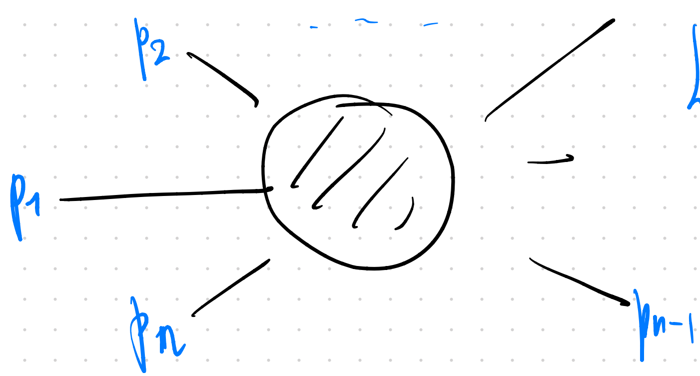
7 - SOFT AND COLLINEAR FACTORISATION

WS 2021

TUM

We have seen that the requirement of LOCALITY implies that scattering amplitudes have poles when intermediate states go on shell and this requirement is enough to fix YANG-MILLS theories for spin 1 particles!

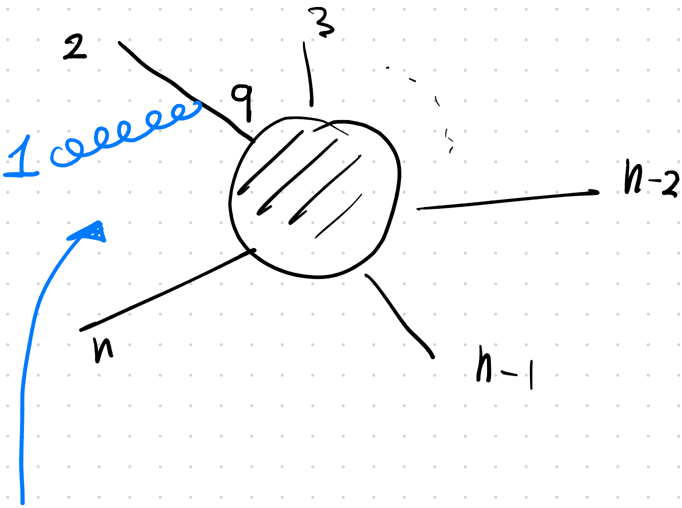
There are more UNIVERSAL properties that scattering amplitudes in YM theory must fulfil, in particular when external particles become SOFT or COLLINEAR



$$\left\{ \begin{array}{ll} p_1 \to 0 & \text{soft} \\ p_1 \parallel p_2 & \text{collinear} \\ & \text{etc} \end{array} \right\}$$

Let's have a closer look, take color ORDERED n-gluon amplitude

$\mathcal{M}(1,2,\dots,n)$ assume p_1 of g_1 becomes soft



$$\frac{1}{q^2} = \frac{1}{(p_1+p_2)^2} = \frac{1}{p_1^2+p_2^2+2p_1\cdot p_2}$$

if $p_2^2=0$ on-shell
$p_1^2=0$

$$\frac{1}{2p_1\cdot p_2} \to \infty$$

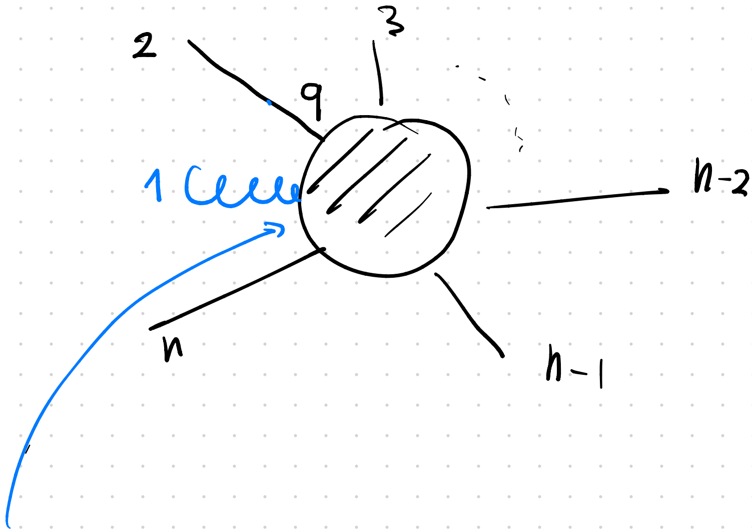
if $p_1 \to 0$

SOFT

when can this happen? COLOR ORDERED !

