5 – More on Color Ordering

Gluon Amplitudes, MHV, Parke Taylor.

WS 2021

TUM

COLOR ORDERING: n-gluon amplitudes

we saw that tree-level amplitude for $q\bar{q} \to gg$ can be decomposed into two "color ordered" ampl.

$$\mathcal{M}_{q\bar{q}\to g_a g_b} = M_1 t^a t^b + M_2 t^b t^a$$

Color ordered amplitudes are simple and from one we can obtain the others by crossings etc.

Let's see how this works for n-gluon amplitudes

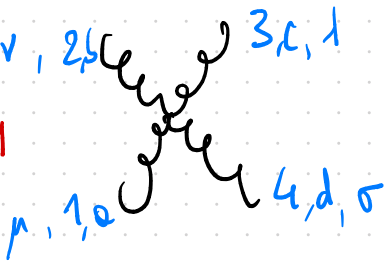
Let's rescale $\quad t^a = \frac{T^a}{\sqrt{2}} \Rightarrow \mathrm{Tr}[T^a T^b] = \delta^{ab}$

which moves factors of $\sqrt{2}$ from color algebra to Feynman rules, irrelevant at the moment.

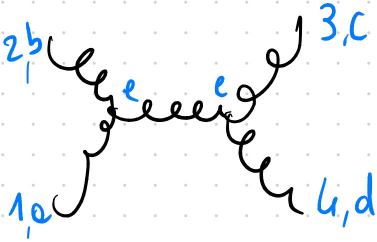
$\Rightarrow$ irrelevant, but follows literature conventions!

$$g^a_\mu(p_1) + g^b_\nu(p_2) + g^c_\lambda(p_3) + g^d_\sigma(p_4) \longrightarrow 0$$

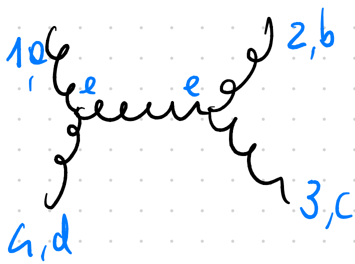
4 Feynman diagrams FOCUS ON COLOR FACTORS

1 

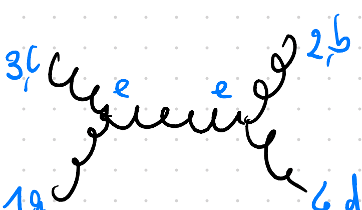
$$\sim -ig^2\Big[f^{abe}f^{cde}\left(g^{\mu\lambda}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\lambda}\right) + f^{ace}f^{bde}\left(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\sigma}g^{\nu\lambda}\right) + f^{ade}f^{bce}\left(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma}\right)\Big]$$

2 

$$\sim f^{abe} f^{cde}$$

3 

$$\sim f^{ade} f^{bce}$$

4 

$$\sim f^{ace} f^{bde}$$

so there are three possible products of f^{abc} etc

use $\mathrm{Tr}[T^a T^b] = \delta^{ab}$ plus

$[T^a, T^b] = i f^{abe} T^e \sqrt{2}$ to write

$$\mathrm{Tr}\left([T^a, T^b] T^c\right) = i\sqrt{2} f^{abe} \mathrm{Tr}[T^e T^c]$$

$$= i\sqrt{2} f^{abc}$$

$$\Rightarrow f^{abc} = -\frac{i\sqrt{2}}{2} \mathrm{Tr}\left([T^a, T^b] T^c\right)$$

$$= -\frac{i}{\sqrt{2}}\left[\mathrm{Tr}(T^a T^b T^c) - \mathrm{Tr}(T^b T^a T^c)\right]$$

$$= -\frac{i}{\sqrt{2}}\left[\mathrm{Tr}(T^a T^b T^c) - \underbrace{\mathrm{Tr}(T^a T^c T^b)}_{\text{cyclicity}}\right]$$

so

$$f^{abe} f^{cde} \propto \left[\mathrm{Tr}\left(T^a T^b T^e\right) - \mathrm{Tr}\left(T^a T^e T^b\right) \right]$$

$$\times \left[\mathrm{Tr}\left(T^c T^d T^e\right) - \mathrm{Tr}\left(T^c T^e T^d\right) \right]$$

