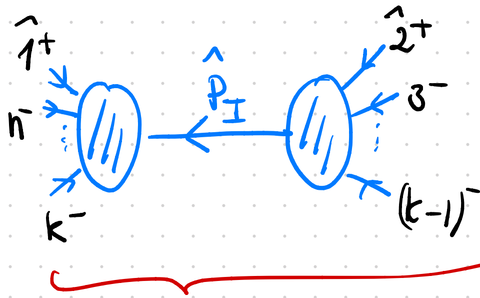
$$|n] \to |\hat{n}] = |n] + z|1]$$

$$\langle 12\rangle \to \langle 12\rangle - z\langle n2\rangle$$

$$\langle n1\rangle \to \langle n1\rangle - z\langle nn\rangle = \langle n1\rangle$$

$$A_n(z) \sim \langle 12\rangle^3 \to z^3 \ ! \quad \text{DIVERGES}$$

So let's assume shift $[1\,2\rangle$ is a good shift here. Then we have (ALL INCOMING!)

$$A_n(1^+ 2^+ 3^- \dots n^-) = \sum_{k=4}^{n}$$ 

COLOR ORDERED !

all possibilities such that : $\hat{P}_I = \hat{p}_2 + p_3 + \dots + p_{k-1}$

1] $\hat{1}$ and $\hat{2}$ are on opposite sides so that P_I depends on z !

2] at least 3-point function on each side otherwise $P_I = \hat{p}_1 \to P_I^2 = 0$ there is no such diagram so

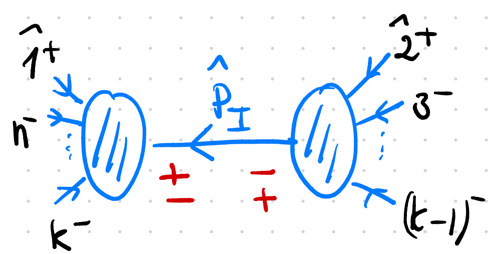
$$A_n(1^+2^+3^-\dots n^-) = \sum_{k=4}^{n}\sum_{h_I=\pm}\Big\{\hat{A}_{n-k+3}(\hat{1}^+\,\hat{P}_I^{h_I}\,k^-\dots n^-)$$

$$\times\frac{1}{P_I^2}\times\hat{A}_{k-1}(-\hat{P}_I, 2^+, 3^-, \dots, (k-1)^-)\Big\}$$

this has opposite helicity compared to

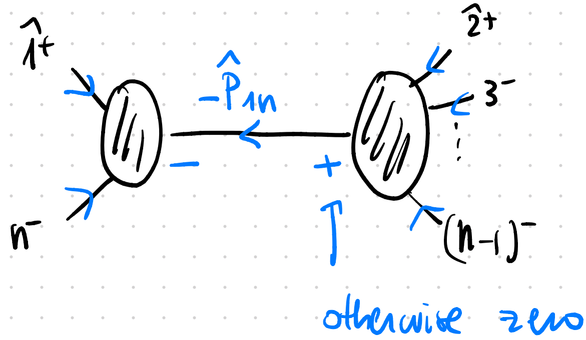
Now notice

1. $A(1^-\dots j^+\dots k^-) = 0$ except for $k = 3$

$$A_n(1^+2^+3^-\dots n^-) = \sum_{k=4}^{n}$$ 

either left or right amplitude always zero except:

$A_n(1^+2^+3^-\dots n^-) =$



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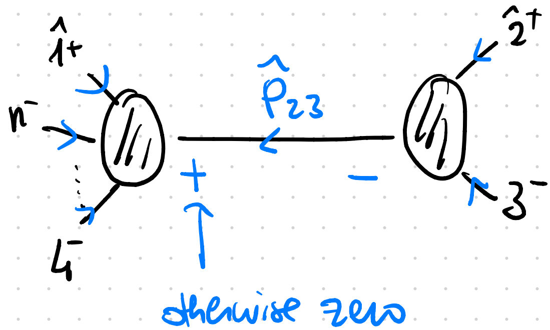


What do the middle propagators look like?

The two $\hat P_I$ are evaluated at the residue $\hat P_I^2 = 0$! there should be a POLE.