either attached on p_2, or on p_n or in between

2

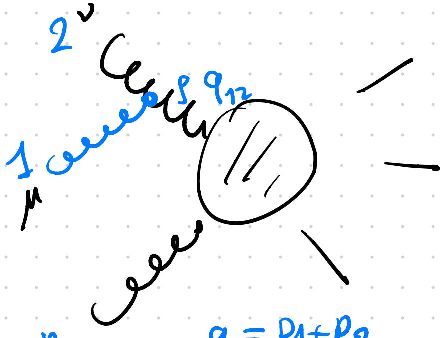


must be attached on an OFF-SHELL particle k^μ

$$\frac{1}{q^2} = \frac{1}{k^2 + 2p_1 \cdot k}$$ if $p_1 \to 0$, no divergence

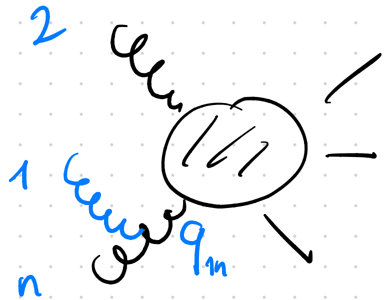
Note also p_1 must be attached through a 3-gluon vertex, otherwise again no divergence $\Rightarrow$ ignore 4-gluon vertices

Let's see how the amplitude behaves in that limit:



$q_{12} = p_1 + p_2$

;



$q_{1n} = p_1 + p_4$

$$\lim_{p_1 \to 0} \mathcal{M}(1 2 \dots n) = -i \frac{V_{3g}(1,2,q_{12})^\rho}{2\, p_1 \cdot p_2} \mathcal{M}_\rho(\underset{\underset{p_2}{\simeq}}{q_{12}}, 3, \dots n)$$

$$- \frac{i V_{3g}(n,1,q_{1n})^\rho}{2 p_1 \cdot p_n} \mathcal{M}_\rho(2, 3, \dots \underset{\underset{n}{\simeq}}{q_{1n}})$$

q_{12} must be taken incoming!

$$V_{3g}^\rho(1 2 q_{12}) =$$

$$\frac{ig}{\sqrt{2}} \left[g^{\mu\nu}(p_1 - p_2)^\rho + g^{\nu\rho}(\underbrace{p_2 + q_{12}}_{p_1 + 2p_2})^\mu + g^{\rho\mu}(\underbrace{-q_{12} - p_1}_{-2p_1 - p_2})^\nu \right] \varepsilon_{1\mu} \varepsilon_{2\nu}$$

remember $\varepsilon_1 \cdot p_1 = \varepsilon_2 \cdot p_2 = 0$

follo

$$V_{3g}^{\rho}(1 2\, q_{12}) = \frac{ig}{\sqrt{2}} \Big[\varepsilon_1 \cdot \varepsilon_2 \, (p_1 - p_2)^{\rho} + 2\, \varepsilon_2^{\rho}\, \varepsilon_1 \cdot p_2$$

$$- 2\, \varepsilon_1^{\rho}\, \varepsilon_2 \cdot p_1 \Big]$$

now assume $p_1 \to 0$ (SOFT)

and use $p_{2\rho}\, M_{\rho}(2, 3, \dots, n) = 0$ (Ward identity!)

