6 - Complex Moments and

Uniqueness of Yang Mills

POLES AND SINGULAR LIMITS OF QCD SCATTERING AMPLITUDES

- We have seen that scattering amplitudes can be written in compact form, by exploiting their transf. properties under POINCARE' ⇒ Little Group ⇒ <u>SPIN HEL</u>
  SU(N) ⇒ Color ordering ⇒ <u>Primitive Amplitudes</u>

these allow us even to constrain their ANALYTIC FORM

still, we are not using full information!
together with Global, Gauge and Space-Time symmetries, QFT is founded on the concepts of <u>UNITARITY</u> AND <u>LOCALITY</u>

LOCALITY : we learn in QFT that locality of interactions implies that POLES in the S-MATRIX are always associated to on-shell intermediate states

start from GREEN FUNCTION

$$\underbrace{G_n(p_1, \dots, p_n)}_{\text{momentum-space Green Function}} = \int d^4x_1\, e^{ip_1\cdot x_1} \cdots \int d^4x_n\, e^{ip_n\cdot x_n} \langle \Omega | T\,\phi(x_1)\dots\phi(x_n)|\Omega\rangle$$

suppose $\exists$ a one-particle state $|\psi\rangle$ with mass m_ψ, and a subset of momenta

$$p^\mu = p_1^\mu + \dots + p_r^\mu = p_{r+1}^\mu + \dots + p_n^\mu \;;\quad p^\mu p_\mu = m_\psi^2$$

such that

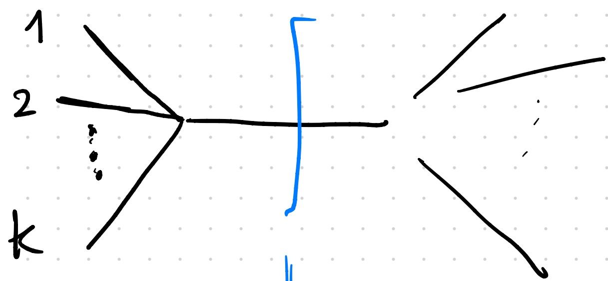
$$\langle\psi|\,\phi(x_1)\ldots\phi(x_r)\,|\Omega\rangle \neq 0 \quad \text{then}$$

G will have a pole at $p^2 = m_\psi^2$ and it will factorise close to that pole as

$$G(p_1 \ldots p_n) \sim \underbrace{M_\psi^{1,r}}_{\substack{\text{matrix element} \\ \text{for } \phi_1 \ldots \phi_r \to \psi}} \frac{1}{p^2 - m_\psi^2 + i\varepsilon} \underbrace{\left(M_\psi^{r+1,n}\right)^\dagger}_{\substack{\text{matrix element for} \\ \psi \to \phi_{r+1} \ldots \phi_n}} + \ldots \text{non dir terms}$$

note that $|\psi\rangle$ must not be elementary in general! Could be some bound state!

If we think about tree-level <u>modern</u> amplitudes,



pole from propagator $\frac{1}{(p_1+p_2+\dots+p_n)^2}$

if amplitude is <u>color ordered</u>, propagators will also contain only <u>consecutive</u> external legs!

LOCALITY @ tree level means = any poles with non-vanishing residue must correspond to the propagator of a physical particle going on-shell $\Rightarrow$ SPURIOUS POLES have ZERO residue.

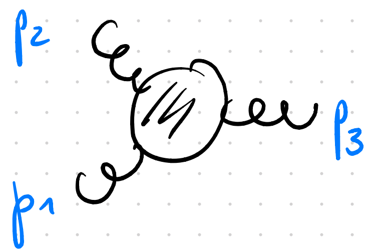
HOW CAN WE USE THIS ?

One step back : THREE POINT "AMPLITUDES"

Till now we considered four-point amplitudes, but there exist simpler things:

- two-point just propagators, nothing interesting to say at tree-level $\Rightarrow$ (OFF SHELL !)

