5- Unitarity 2/2

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TUM

GENERALISED UNITARITY – the OPP Algorithm

In previous lecture we have seen how to use information from double cuts to reconstruct the coefficients of the BOX master integral for $gg \to gg$ at one loop.

We have also seen that in $N=4$ SYM something special happens and all contributions from TRIANGLES and BUBBLES cancel!

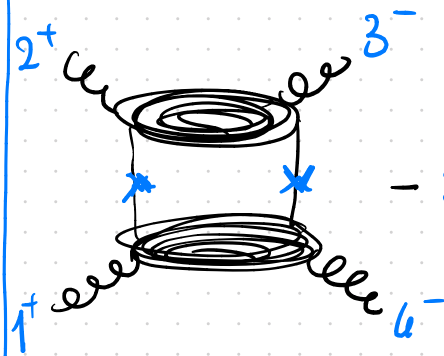
In general theory, and in particular in pure YM or in QCD, they DON'T CANCEL!

We can, at least in principle, compute them following the same idea as for the BOX, but with a bit more effort, if we use only double cuts $\Rightarrow$

We could proceed as follows

1) we know that there are no triangles or bubbles in the s_{12} channel

2) In the s_{23} channel, instead, we have, using results of previous lecture



$$\left[\;\cdots\; - s_{12}\, s_{23}\, A_4^{(\text{tree})}(1^+ 2^+ 3^- 4^-)\, I_4(s_{12}, s_{23}) \right]$$

$$= \underbrace{C_{32}\, I_{3,2}(s_{2,3})}_{\text{triangle}} + \underbrace{C_{22}\, I_{2,2}(s_{2,2})}_{\text{Bubble}} + \underbrace{R}_{\text{Rational part}}$$

and by cutting in s_{23} we get at the INTEGRAND LEVEL

$$c_{3,2}\,\mathrm{Cut}_{s_{23}}(I_{3,2}) + c_{2,2}\,\mathrm{Cut}_{s_{23}}(I_{2,2})$$

$$= \frac{\langle \ell_2 1\rangle\langle \ell_2 2\rangle}{\langle 23\rangle\langle 41\rangle\langle 3\ell_2\rangle\langle \ell_2 4\rangle\langle \ell_1\ell_2\rangle^2}\Big\{ (4-n_f)\left(\langle\ell_1 1\rangle^2\langle\ell_2 2\rangle^2+\langle\ell_1 2\rangle^2\langle\ell_2 1\rangle^2\right) + (n_s-6)\left(\langle \ell_1 2\rangle\langle \ell_2 2\rangle\langle \ell_2 1\rangle\langle \ell_1 1\rangle\right)\Big\}$$

and we need now to look at integrands of triangle and bubble, do a bit of gymnastic, and try to identify them ...

this procedure becomes complicated in general,
it can be "automated", or at least organized,
by the so-called OPP- Algorithm
Ossola, Papadopoulos, Pittau
2007
hep-ph/0609007

there are many variations of this algorithm,
but conceptually they are all very similar
Based on idea of GENERALISED CUTS

$\Rightarrow$ why limiting at cutting 2 propagators ?
we can cut as many props, as many
components of loop momentum $\Rightarrow$ 4 in 4 dim!
(every cut imposes a constraint !)

this allows us to fix coefficients of all master integrals, except the rational term R, which is, by definition, not cut constructible

GO BACK TO GENERIC DEFINITION OF INTEGRAND IN $D = 4 - 2\varepsilon$

We proved that

$$A_N^{1\ell}(\ell) = \sum_{1 \le i_1 < i_2 < i_3 < i_4 \le N} \frac{d_{i_1 i_2 i_3 i_4}(\ell)}{D_{i_1} D_{i_2} D_{i_3} D_{i_4}}$$

loop momentum

$$+ \sum_{1 \le i_1 < i_2 < i_3 \le N} \frac{c_{i_1 i_2 i_3}(\ell)}{D_{i_1} D_{i_2} D_{i_3}} + \sum_{1 \le i_1 < i_2 \le N} \frac{b_{i_1 i_2}(\ell)}{D_{i_1} D_{i_2}}$$