$$V_{3g}^{\rho}(1 2\, q_{12}) \sim \frac{ig}{\sqrt{2}} \Big[2\, \varepsilon_1 \cdot p_2 \Big] \varepsilon_2^{\rho}$$

similarly

$$V_{3g}^{\rho}(n 1\, q_{1n}) \overset{p_1 \to 0}{\sim} \frac{ig}{\sqrt{2}} \Big[\varepsilon_n \cdot \varepsilon_1\, p_n^{\rho} - 2 \varepsilon_n^{\rho}\, \varepsilon_1 \cdot p_n \Big]$$

$$\sim -\frac{ig}{\sqrt{2}} \Big[2\, \varepsilon_1 \cdot p_n \Big] \varepsilon_2^{\rho}$$

So the amplitude becomes:

$$\lim_{p_1 \to 0} \mathcal{M}(12\ldots n) \sim \frac{g}{\sqrt{2}} \left(\frac{\varepsilon_1 \cdot p_2}{p_1 \cdot p_2} - \frac{\varepsilon_1 \cdot p_n}{p_1 \cdot p_n} \right) \varepsilon_2^{\rho} \mathcal{M}_{\rho}(2\ldots n)$$

this is nothing but Amplitude for scattering of n-1 gluons

SOFT FACTOR $\left[\frac{\varepsilon_1 \cdot p_2}{p_1 \cdot p_2} - \frac{\varepsilon_1 \cdot p_n}{p_1 \cdot p_n} \right]$ Universal

And it is the same even if soft gluon is emitted by quarks! We'll show this later

Let's write this in spinor - helicity formalism!

$$\varepsilon_1(p_1, p_n) \longrightarrow \boxed{\varepsilon_1 \cdot p_n = 0}$$

$\uparrow$ gauge momentum

$$\varepsilon_1^{+\mu} = -\frac{[n\,\gamma^\mu\,1\rangle}{\sqrt{2}\,[n\,1]} \qquad \varepsilon_1^{-\mu} = \frac{\langle n\,\gamma^\mu\,1]}{\sqrt{2}\,\langle n1\rangle}$$

so for the two helicities

$$\frac{\varepsilon_1^+ \cdot p_2}{p_1 \cdot p_2} = + \frac{[n\,2]\cancel{\langle 2\,1\rangle}}{\sqrt{2}\,[n1]}\,\frac{2}{\cancel{\langle 12\rangle}[21]} = -\frac{\sqrt{2}\,[n2]}{[n1][12]}$$

$$\frac{\varepsilon_1^- \cdot p_2}{p_1 \cdot p_2} = \frac{\langle n2\rangle\cancel{[21]}}{\sqrt{2}\,\langle n1\rangle}\,\frac{2}{\langle 12\rangle\cancel{[21]}} = \frac{\sqrt{2}\,\langle n2\rangle}{\langle n1\rangle\langle 12\rangle}$$

so

$$\lim_{p_1\to 0} \mathcal{M}(\overset{+}{1}2\ldots n) \sim -g\,\frac{[n2]}{[n1][12]}\,\mathcal{M}(23\ldots n)$$

$$\lim_{p_1\to 0} \mathcal{M}(\overset{-}{1}2\ldots n) \sim +g\,\frac{\langle n2\rangle}{\langle n1\rangle\langle 12\rangle}\,\mathcal{M}(23\ldots n)$$

[result in gauge $\varepsilon_1 \cdot p_n = 0$ $\Rightarrow$ valid in any gauge!]

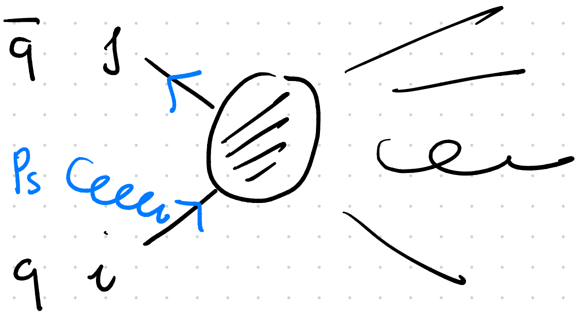
in general we write for <u>UNIVERSAL SOFT FACTORISATION</u>

$$\lim_{p_j\to 0} \mathcal{M}(12\ldots j^{\lambda}\ldots n) = S(j+1, j^{\lambda}, 1)\,\mathcal{M}(12\ldots j-1, j+1\ldots n)$$