<u>4 terms</u> - pick the first

$$A = \mathrm{Tr}\left(T^a T^b T^e\right) T\left(T^c T^d T^e\right) =$$

$$= T^a_{ij} T^b_{jk} T^e_{ki} \; T^c_{em} T^d_{mn} T^e_{ne}$$

$$\Rightarrow \quad T^e_{ki} T^e_{ne} \overset{\text{Fierz}}{=} \delta_{kl}\delta_{in} - \frac{1}{N}\delta_{ki}\delta_{ne}$$

$$= T^a_{ij} T^b_{jk} T^c_{km} T^d_{mi} - \frac{1}{N} T^a_{ij} T^b_{ji} T^c_{em} T^d_{me}$$

$$= \mathrm{Tr}\left(T^a T^b T^c T^d\right) - \frac{1}{N} \mathrm{Tr}\left(T^a T^b\right) \mathrm{Tr}\left(T^c T^d\right)$$

$$B = -\mathrm{Tr}(T^a T^b T^e)\,\mathrm{Tr}(T^c T^e T^d)$$

$$= \dots = -\mathrm{Tr}(T^a T^b T^d T^c) + \frac{1}{N}\mathrm{Tr}(T^a T^b)\,\mathrm{Tr}(T^d T^c)$$

$$C = -\mathrm{Tr}(T^a T^e T^b)\,\mathrm{Tr}(T^c T^d T^e)$$

$$= -\mathrm{Tr}(T^b T^a T^c T^d) + \frac{1}{N}\mathrm{Tr}(T^a T^b)\,\mathrm{Tr}(T^c T^d)$$

cyclic T^a beginning

$$D = +\mathrm{Tr}(T^a T^e T^b)\,\mathrm{Tr}(T^c T^e T^d)$$

$$= \mathrm{Tr}(T^b T^a T^d T^c) - \frac{1}{N}\mathrm{Tr}(T^a T^b)\,\mathrm{Tr}(T^c T^d)$$

cyclic T^a beginning

summing $A+B+C+D$

↳ the $\frac{1}{N}$ part cancels ! $\Rightarrow$ PHOTON DECOUPLING !

$$f^{abe} f^{cde} \sim \mathrm{Tr}(\overset{1}{T^a}\overset{2}{T^b}\overset{3}{T^c}\overset{4}{T^d}) - \mathrm{Tr}(\overset{1}{T^a}\overset{2}{T^b}\overset{4}{T^d}\overset{3}{T^c})$$

$$- \mathrm{Tr}(\overset{1}{T^a}\overset{3}{T^c}\overset{4}{T^d}\overset{2}{T^b}) + \mathrm{Tr}(\overset{1}{T^a}\overset{4}{T^d}\overset{3}{T^c}\overset{2}{T^b})$$

$$\Rightarrow \{\mathrm{Tr}(1234);\ \mathrm{Tr}(1243);\ \mathrm{Tr}(1342);\ \mathrm{Tr}(1432)\}$$

similary for the other two we find

$$M_{gggg} = M_1 \mathrm{Tr}(1234) + \text{all permutations of } (2,3,4)$$

$$\Rightarrow \underline{\underline{3! = 6}}$$

$$= \sum_{\sigma \in P_3} M_4[1\,\sigma(234)]\,\mathrm{Tr}(1\,\sigma(234))$$

$\Uparrow$

4 gluons

color ordered amplitudes!

START FROM HERE

REMARK 1.

In order to expose color traces, we have used only two identities valid for SU(N)

- $f^{abc} = -\frac{i}{\sqrt{2}}\left(Tr(T^a T^b T^c) - Tr(T^a T^c T^b)\right)$

- $T^a_{ij} T^a_{ke} = \delta_{ie}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{ke}$ FIERZ

this last identity is the statement that SU(N) generators form the complete set of traceless and hermitian $N \times N$ matrices

$\Downarrow$

Contract δ_{ij}

$$T^a_{ii} T^a_{ke} = \delta_{ie}\delta_{ik} - \delta_{ke} = 0 \; !$$

$\Rightarrow$ $T^a_{ii} = 0$ thanks to second term 7 if we consider $U(N)$ instead of $SU(N)$

$U(N) = SU(N) \times U(1)$

↳ additional generator $\propto \mathbb{1}$

$$T^{a_{U(1)}}_{ij} = \frac{1}{\sqrt{N}} \delta_{ij}$$

if we add this to set of $SU(N)$ generators

$$\Rightarrow T^a_{ij} T^a_{kl} = \delta_{il} \delta_{jk} \quad \text{for } U(N) \, !$$

in jargon, extra generator is called "photon"

since it commutes with all $SU(N)$ generators, it does not couple to gluons !

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{ig}{\sqrt{2}} [A_\mu, A_\nu]$$ Field strength

↳ self-coupling !

IN GENERAL, this "photon" could couple to fermions instead ! $\sim \bar{\psi} \gamma^\mu \psi A_\mu^{U(1)}$!