$$+ \sum_{1 \le i_1 \le n} \frac{a_{i_1}(\ell)}{D_{i_1}}$$

up to $\underline{\underline{O(\varepsilon)}}$ $\blacksquare$

$$d_{i_1 i_2 i_3 i_4}(\ell) = \underbrace{\overset{l_1 \cdots l_4}{d_0}}_{C_{4,i}\ \text{integral level}} + \overset{\overset{\text{hidden}}{l_1 \cdots l_4}}{d_1}(\ell \cdot n_4)$$

$$C_{i_1 i_2 i_3}(\ell) = \overbrace{C_0}^{C_{3,i}} + c_1(\ell \cdot n_3) + c_2(\ell \cdot n_4)$$

$$+ c_4\left[(\ell \cdot n_3)^2 - (\ell \cdot n_4)^2\right] + c_5(\ell \cdot n_3)(\ell \cdot n_4)$$

$$+ c_6(\ell \cdot n_3)^2 + c_7(\ell \cdot n_4)^2$$

$$b_{i_1 i_2}(\ell) = \overbrace{b_0}^{C_{2,i}} + b_1(\ell \cdot n_2) + b_2(\ell \cdot n_3) + b_3(\ell \cdot n_4)$$

$$+ b_5\left[(\ell \cdot n_2)^2 - (\ell \cdot n_4)^2\right] + b_6\left[(\ell \cdot n_3)^2 - (\ell \cdot n_4)^2\right]$$

$$+ b_7(\ell \cdot n_2)(\ell \cdot n_3) + b_8(\ell \cdot n_2)(\ell \cdot n_4)$$

$$+ b_9(\ell \cdot n_3)(\ell \cdot n_4)$$

$$a_{i_1}(\ell) = \overbrace{a_0}^{C_{1,i}} + a_1(\ell \cdot n_1) + a_2(\ell \cdot n_2) + a_3(\ell \cdot n_3) + a_4(\ell \cdot n_4)$$

n_i^μ are vectors required to span physical space

we know that at the INTEGRAL level, all the integrals proportional to $(\ell\cdot n_j)^{2n+1}$ drop due to rotational invariance in transverse space.

BUT we CANNOT drop them if we want to use unitarity at the INTEGRAND level!

Clearly, physically, we only need to determine value of coefficients d_0, c_0, b_0, a_0

1. start performing quadruple cuts to determine d_0 (and d_1)

2. Do triple cuts, subtracting $d_0 + d_1(\ell\cdot n_4)$, to determine c_0 (and other c_j)

3. Continue with double cuts, single cuts ...

but at each step we need to know all unphysical coefficients to properly subtract them from the integrand!

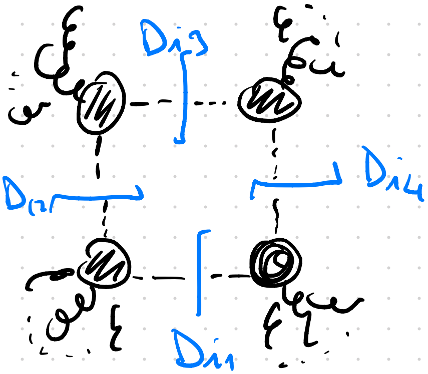
Let's see how to determine d_0, c_0, b_0, a_0 using generalised cuts

QUADRUPLE (MAXIMAL) CUT

loop momentum in $D=4$ has 4 components, therefore we can perform up to 4 cuts @ 1 loop

each possible quadruple cut (i.e. each choice of 4 propagators put on-shell) selects the coefficient d_0 of the corresponding box integral

Let's choose $\left[\dfrac{d_{i_1 i_2 i_3 i_4}}{D_{i_1} D_{i_2} D_{i_3} D_{i_4}} \right]$ for fixed i_j



$$\left.\begin{array}{l} \ell^2 = 0 \\ (\ell+q_1)^2 = 0 = 2\ell\cdot q_1 + q_1^2 = 0 \\ (\ell+q_2)^2 = 0 = 2\ell\cdot q_2 + q_2^2 = 0 \\ (\ell+q_3)^2 = 0 = 2\ell\cdot q_3 + q_3^2 = 0 \end{array}\right\} (*)$$

so we have $2\ell\cdot q_i = -q_i^2 \quad \forall\, i = 1,2,3$

now remember Van Neerven - Vermaseren BASIS

$$\ell^\mu = \sum_{j=1}^{3} (\ell\cdot q_j)\, v_j^\mu + (\ell\cdot n_4)\, n_4^\mu$$