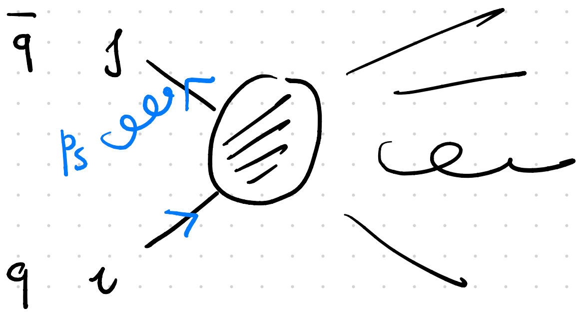
$$S(a, s^{\lambda}, b) = \begin{cases} -g\,\dfrac{[ab]}{[as][sb]} & \lambda = + \\[2ex] +g\,\dfrac{\langle ab\rangle}{\langle as\rangle\langle sb\rangle} & \lambda = - \end{cases}$$

UNIVERSALITY OF SOFT FACTOR

(Emission of soft gluon from a quark line)



color ordered vertex depends on q or $\bar q$



$$V^\mu_{q\bar q g} = \pm \frac{ig}{\sqrt{2}}\gamma^\mu$$

$$\frac{ig}{\sqrt{2}}\bar u_j(p_j)\left\{[\mathcal{M}]_{jk}\frac{(\not p_s+\not p_i)}{(p_s+p_i)^2}\gamma^\mu - \gamma^\mu\frac{(\not p_s+\not p_j)}{(p_s+p_j)^2}[\mathcal{M}]_{ki}\right\}u_i(p_i)\varepsilon^s_\mu$$

SOFT GLUON

in $p_s \to 0$ limit

$$\sim \frac{ig}{\sqrt{2}} \bar{u}_j(p_j) \left\{ M_{ji} \frac{\cancel{p}_i \cancel{\varepsilon}_s^{+}}{2 p_i \cdot p_s} - \frac{\cancel{\varepsilon}_s^{+} \cancel{p}_j}{2 p_s \cdot p_j} M_{ji} \right\} u_i(p_i)$$

use

$$\cancel{p}_i \cancel{\varepsilon}_s^{+} = 2 p_i \cdot \varepsilon_s^{+} - \cancel{\varepsilon}_s^{+} \cancel{p}_i \quad \& \quad \cancel{p}_i u_i(p_i) = 0 \quad \text{etc}$$

$$= \frac{ig}{\sqrt{2}} \bar{u}_j(p_j) \left\{ M_{ji} \frac{2 \varepsilon_s^{+} \cdot p_i}{2 p_i \cdot p_s} - \frac{2 \varepsilon_s^{+} \cdot p_j}{2 p_s \cdot p_j} M_{ji} \right\} u_i$$

$$i\mathcal{M} = \frac{ig}{\sqrt{2}} \underbrace{\left(\frac{\varepsilon_s^{+} \cdot p_i}{p_i \cdot p_s} - \frac{\varepsilon_s^{+} \cdot p_j}{p_s \cdot p_j} \right)}_{} \bar{u}_j(p_j) M_{ji} \, u_i(p_i)$$

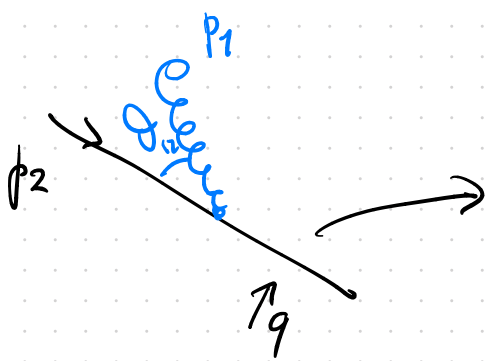
SAME AS FOR GLUONS !