- three-point

 $= \; ?$

if all particles are on-shell, $p_i^2 = 0$, kinematically forbidden!

$$p_1^2 = p_2^2 = 0 \qquad p_3^2 = (p_1+p_2)^2 = 0 \Rightarrow 2p_1 \cdot p_2 = 0$$

Let's think about this amplitude more generally in spinor helicity: it must depend on ε_j^μ ; p_j^μ

$$\Rightarrow \{ |1\rangle, |2\rangle, |3\rangle \text{ and } |1], |2], |3] \}$$

$$p_1+p_2+p_3 = 0 \Rightarrow \begin{cases} |1\rangle[1| + |2\rangle[2| + |3\rangle[3| = 0 \\ |1]\langle 1| + |2]\langle 2| + |3]\langle 3| = 0 \end{cases}$$

$$\langle 1| \left(|1\rangle[1| + |2\rangle[2| + |3\rangle[3| \right)$$

$$= \langle 12\rangle[2| + \langle 13\rangle[3| = 0$$

$$\langle 2| \left(|1\rangle[1| + |2\rangle[2| + |3\rangle[3| \right)$$

$$= \langle 21\rangle[1| + \langle 23\rangle[3| = 0$$

so we have $\begin{cases} \langle 12\rangle[2| = -\langle 13\rangle[3| \\ \langle 12\rangle[1| = +\langle 23\rangle[3] \end{cases}$

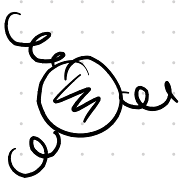
either $\langle 12\rangle = 0 = \langle 13\rangle = \langle 23\rangle$

or
$$\begin{cases} [2| = -\dfrac{\langle 13\rangle}{\langle 12\rangle}[3| \\[2ex] [1| = +\dfrac{\langle 23\rangle}{\langle 12\rangle}[3| \end{cases} \Rightarrow [1| \,/\!/\, [2| \,/\!/\, [3|$$

which implies $[12] = [13] = [23] = 0$

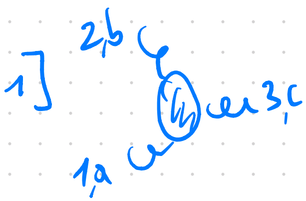
so either all $\langle ij\rangle = 0$ or $[ij] = 0$

now for <u>real momenta</u> $\langle ij\rangle = [ji]^*$ so they are all zero!

 $\equiv 0$ there is nothing it can depend on.

Let's relax our assumption on momenta and study amplitude for COMPLEX MOMENTA

then $\langle ij \rangle \neq [ji]^*$ and we have two possibilities:



$$= f\left(\langle 12\rangle, \langle 13\rangle, \langle 23\rangle\right) \qquad [ij] = 0$$

$$= g\left([12], [13], [23]\right) \qquad \langle ij \rangle = 0$$

LITTLE GROUP scaling (incoming momenta)

to study various helicity possibilities

$$|p\rangle \to z|p\rangle \qquad \varepsilon_+^\mu \longrightarrow z^2 \varepsilon_+^\mu$$

$$|p] \to \frac{1}{z}|p] \qquad \varepsilon_-^\mu \longrightarrow \frac{1}{z^2}\varepsilon_-^\mu$$

$$M(1^{h_1}2^{h_2}3^{h_3}) = \begin{cases} c^{abc}\langle 12\rangle^A\langle 23\rangle^B\langle 31\rangle^C & (1) \\ c^{abc}[12]^A[23]^B[31]^C & (2) \end{cases}$$

(1) $$\begin{cases} A+B=2\lambda_2; & -2C=(\lambda_2-\lambda_1-\lambda_3)2 \\ A+C=2\lambda_1; & -2A=(\lambda_3-\lambda_1-\lambda_2)2 \\ B+C=2\lambda_3; & -2B=(\lambda_1-\lambda_2-\lambda_3)2 \end{cases}$$

$$\mathcal{M}(1^{\lambda_1}2^{\lambda_2}3^{\lambda_3}) =$$

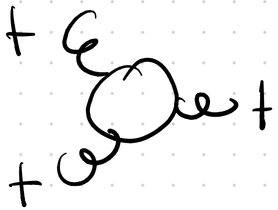
$$c^{abc}\,\langle 12\rangle^{\lambda_1+\lambda_2-\lambda_3}\,\langle 23\rangle^{\lambda_2+\lambda_3-\lambda_1}\,\langle 31\rangle^{\lambda_1+\lambda_3-\lambda_2}$$

similarly for case (2):

$$= c^{abc}\,[12]^{\lambda_3-\lambda_1-\lambda_2}\,[23]^{\lambda_1-\lambda_2-\lambda_3}\,[31]^{\lambda_2-\lambda_1-\lambda_3}$$

helicity structure uniquely fixes structure up to c^{abc}.