$$= -\frac{1}{2} \sum_{j=1}^{3} q_j^2\, v_j^\mu + (\ell\cdot n_4)\, n_4^\mu$$

and now use $\ell^2 = 0$ from $(*)$

$$0 = \ell^2 = \frac{1}{4}\left(\sum_{j=1}^{3} q_j^2\, v_j^\mu\right)^2 + (\ell\cdot n_4)^2 \qquad \left\{\begin{array}{l} v_j\cdot n_4 = 0 \\ n_4\cdot n_4 = 1 \end{array}\right.$$

and obtain finally

$$(\ell\cdot n_4)^2 = -\frac{1}{4}\left(q_1^2 v_1^\mu + q_2^2 v_2^\mu + q_3^2 v_3^\mu\right)^2 =$$

in general 2 complex solutions:

$$\bar{\ell}^\mu_\pm = -\frac{1}{2}\sum_{j=1}^{3} q_j^2 v_j^\mu \pm \frac{1}{2}\sqrt{-\left(q_1^2 v_1^\mu + q_2^2 v_2^\mu + q_3^2 v_3^\mu\right)^2}$$

In special case of 4-gluon scattering, 1 box only:

$$(\ell\cdot n_4)^2 = \frac{1}{8} s_{12}^2 s_{23}^2$$

using $q_1 = p_1$, $q_2 = p_1+p_2$, $q_3 = p_1+p_2+p_3$

$p_i^2 = 0$; etc

so we have

$$(\ell\cdot n_4) = \pm\frac{1}{\sqrt{2}}\left(\frac{s_{12}s_{23}}{2}\right)$$

$$\bar{\ell}^\mu_\pm = \underbrace{-\frac{1}{2}\sum_{j=1}^{3} q_j^2 v_j^\mu}_{} \pm \frac{1}{\sqrt{2}}\left(\frac{s_{12}s_{23}}{2}\right) n_4^\mu$$

$q_1^2 = 0$ $\quad q_2^2 = (p_1+p_2)^2 = s_{12}$ $\quad q_3^2 = (p_1+p_2+p_3)^2 = p_4^2 = 0$

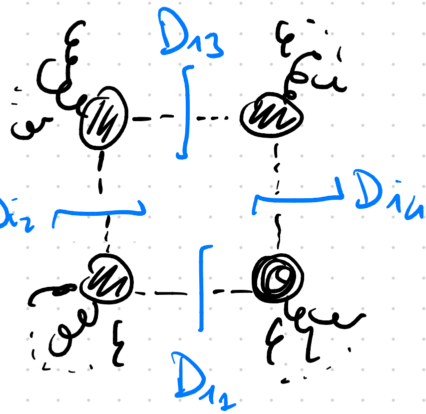
$$\bar{\ell}^\mu_\pm = -\frac{1}{2} s_{12} v_2^\mu \pm \frac{1}{\sqrt{2}}\left(\frac{s_{12}s_{23}}{2}\right) n_4^\mu$$

Now let's go back to general case, keep boxes:

$$A_N^{1\text{loop}} = \sum_{1 \le i_1 < i_2 < i_3 < i_4 < N} \frac{d_{i_1 \dots i_4}(\ell)}{D_{i_1} D_{i_2} D_{i_3} D_{i_4}} + \text{lower point}$$

Computing the 4-cut, we select 1 special box and get

$$A_N^{1\text{loop}}(\bar{\ell}_\pm)_{\text{cut}} =$$



$$= A_1^{\text{tree}}(\bar{\ell}_\pm)\, A_2^{\text{tree}}(\bar{\ell}_\pm)\, A_3^{\text{tree}}(\bar{\ell}_\pm)\, A_4^{\text{tree}}(\bar{\ell}_\pm)$$

$$= \text{from decomposition} = \underline{\underline{d_{i_1 i_2 i_3 i_4}(\bar{\ell}_\pm)}}$$

Fixed indices now, 1 box!

remember that

$$d_{i_1 \cdots i_4}(\bar{\ell}_\pm) = d_0 + d_1 \underbrace{(\bar{\ell}_\pm \cdot n_4)}_{\text{computed before}}$$

[general integrand for box]

$$= d_0 \pm d_1 \frac{1}{2} \sqrt{-(q_1^2 v_1^\mu + q_2^2 v_2^\mu + q_3^2 v_3^\mu)^2} \equiv \text{prod 4 tree-level amplitudes}$$