choose gauge $\varepsilon_s^{+} \cdot p_j = 0$ then

$$\frac{\varepsilon_s^{+} \cdot p_i}{p_i \cdot p_s} = - \frac{\sqrt{2} [j i]}{[j s][s i]} \qquad \frac{\varepsilon_s^{-} \cdot p_i}{p_i \cdot p_s} = \frac{\sqrt{2} \langle j i \rangle}{\langle j s \rangle \langle s i \rangle}$$

COLLINEAR LIMITS

let us take an amplitude with quarks and gluons, as before, and study what happens when a gluon becomes collinear to a quark



$$\frac{i\not{q}}{q^2+i\varepsilon} = \frac{i(\not{p}_1+\not{p}_2)}{2p_1\cdot p_2+i\varepsilon}$$

$$2p_1\cdot p_2 = 2E_1E_2\left(1-\frac{2|\vec{p}_1||\vec{p}_2|}{2E_1E_2}\cos\theta_{12}\right)$$

if p_1 and p_2 are BOTH massless

$$2p_1\cdot p_2 = 2E_1E_2\left(1-\cos\theta_{12}\right) \longrightarrow \theta_{12}^2 \text{ if } \theta_{12}\to 0$$

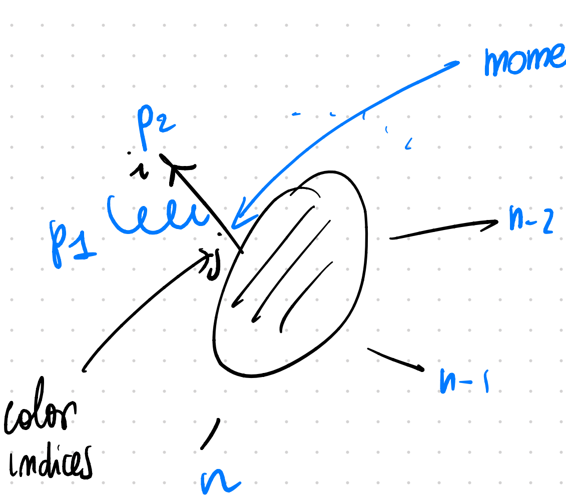
COLLINEAR SINGULARITY

we'll see that real DIV is milder!

we observe immediately something important:

COLLINEAR SINGULARITIES wrt SOFT ones

are always associated to 1 external parton only!



$\sim \mathcal{M}(1 \dots n)$

$$\mathcal{M}(1 \dots n) = g\, T^a_{ij}\, \bar{u}(p_2)\, \gamma^\mu \frac{(\not{p}_1 + \not{p}_2)}{(p_1+p_2)^2}\, \mathcal{M}_j(p_{12}, p_3, \dots p_n)$$

+ other Feynman Diagrams that are not singular in collinear limit to study the collinear limit, it's convenient to introduce a so-called Sudakov Decomposition

Given a light-like direction p^μ $(p^2 = 0)$

we parametrize the two "collinear" momenta

$$\begin{cases} p_1 = x_1 \, p + y_1 \, \bar{p} + p_\perp^\mu \\ p_2 = x_2 \, p + y_2 \, \bar{p} - p_\perp^\mu \end{cases}$$

choose a reference frame where $p_{1\perp}^\mu = -p_{2\perp}^\mu$!

where $\bar{p}$ is a complementary light-like momentum such that $p \cdot \bar{p} \neq 0$

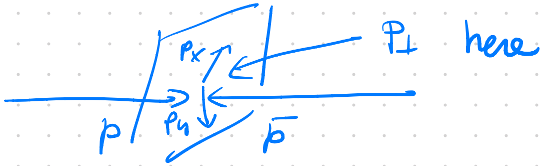
$p_\perp$ is momentum in orthogonal plane $\Rightarrow$ $\boxed{p \cdot p_\perp = \bar{p} \cdot p_\perp = 0}$