Let's look at the amplitude in first diagram

$$\hat A_3(\hat 1^+, -\hat P_{1n}^-, n^-) = -\frac{[\hat P_{1n}\, n]^3}{[n\,\hat 1]\,[\hat 1\, \hat P_{1n}]}$$

$|-\hat P_{1n}] = i\,|\hat P_{1n}]$

etc

In fact, remember we derived:

$$A(1^- 2^+ 3^+) = \frac{\langle 23 \rangle^3}{\langle 12 \rangle \langle 31 \rangle}$$

$$A(1^+ 2^- 3^-) = \frac{[23]^3}{[12][31]} = \frac{[-\hat{P}_{1n}\, n]^3}{[1(-\hat{P}_{1n})][n\hat{1}]}$$

$$= \frac{i^3 [\hat{P}_{1n}\, n]^3}{i\, [1 \hat{P}_{1n}][n\hat{1}]} = - \frac{[\hat{P}_{1n}\, n]^3}{[1 \hat{P}_{1n}][n\hat{1}]}$$

now $\hat{P}_{1n}$ must be on-shell

$$\hat{P}_{1n}^2 = 2\hat{p}_1 \cdot p_n = \langle \hat{1} n \rangle [n \hat{1}] = \langle 1 n \rangle [n \hat{1}]$$

only "z" shifted one

so on the pole z such that $[n\hat{1}] = 0$

similarly

$$|\hat{P}_{1n}\rangle[\hat{P}_{1n}\, n] = \cancel{\hat{P}}_{1n}\, |n] = (\cancel{\hat{p}}_1 + \cancel{p}_n)\, |n] = \cancel{\hat{p}}_1\, |n]$$

$$= |\hat{p}_1\rangle[\hat{1} n] = 0 \quad \text{again}$$

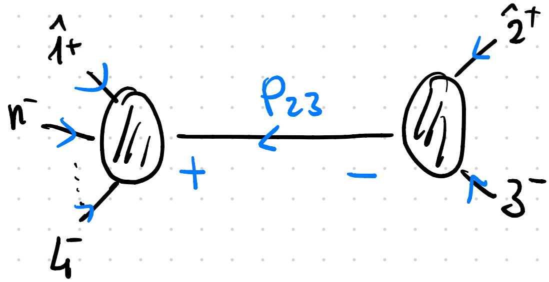
so $[\hat{P}_{1n}\, n] \sim 0$ as $[\hat{1} n]$ does!

similarly also $[\hat{1}\, \hat{P}_{1n}] \to 0$ as $[\hat{1} n]$

so $A_3(\hat{1}^{+}, -\hat{P}_{1n}^{-}, n^{-}) = -\dfrac{[\hat{P}_{1n}\, n]^3}{[n \hat{1}][\hat{1}\, \hat{P}_{1n}]} \longrightarrow \underline{[\hat{1} n] \to 0}$

$\Rightarrow$ imposing momentum conservation, this amplitude is zero!

We are left with one term only in our recursion !

$A_n(1^+2^+3^-\dots n^-) =$ 

$$= \hat{A}_{n-1}(\hat{1}^+\ \hat{P}_{23}^+\ 4^-\dots n^-)\ \frac{1}{P_{23}^2}\ \underbrace{A_3(-\hat{P}_{23}^-, 2^+, 3^-)}$$

$$= \frac{[3\hat{P}_{23}]^3}{[\hat{P}_{23}\hat{2}][\hat{2}3]} \neq 0!$$

$\Downarrow$

but $|\hat{2}] = |2]$; $|2\rangle = |2\rangle - z|1\rangle$

so $[\hat{2}3]$ etc

don't change,

no on-shell condition applied that kills them!

NOTE AGAIN:

$$A(1^+2^-3^+) = \frac{\langle 31\rangle^3}{\langle 12\rangle\langle 23\rangle}$$

$$A(1^-2^+3^-) = \frac{[31]^3}{[12][23]}$$

$$A_3(-\hat{P}_{23}^-, 2^+, 3^-) = \frac{[3(-\hat{P}_{23})]^3}{[-\hat{P}_{23}\hat{2}][\hat{2}3]} = -\frac{[3\hat{P}_{23}]^3}{[\hat{P}_{23}\hat{2}][\hat{2}3]}$$

change sign

$$A_n(1^+2^+3^-\dots n^-) =$$

$$\underbrace{\frac{\langle\hat{1}\hat{P}_{23}\rangle^4}{\langle\hat{1}\hat{P}_{23}\rangle\langle\hat{P}_{23}4\rangle\langle 45\rangle\dots\langle n\hat{1}\rangle}}_{\text{induction } (n-1)\text{-point}} \; \underbrace{\frac{1}{\langle 23\rangle[32]}}_{\text{POLE} = \frac{1}{P_{23}^2}} \; \underbrace{\frac{[3\hat{P}_{23}]^3}{[\hat{P}_{23}\hat{2}][3\hat{2}]}}_{3\text{-point}}$$

$$= \frac{\langle \hat{1} \hat{P}_{23} \rangle^3 \, [3 \hat{P}_{23}]^3}{\langle 23 \rangle \langle 45 \rangle \cdots \langle n \hat{1} \rangle [32] [3\hat{2}] \langle \hat{P}_{23} 4 \rangle [\hat{P}_{23} \hat{2}]}$$