FIRST CASE : ALL PLUS AMPLITUDE



$$= C^{abc}\langle 12\rangle\langle 23\rangle\langle 31\rangle$$

$$C^{abc}\frac{1}{[12][23][31]}$$

for real momenta diverges

cannot be!

$$\sim M(1^+2^+3^+) = C^{abc}\langle 12\rangle\langle 23\rangle\langle 31\rangle$$

now let's do some dimensional analysis.

it's easy to see that an amplitude with n-legs must have dimension $[E]^{4-n}$ in $D=4$

$\Rightarrow$ cross-section must $[\sigma]$ = AREA

$$= \frac{1}{[E]^2}$$

$\langle ij \rangle \sim \sqrt{2 p_i \cdot p_j} \sim E$ so $[C^{abc}] = -2$!

this C^{abc} must include coupling constant

$\Longrightarrow$ if we work with renormalisable theories only solution is $\underline{C^{abc} = 0}$

no scattering amplitude for equal helicities even for complex momenta!

- Now consider

$$\mathcal{M}(1^+ 2^+ 3^-) = C^{abc} \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle}$$

$$\cancel{= C^{abc} \frac{[23][31]}{[12]^3}}$$

$\rightarrow$ diverges for real momenta

$$\rightarrow \mathcal{M}(1^+2^+3^-) = c^{abc} \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}$$

$$\mathcal{M}(1^+2^-3^+) = c^{abc} \frac{\langle 31\rangle^3}{\langle 12\rangle\langle 23\rangle}$$

$$\mathcal{M}(1^-2^+3^+) = c^{abc} \frac{\langle 23\rangle^3}{\langle 12\rangle\langle 31\rangle}$$

equal bosons!

$1 \Leftrightarrow 2$, $1 \Leftrightarrow 3$; $2 \Leftrightarrow 3$ symmetric !!

Spinor products are antisymmetric for all exchanges

which implies c^{abc} = totally antisymmetric under exchanges of all pairs !

Can we say more about this c^{abc} from general arguments?

Let us consider 4-gluon scattering again (MHV)

$$M(1^+2^+3^-4^-) = \underbrace{\langle 12\rangle^2 [34]^2}_{\substack{\text{scales properly} \\ \text{under little group} \\ \text{it's not unique!}}} F^{abcd}(s,t,u)$$

PARKE TAYLOR:

$$\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$ scales in the same way!

notice that ratio of the two has no spinor phase

$$\cancel{\langle 12\rangle^2}[34]^2 \quad \frac{\cancel{\langle 12\rangle}\langle 23\rangle\langle 34\rangle\langle 41\rangle}{\langle 12\rangle^4} =$$ is "helicity free"

$$\Rightarrow \frac{[34]^2\langle 23\rangle\langle 34\rangle\langle 41\rangle}{\langle 12\rangle} = \text{all as } z_j^0 \,!$$

so it must be a combination of s, t, u !

$$= -\frac{\overbrace{[21]\langle 14\rangle[43]\langle 32\rangle \; [43]\langle 34\rangle}^{S_{12}}}{\underbrace{[21]\langle 12\rangle}_{S_{12}}}$$

$$= + \frac{[21]\langle 14\rangle[41]\langle 12\rangle \cancel{S_{12}}}{\cancel{S_{12}}} = S_{12}S_{23} = Su\,!$$