$p_\perp^\mu p_{\perp\mu} = -p_\perp^2$

it's Euclidean vector

EXAMPLE [z-axis is collinear direction]

$p^\mu = (E, 0, 0, E)$; $\bar{p}^\mu = (E, 0, 0, -E)$; $p_\perp = (0, p_x, p_y, 0)$



this is convenient because for both momenta:

$$p_i^\mu p_{i\mu} = -p_\perp^2 - 2x_i y_i\, p\cdot\bar p = 0$$

notice the sign

$$y_i = -\frac{p_\perp^2}{2x_i\, p\cdot\bar p}$$ and also

$$(p_1+p_2)^2 = \left[(x_1+x_2)p + (y_1+y_2)\bar p\right]^2 = 2(x_1+x_2)(y_1+y_2)\,\bar p\cdot p$$

and using expressions for $y_{1,2}$

$$(p_1+p_2)^2 = -(x_1+x_2)\left[\frac{1}{x_1}+\frac{1}{x_2}\right]\frac{p_\perp^2}{p\cdot\bar p}\cdot p\cdot\bar p$$

$$= -\frac{(x_1+x_2)^2}{x_1x_2}\,p_\perp^2$$

which is the invariant mass of two particles so now collinear limit clearly corresponds

to $p_\perp \to 0$ because then $y_1 \sim O(p_\perp^2)$

$p_{12}^2 \sim O(p_\perp^2)$

$p_1 \sim x_1 p \qquad p_2 \sim x_2 p \Rightarrow \boxed{p_1+p_2 = (x_1+x_2)\, p}$.

NOTE ALSO: I choose decomposition such that,

in strict collinear limit, $x_1+x_2 = 1$

$p_1+p_2 \longrightarrow p$ in collinear limit

Use now SUDAKOV DECOMPOSITION (keep only DIVERGENT TERMS)

$$\mathcal{M}(1\ldots n) \sim g\, T^a_{ij}\, \bar{u}(p_2)\gamma^\mu \frac{\not{p}_1+\not{p}_2}{(p_1+p_2)^2} \mathcal{M}_j(p_{12}, p_3, \ldots p_n)$$

$$\sim g\, T^a_{ij}\, \bar{u}(p_2)\gamma^\mu \frac{(x_1+x_2)\not{p}}{p_{12}^2} \mathcal{M}_j(p\, x_{12}, p_3, \ldots p_n)$$

Left handed Quark, gluon + or −

$$\mathcal{M}_L^\lambda \sim g\, T^a_{ij} \frac{\langle 2 | \not\epsilon_1^\lambda p] (x_1+x_2)}{p_{12}^2} \langle p | \mathcal{M}_j (p x_{12}, \dots p_n)$$

$\uparrow$ helicity gets FIXED !

$$\langle 2\, \not\epsilon_1^+\, p] = -\frac{\langle 2 \gamma^\mu p] [r_1 \gamma_\mu 1 \rangle}{\sqrt{2}\, [r_1 1]} = -\frac{2 \langle 21 \rangle [r_1 p]}{\sqrt{2}\, [r_1 1]}$$

$$\langle 2\, \not\epsilon_1^-\, p] = +\frac{\langle 2 \gamma^\mu p] \langle r_1 \gamma_\mu 1]}{\sqrt{2}\, \langle r_1 1 \rangle} = \frac{2 \langle 2 r_1 \rangle [1 p]}{\sqrt{2}\, \langle r_1 1 \rangle}$$

And therefore

$$\mathcal{M}_L^+ \sim -g\, T^a_{ij} \frac{2 \langle 21 \rangle [r_1 p]\, (x_1+x_2)}{\sqrt{2}\, [r_1 1] \underbrace{\langle 12 \rangle [21]}_{p_{12}^2\, !}} \langle p\, \mathcal{M}_j (p x_{12} \dots p_n)$$

$\sim \theta_{12}^2 \to 0$!