Let's give it a name

$$D_\pm = A_1^{tree}(\bar{\ell}_\pm)\, A_2^{tree}(\bar{\ell}_\pm)\, A_3^{tree}(\bar{\ell}_\pm)\, A_4^{tree}(\bar{\ell}_\pm)$$

$$\Rightarrow \quad d_0 = \frac{(D_+ + D_-)}{2}$$ physical coefficient

$$d_1 = \frac{1}{2} \frac{(D_+ - D_-)}{\sqrt{-(q_1^2 v_1^\mu + q_2^2 v_2^\mu + q_3^2 v_3^\mu)^2}}$$ "non-physical" coefficient

we need d_1 to determine triangle coefficients!

In fact let's imagine we want to focus now on a particular triangle contribution. Following the UNITARITY APPROACH, we can isolate this contribution using a triple cut → Of course, there will be in general various boxes that will have support on that cut; if we have computed all boxes at previous step, we can subtract them!

Focus on triangle $\frac{C_{i_1 i_2 i_3}}{D_{i_1} D_{i_2} D_{i_3}}$ ⇒ For fixed i_j!

$$\left(A_N^{1\,loop}(\ell) - \sum_{i_4} \left[\frac{d_{i_1 i_2 i_3 i_4}}{D_{i_4}} \right] \right) = \text{subtracts contamination from boxes}$$

all boxes that contain that triangle!

by computing the triple cut we write

$$\left[A_N^{1loop} - \sum_{i_4} \frac{d_{i_1 i_2 i_3 i_4}}{D_{i_4}} \right] \Bigg|_{Cut_{i_1 i_2 i_3}} = C_{i_1 i_2 i_3}$$

$$= C_0 + C_1 (\ell \cdot n_3) + C_2 (\ell \cdot n_4)$$

$$+ C_4 \left[(\ell \cdot n_3)^2 - (\ell \cdot n_4)^2 \right] + C_5 (\ell \cdot n_3)(\ell \cdot n_4)$$

$$+ C_6 (\ell \cdot n_3)^2 + C_7 (\ell \cdot n_4)^2 \quad \Big\} \; 7 \text{ coefficients}$$

on triple cut

$$\left[A_1^{tree}(\bar{\ell}) A_2^{tree}(\bar{\ell}) A_3^{tree}(\bar{\ell}) - \sum_{i_4} \frac{d_{i_1 \dots i_4}(\bar{\ell})}{D_{i_4}} \right] \quad (**)$$

TRIPLE & DOUBLE CUT

triple-cut condition freezes 3 of the 4 components of the loop momentum

$$\ell^2 = 0$$
$$(\ell + q_1)^2 = 0$$
$$(\ell + q_2)^2 = 0$$

general solution still depends on 1 free parameter α

free parameter!

$$\bar{\ell}^\mu_\pm(\alpha) = -\frac{1}{2}\left(q_1^2 v_1^\mu + q_2^2 v_2^\mu\right) \pm \frac{1}{2}\sqrt{-\left(q_1^2 v_1^\mu + q_2^2 v_2^\mu\right)^2}\left(\cos(\alpha)\, n_3^\mu + \sin(\alpha)\, n_4^\mu\right)$$

$\Uparrow$ transverse space

to fix all 7 coefficients you can evaluate eq (**) on 7 choices for $\bar{\ell}^\mu_\pm(\alpha)$!

Similarly, you can perform double cuts, and the loop momentum will be parametrized as $\bar{l}^\mu\underbrace{(\alpha,\beta)}_{\text{two degrees of freedom}}$

by evaluating this for random numerical values of α, β, you get a system of equations that you can solve to determine all coefficients b_i of bubbles!

Numerical approach is very efficient $\Rightarrow$ $\boxed{\text{NUMERICAL UNITARITY}}$

You can repeat the numerical "fitting" of the d_i, c_i, b_i, a_i for every "phase space" point (value of s_{ij}, kinematics!)

Used often to perform complicated 1-loop calculations numerically in a <u>completely automated way</u>!

this concludes first examples on applications of Unitarity @ 1 loop.

We have <u>not</u> discussed yet how to compute **RATIONAL PARTS** ; we'll see an example in few lectures, after we have learnt how to compute **MASTER INTEGRALS** .