$\Uparrow$

it can be hidden is

$F^{abcd}(s, t, u)$

So let's write

$$A(1^+2^+3^-4^-) = \langle 12\rangle^2 [34]^2 \, F^{abcd}(s,t,u)$$

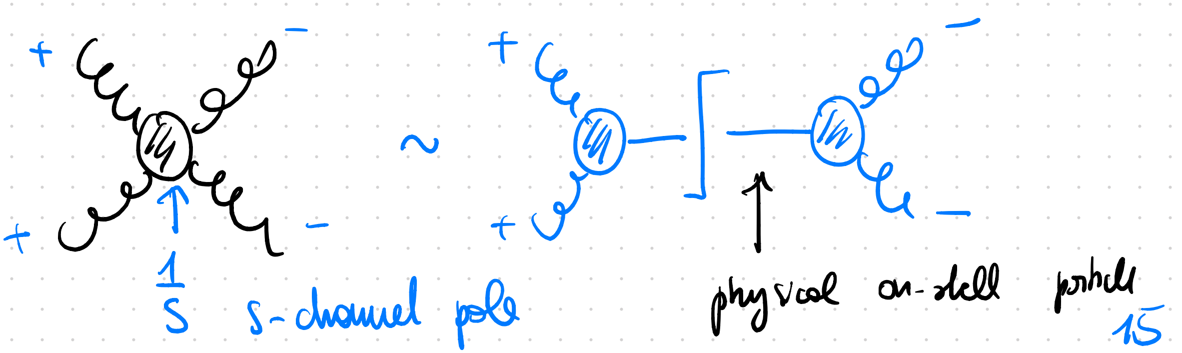
Dim analysis $[A] = 0 = 4-4$!

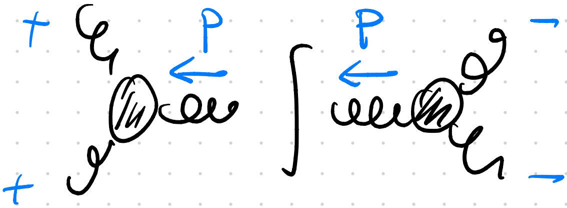
$$\Rightarrow F^{abcd} \sim \frac{1}{[E]^4} \Rightarrow$$ it must have some poles.

Can we constrain it from general arguments?

$\Rightarrow$ [UNITARITY implies that poles in the S-matrix must correspond to physical on-shell particles!]

Let's use this:



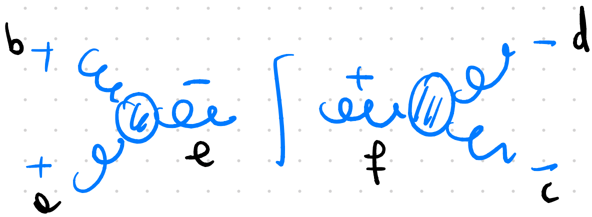


$$p^\mu = -(p_1+p_2)^\mu = +(p_2+p_3)^\mu$$

$12 \to P \qquad P \to 34$

$P\,+$ vanishes due to left-amplitude

$P\,-$ non zero $\Rightarrow$ means $+$ on the right because it is outgoing



$$\lim_{s\to 0} M(1^+2^+3^-4^-) = \frac{\delta^{ef}}{p^2} M(1^+2^+P^-)\, M(3^-4^-(-P)^+)$$

from complex conj

$$M(1^+2^+3^-) = C^{abc}\,\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}$$

$$\lim_{\delta\to 0} M(1^+2^+3^-4^-) = \frac{c^{abe}c^{cde}}{s}\,\frac{\langle 12\rangle^3}{\langle 2P\rangle\langle P1\rangle}\,\frac{[34]^3}{[3(-P)][(-P)4]}$$

notice that we can always choose

$$|-p\rangle = i|p\rangle \qquad \langle -p| = i\langle p|$$

$$|-p] = i|p] \qquad [-p| = i[p|$$

(see next page)

$$\lim_{\delta\to 0} M(1^+2^+3^-4^-) = -\frac{c^{abe}c^{cde}}{s}\,\frac{\langle 12\rangle^3[34]^3}{\langle 2P\rangle[P4]\,[3P]\langle P1\rangle}$$

$$P = -p_1 - p_2 = p_3 + p_4$$

$$\langle 2P\rangle[P4] = -\langle 21\rangle[14] = \langle 12\rangle[14]$$

$$[3P]\langle P1\rangle = [34]\langle 41\rangle$$

$$\lim_{\delta\to 0} M(1^+2^+3^-4^-) = -\frac{c^{abe}c^{cde}}{s}\,\frac{\langle 12\rangle^{\cancel{3}\,2}[34]^{\cancel{3}\,2}}{\cancel{\langle 12\rangle}\cancel{[34]}\,s_{14}}$$