notice that $\langle 12 \rangle$ cancels from num and den

which softens the collinear divergence! $\Rightarrow$ ANGULAR MOM CONS

$$\mathcal{M}_L^+ \sim -g\, T^a_{1j} \frac{\sqrt{2}\,[r_1 p]}{[r_1 1]} \frac{1}{[12]} \langle p(x_1+x_2) \mathcal{M}_j(x_{12}p \dots p_n)$$

$+$ non divergent terms

now divergence

$$\sim \frac{1}{[12]} \sim \frac{1}{\sqrt{2p_1\cdot p_2}} \sim \frac{1}{\theta_{12}} \quad \text{only!}$$

At leading order, we can take limit everywhere and compute residue at $\theta_{12}=0$; $x_1+x_2=1$

$$\frac{[r_1 p]}{[r_1 1]} \simeq \frac{\sqrt{2 r_1\cdot p}}{\sqrt{2r_1\cdot p}} \frac{1}{\sqrt{x_1}} = \frac{1}{\sqrt{x_1}}$$

$p_1 \to x_1 p$

$$\mathcal{M}_L^+ \sim -g\, T^a_{1j} \frac{\sqrt{2}}{\sqrt{x_1}} \frac{1}{[12]} \underbrace{\mathcal{M}^L_j(P, \dots, p_n)}_{\langle p\, \mathcal{M}(p \dots p_n)}$$

$$|$$

since $x_1 + x_2 = 1$; $x_1 = 1 - z \quad x_2 = z$ mom fractions!

$$|$$

helicity conserved

$$\mathcal{M}_L^+ \sim -g\, T^a_{1j} \frac{\sqrt{2}}{\sqrt{1-z}} \frac{1}{[12]} \mathcal{M}^L_j(P, \dots P_n)$$

Divergence $\sim \frac{1}{\sqrt{2p_1 \cdot p_2}}$! $\sim \frac{1}{\theta_{12}}$

WHAT ABOUT THE OTHER GLUON HELICITY ?

$$\mathcal{M}_L^- \sim +g\, T^a_{1j} \frac{\sqrt{2}\langle 2 r_1\rangle [1p]}{\cancel{\sqrt{2}}\, \langle r 1\rangle \langle 12\rangle [21]} \langle p_{12} \mathcal{M}_j(p_{x12} \dots p_n)$$

here we should also see cancellation of $\frac{1}{\theta_{12}^2}$ divergence to leave $\frac{1}{\theta_{12}}$;

$[1p] \to 0$ if $1 \parallel p$ so let's get cancellation:

$$\frac{[1p]\,\langle 12\rangle}{[21]\,\langle 12\rangle} = -\frac{\langle 21\rangle[1p]}{2p_1\cdot p_2} = -\frac{\langle 2\,1\,p]}{2p_1\cdot p_2}$$

use $p_1 + p_2 = (x_1+x_2)\,p + (y_1+y_2)\,\bar{p}$

$$= -\frac{\langle 2\left((x_1+x_2)\cancel{p} + (y_1+y_2)\bar{p} - \cancel{p_2}\right)p]}{2p_1\cdot p_2}$$

$$= -\frac{(y_1+y_2)}{2p_1\cdot p_2}\langle 2\,\bar{p}\rangle[\bar{p}\,p]$$

$$\boxed{\begin{array}{l}\text{use } p_1\cdot p_2 = \\ (x_1+x_2)(y_1+y_2)\,\bar{p}\cdot p\end{array}}$$

$$= -\frac{\cancel{(y_1+y_2)}}{\cancel{(y_1+y_2)}(x_1+x_2)\,2p\cdot\bar{p}}\langle 2\bar{p}\rangle[\bar{p}\,p] = -\frac{\langle 2\,\bar{p}\rangle\cancel{[\bar{p}\,p]}}{(x_1+x_2)\langle p\bar{p}\rangle\cancel{[\bar{p}\,p]}}$$

$$\Rightarrow \frac{[1p]}{[21]} = -\frac{\langle 2\bar{p}\rangle}{\langle p\bar{p}\rangle}\frac{1}{\underbrace{(x_1+x_2)}_{\downarrow\, 1}} \rightarrow -\frac{\langle 2\bar{p}\rangle}{\langle p\bar{p}\rangle}$$