Use now

- $\langle \hat{1} \hat{P}_{23} \rangle [3 \hat{P}_{23}] = - \langle \hat{1} \hat{P}_{23} \rangle [\hat{P}_{23} 3]$

$$= - \langle 1 (\hat{P}_2 + P_3) 3] = - \langle 1 \hat{2} \rangle [\hat{2} 3]$$

and since:

$$|\hat{2}\rangle = |2\rangle - z|1\rangle$$

$$= - \langle 12 \rangle [23]$$

• $\langle \hat P_{23}\, 4\rangle [\hat P_{23}\, \hat 2] = -[\hat 2\, \hat P_{23}]\langle \hat P_{23}\, 4\rangle$

$$= -[\hat 2\,(\not{\hat p}_2 + \not p_3)\, 4\rangle = \cancel{-[\hat 2\, \hat 2]\langle \hat 2\, 4\rangle} - [23]\langle 34\rangle$$

$$= -[23]\langle 34\rangle = +[32]\langle 34\rangle$$

so combining everything we get

$$A_n(1^+2^+3^-\dots n^-) =$$

$[3\hat 2] = [32]$

$\hat 2]$ not shifted!

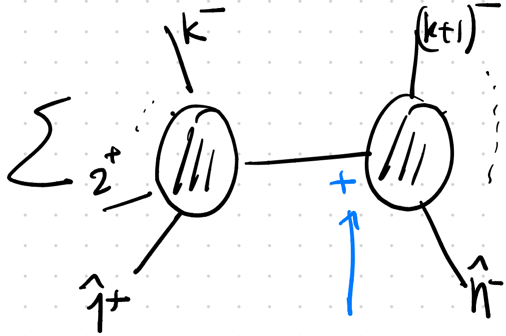
$$= \frac{\langle 12\rangle^3\, \cancel{[32]^3}}{\langle 23\rangle\langle 45\rangle\cdots\underbrace{\langle n\hat 1\rangle}_{=\langle n1\rangle\ \text{similarly}}\, \cancel{[32]}\,\cancel{[3\hat 2]}\,\cancel{[32]}\,\langle 34\rangle}$$

$$= \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\cdots\langle n1\rangle} = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}$$

is this the only way? No of course, we can try another shift, say $[1\, n\rangle$

$$\begin{cases} |1] \to |1] + z|n] \\ |n\rangle \to |n\rangle - z|1\rangle \end{cases}$$

$$A_n(1^+ 2^+ \dots n^-) = \sum$$



must be + and the right-hand amplitude must be a 3 point!

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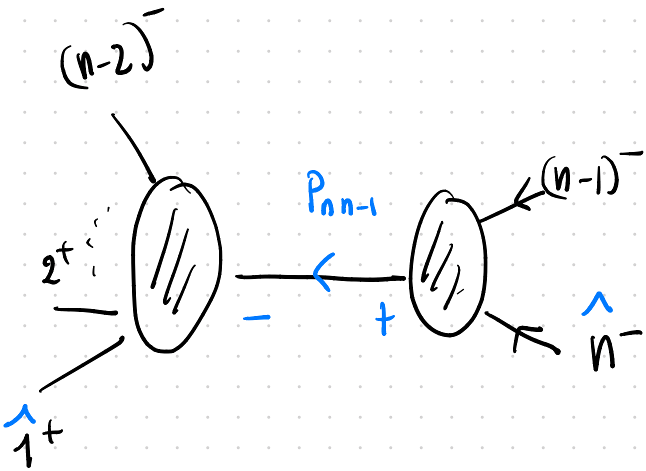