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About $|-p\rangle \to i|p\rangle$ etc, we have :

$$\not{p} = |p\rangle[p| + |p]\langle p|$$

$$p^\mu \to -p^\mu$$

$$-\not{p} = -\left(|p\rangle[p| + |p]\langle p|\right)$$

also for two spinor components separately

$$\langle q\not{p} = \langle qp\rangle[p|$$

$$[q\not{p} = [qp]\langle p|$$

$$\langle q(-\not{p}) = -\langle qp\rangle[p|$$

$$[q(-\not{p}) = -[qp]\langle p|$$

$$|-p\rangle[-p| = -|p\rangle[p|$$

$$|-p]\langle -p| = -|p]\langle p|$$

$$\Rightarrow |-p\rangle = i|p\rangle$$

$$|-p] = i|p] \text{ etc}$$

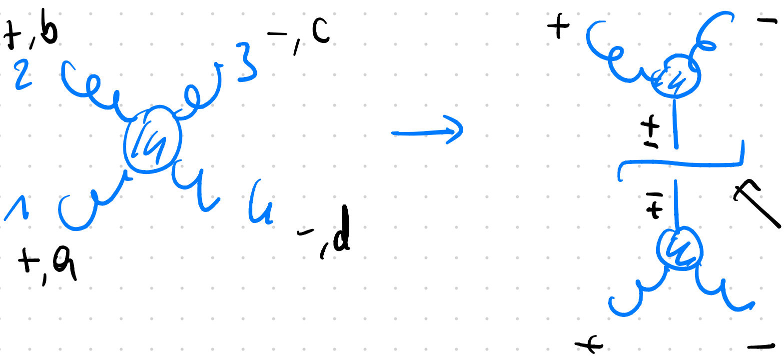
$$\lim_{s\to\infty} M(1^+2^+3^-4^-) = -\frac{c^{abe}c^{cde}}{s\,u}\langle 12\rangle^2[34]^2$$

$= $ wanted to be $F^{abcd}(s,t,u)\langle 12\rangle^2[34]^2$

so we find

$$\boxed{s u\, F^{abcd}(s,t,u) = -\, c^{abe}c^{cde}}$$

the s-channel is not the only one!



u-channel

here gluon can have two helicities!

$$\lim_{u\to 0} \mathcal{M}(1^+2^+3^-4^-) = C^{ade}C^{bce}$$

$$\otimes \frac{1}{u}\left[\frac{\langle 1P\rangle^3[3P]^3}{\langle 14\rangle\langle 4P\rangle[32][2P]} + \frac{[4P]^3\langle 2P\rangle^3}{[41][1P]\langle 23\rangle\langle 3P\rangle}\right]$$

$P = p_2 + p_3$

$$= \frac{C^{ade}C^{bce}}{u}\left[\frac{\langle 41\rangle[34]^3}{[32][21]} + \frac{[14]\langle 21\rangle^3}{\langle 23\rangle\langle 34\rangle}\right]$$

u pole is $\langle 14\rangle[41] \to 0$ with complex momenta, only one of the two goes to zero:

$\Rightarrow$ we can choose! choose $[14]=0$ then:

$$= \frac{C^{ade}C^{bce}}{u}\left\{\frac{\langle 41\rangle[34]^3\langle 12\rangle\langle 23\rangle}{[32][21]\langle 12\rangle\langle 23\rangle} = \frac{[34]\langle 41\rangle\langle 23\rangle[34]\,\langle 12\rangle[34]}{s\,u}\right.$$

$[32]\langle 21\rangle$ mom cons

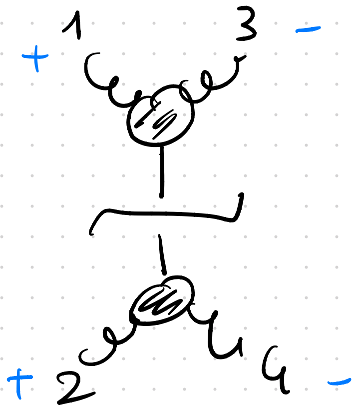
$$\left. = \frac{\cancel{u}\,\langle 12\rangle^2[34]^2}{s\,\cancel{u}}\right\}$$