{ collinear limit!
Convergent

so amplitude becomes $(x_1+x_2 = 1\,!)$

$$M_L^- \sim -g T^a_{ij} \frac{\sqrt{2}\langle 2r_1\rangle \langle 2\bar{p}\rangle}{\langle r_1 1\rangle\langle 12\rangle\langle p\bar{p}\rangle} \langle p\, M_j(p_{x12}\ldots p_n)$$

↑ helicity FIXED

now in collinear limit $p_2 \to x_2 p$

$$\frac{\langle 2\bar{p}\rangle}{\langle p\bar{p}\rangle} \to \sqrt{x_2}\,; \qquad \frac{\langle 2r_1\rangle}{\langle r_1 1\rangle} \to -\sqrt{\frac{x_2}{x_1}}$$

$$M_L^- \sim g T^a_{ij} \sqrt{2}\sqrt{\frac{x_2}{x_1}}\sqrt{x_2}\frac{1}{\langle 12\rangle} M_j^L(p_1\ldots p_n)$$

$$\mathcal{M}_L^- \sim g\, T^a_{ij} \frac{\sqrt{2}\; z}{\sqrt{1-z}} \frac{1}{\langle 12 \rangle} \mathcal{M}_j^{L}(p \ldots p_n)$$

Let's compute now sum over gluon polarisations:

$$\sum_{\substack{pol \\ tg}} |\mathcal{M}|^2 = \left(|\mathcal{M}_L^-|^2 + |\mathcal{M}_L^+|^2 \right)$$

$$= g^2 \left(T^a_{ik} T^a_{kj} \right) \left[\frac{2z^2}{1-z} \frac{1}{p_{12}^2} + \frac{2}{1-z} \frac{1}{p_{12}^2} \right] \sum_{tq} |\mathcal{M}|^2$$

$$= 2g^2 \frac{\delta_{ij}\, C_F}{p_{12}^2} \left[\frac{1+z^2}{1-z} \right] \sum_{tq} |\mathcal{M}|^2$$

$\Uparrow$

Splitting Function

$$P_{qq}(z) = \frac{1+z^2}{1-z}$$

probability that after splitting, daughter parton carries fraction "z" of parent momentum

$$q \to q\,g \begin{cases} \text{quark carries } z = x_2 \\ \text{gluon carries } 1-z = x_1 \end{cases}$$

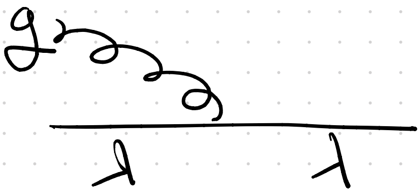
$P_{qq}(z)$ is probability for a quark to emit a quark of mom z

$$P_{qq}(1-z) = \frac{1+(1-z)^2}{z} = P_{qg}(z)$$

prob of a quark to emit a gluon of mom z

$P_{gg}(z)$ different!

Depends on type of partons



[Splitting q-g
helicity quark is
conserved along
quark line]

not always true! In general we need to sum

$$A_n(1 \dots a^{\lambda_a}, b^{\lambda_b} \dots n) \to$$

$$\xrightarrow{a \parallel b} \sum_{\lambda_P = \pm} \mathrm{Split}_{-\lambda_P}(a^{\lambda_a} b^{\lambda_b}; z)\, A_{n-1}(\dots P^{\lambda_P} \dots)$$

$q = a$
$g = b$

$$\mathrm{Split}_R(q^L, g^-, z) = -\frac{z}{\sqrt{1-z}} \frac{1}{\langle qg \rangle}$$

$R = -L$
outgoing/
incoming

$$\mathrm{Split}_R(q^L, g^+, z) = \frac{1}{\sqrt{1-z}} \frac{1}{[qg]}$$