$A_3(\hat{P}^+_{nn-1}, (n-1)^-, \hat{n}^-)$

$$= \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle \dots \langle n-2\; P_{nn-1}\rangle\langle P_{nn-1}\; 1\rangle} \frac{1}{\langle n\; n-1\rangle [n-1\; n]} \frac{[n-1\; \hat{n}]^3}{[-\hat{P}_{nn-1}(n-1)][\hat{n}(-\hat{P}_{nn-1})]}$$

$$= - \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle \cdots \langle n-2\, \hat{P}_{nn-1}\rangle\langle \hat{P}_{nn-1}\, 1\rangle} \frac{1}{\langle n-1\, n\rangle [n\, n-1]} \frac{[n-1\, \hat{n}]^3}{[\hat{P}_{nn-1}\, (n-1)][\hat{n}\, \hat{P}_{nn-1}]}$$

$$\begin{cases} \langle n-2\, \hat{P}_{nn-1}\rangle [\hat{n}\, \hat{P}_{nn-1}] = - \langle n-2\, \hat{P}_{nn-1}\rangle [\hat{P}_{nn-1}\, \hat{n}] \\ \quad = - \langle n-2 \,(\cancel{\hat{p}_n} + p_{n-1})\, \hat{n}] \\ \quad = - \langle n-2\;\, n-1\rangle [n-1\;\, \hat{n}] \\ \\ \langle \hat{P}_{nn-1}\, 1\rangle [\hat{P}_{nn-1}\;\, n-1] = - \langle 1 \,(\hat{p}_n + \cancel{p_{n-1}})\;\, n-1] \\ \quad = - \langle 1\, \hat{n}\rangle [n\;\, n-1] \\ \quad = + \langle n\, 1\rangle [n\;\, n-1] \end{cases}$$

so finally

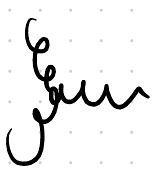
$$= \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle \cdots \langle n-2\;\, n-1\rangle\langle n-1\;\, n\rangle\langle n 1\rangle} \cancel{\frac{[n-1\;\, n]^3}{[n\;\, n-1]^2 [n-1\;\, n]}} \checkmark$$

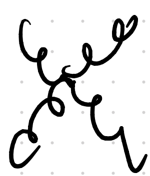
COMMENTS ON VALIDITY OF BCFW recursion

We have been able to derive amplitudes for n-gluon scattering, only using as information 3-gluon <u>on-shell</u> amplitudes...

But in YM theory there are two types of interactions :

$$\{ A^2 \partial A \; ; \; A^4 \} \subset \mathcal{L}$$





A^4 term required by <u>off-shell</u> gauge invariance of $\mathcal{L}$

why does it not matter for amplitudes ?

$\Rightarrow$ the point is exactly that 3-point amplitude we start from is on-shell and gauge invariant

it somehow comes inside the information (REDUNDANT!) contained in 4-gluon vertex – we never need it!

this becomes impressively powerful in gravity

Graviton interaction comes from

$$\mathcal{L} = \frac{1}{2\kappa^2}\int d^4x \sqrt{-g}\, R \qquad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

expanding in κ, we need infinitely many vertices to compute all tree-level amplitudes!

On the other hand, one can prove that

BCFW is valid for gravity at tree-level and

All infinite vertices are "redundant"!

$\Longrightarrow$ all tree-level <u>on-shell</u> amplitudes can be derived from

3-graviton scattering!

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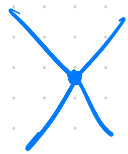
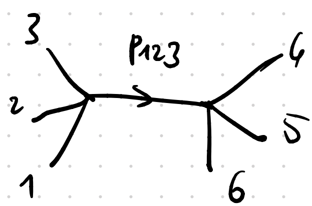
what about $\lambda\phi^4$? "Simplest" QFT?

simplest amplitude is $A_4 =$



$= \lambda$

next-to-simplest $A_6 =$



+ all permutations

but $A_6 = \lambda^2 \left[\frac{1}{s_{123}} + \cdots \right] = \lambda^2 \left[\frac{1}{(p_1+p_2+p_3)^2} + \cdots \right]$

$$= \lambda^2 \left[\frac{1}{\left(\langle 12\rangle[21] + \langle 13\rangle[31] + \langle 23\rangle[32] \right)} + \cdots \right]$$

whatever shift I do, there is always terms that do not contain those two momenta

$A_6(z) \longrightarrow O(z^0)$ when $z \to \infty$

there are always boundary terms and BCFW does not work for $\lambda\phi^4$!

$\Longrightarrow$ Gauge theories in this sense are definitely simpler than $\lambda\phi^4$.