$$\lim_{u \to 0} M(1^+ 2^+ 3^- 4^-) = \frac{C^{ade} C^{bce}}{s u} \langle 12 \rangle^2 [34]^2$$

which implies

$$\lim_{u \to 0} u\, s\, F^{abcd}(s,t,u) = C^{ade} C^{bce}$$

Similarly t - channel



$$\lim_{t \to 0} t\, s\, F^{abcd}(s,t,u) = C^{ace} C^{bde}$$

So we have found the following constraints

$$F^{abcd}(s,t,u) \begin{cases} \lim\limits_{s\to 0} = -\dfrac{C^{abe}C^{cde}}{su} \\[2ex] \lim\limits_{u\to 0} = \dfrac{C^{ade}C^{bce}}{su} \\[2ex] \lim\limits_{t\to 0} = \dfrac{C^{ace}C^{bde}}{st} \end{cases}$$

Since $s+t+u=0$ we can write

$$F^{abcd}(s,t,u) = \frac{1}{su} f_1^{abcd}\left(\frac{s}{u}\right) \leftarrow \text{poles in } s,t$$

$$+ \frac{1}{tu} f_2^{abcd}\left(\frac{t}{u}\right) \leftarrow \text{poles in } t,u$$

$$F^{abcd} = \frac{1}{su}\sum_{n=0}^{\infty} a_n^{abcd}\left(\frac{s}{u}\right)^n + \frac{1}{tu}\sum_{n=0}^{\infty} b_n^{abcd}\left(\frac{t}{u}\right)^n$$

$\Uparrow$ Could have high poles in $\frac{1}{u}$

$\Uparrow$ could have high poles in $\frac{1}{u}$

Let's study limits:

$\boxed{s \to 0}$

$$F^{abcd} \to -\frac{C^{abe}C^{cde}}{su} \sim \frac{1}{su} a_0^{abcd} + O(1)$$

$\boxed{t \to 0}$ $(u \to -s)$

$$F^{abcd} \to \frac{C^{ace}C^{bde}}{st} \sim \frac{1}{tu} b_0^{abcd} + O(1)$$

$$\Rightarrow \begin{cases} a_0^{abcd} = -C^{abe}C^{cde} \\ b_0^{abcd} = -C^{ace}C^{bde} \quad \left[\frac{u}{s} = -1 !\right] \end{cases}$$

$u \to 0$ is more delicate $(t \to -s)$

$$F^{abcd} \to \frac{C^{ade} C^{bce}}{su} =$$

$$= \frac{1}{su} \sum_{n=0}^{\infty} a_n^{abcd} \left(\frac{s}{u}\right)^n + \frac{1}{tu} \sum_{n=0}^{\infty} b_n^{abcd} \left(\frac{t}{u}\right)^n$$

$$\Rightarrow C^{ade} C^{bce} = \sum_{n=0}^{\infty} a_n^{abcd} \left(\frac{s}{u}\right)^n - \sum_{n=0}^{\infty} b_n^{abcd} \left(-\frac{s}{u}\right)^n$$

$$= \sum_{n=0}^{\infty} \left(a_n^{abcd} - (-1)^n b_n^{abcd}\right) \left(\frac{s}{u}\right)^n$$

we don't want $C^{ade} C^{bce}$ to diverge, it should be independent of kinematics!

$$\Rightarrow \underline{a_n^{abcd} = (-1)^n b_n^{abcd}} \quad \forall\, n > 0!$$

so we are left with

$$C^{ade}C^{bce} = a_0^{abcd} - b_0^{abcd}$$ from previous results:

$$= -C^{abe}C^{cde} + C^{ace}C^{bde}$$

$\Longrightarrow$

$$C^{ade}C^{bce} + C^{abe}C^{cde} - C^{ace}C^{bde} = 0$$

using antisymmetry

$$C^{abe}C^{cde} + C^{cae}C^{bde} + C^{ade}C^{bce} = 0$$

this is the JACOBI IDENTITY

$\Longrightarrow$ Gauge theory based on Lie Algebra are the UNIQUE SOLUTION for massless spin 1

$\Rightarrow$ so we see that just using the requirement of locality, namely that scattering amplitudes factorise properly, we could prove that YANG-MILLS are the only viable solution for spin-1 particles //