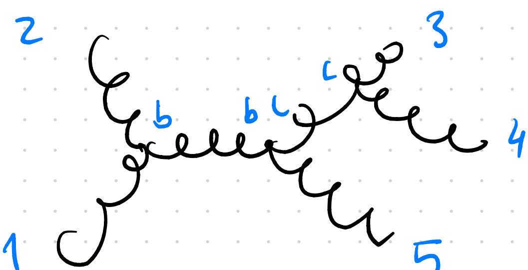
the fact that all terms coming from $\frac{1}{N} \delta_{ij} \delta_{kl}$ of Fierz rearrangement falls in 4-gluon at tree-level is that for these amplitudes we can use $U(N)$, because there are no fermions involved where the extra photon could attach !

REMARK 2.

As long as we consider n-gluon amplitudes at tree-level in QCD, fermions never appear in Feynman diagrams $\Longrightarrow$ Fermion line must close on a loop !

$\Longrightarrow$ IMPORTANT CONSEQUENCE :

@ tree level all products of $f^{12b} f^{bc5} f^{c34} \dots$



$\Longrightarrow$ can all be transformed using

$f^{abc} \propto Tr([T^a, T^b] T^c)$ and then Fierzed

as $T^a_{ij} T^a_{ke} = \delta_{ie}\delta_{jk}$

$\Rightarrow$ this produces only single traces of the type

$\mathrm{Tr}(T^1 T^2 \dots T^n)$ and permutations thereof

$= \mathrm{Tr}(1 2 \dots n)$ $(n-1)!$

Color ordered amplitudes

$$\mathcal{M}_{ng} = \sum_{\sigma \in S_{n-1}} \mathcal{M}_n(1\,\sigma(2 \dots n))\, \mathrm{Tr}(1\,\sigma(2 \dots n)))$$

cyclicity !

<u>IMPORTANT</u> : at loop-level $SU(N) \neq U(N)$

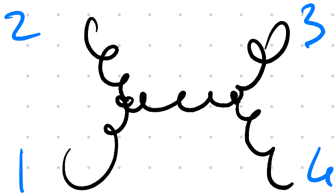
similarly also when fermions are <u>involved</u>.

4g @ 1-loops $-\ \mathrm{Tr}(1\ \sigma(234))$

$$- \mathrm{Tr}(1\,j)\,\mathrm{Tr}(k\,\ell) = \begin{cases} \mathrm{Tr}(12)\,\mathrm{Tr}(34) \\ \mathrm{Tr}(13)\,\mathrm{Tr}(24) \\ \mathrm{Tr}(14)\,\mathrm{Tr}(23) \end{cases}$$

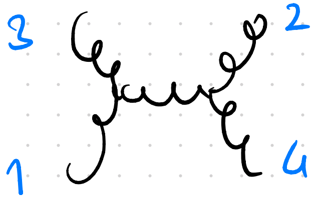
• WHICH FEYNMAN DIAGRAMS contribute to each trace (at tree-level!) ?

We have seen that $\mathrm{Tr}(1234)$ receives contribution from color algebra of diagram 2



$\Rightarrow$ Follow right order!

DIAG 4

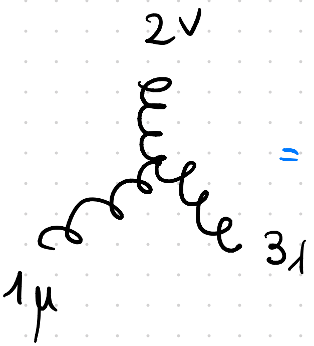


$\sim f^{13e} f^{24e}$

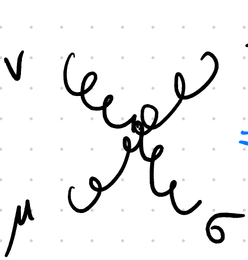
same with $2 \leftrightarrow 3$

$\Rightarrow \mathrm{Tr}(1324), \mathrm{Tr}(1342), \mathrm{Tr}(1243), \mathrm{Tr}(1423)$

gluons are not properly ordered, so this diagram does not contribute it is convenient to introduce $\boxed{\text{COLOR ORDERED FEYNMAN RULES}}$



$$= \frac{ig}{\sqrt{2}}\left[g^{\mu\nu}(p_1-p_2)^\lambda + g^{\nu\lambda}(p_2-p_3)^\mu + g^{\lambda\mu}(p_3-p_1)^\nu\right]$$



$$= \frac{ig^2}{2}\left[2g^{\mu\lambda}g^{\nu\sigma} - g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\sigma}g^{\nu\lambda}\right]$$

ALL and ONLY graphs which are properly ordered should be included when a process is computed with these Feynman Rules

REMARK 3

1) Tree basis is clearly overcomplete!

$$M_{ng} = \sum_{\sigma \in S_{n-1}} M_n(1\,\sigma(2\cdots n))\, Tr(1\,\sigma(2\cdots n))$$

$\Rightarrow$ 4 gluons, there are 6 traces, but we only started from 3 color factors!

$f^{abe} f^{cde}$ $\quad$ $f^{ace} f^{bde}$ $\quad$ $f^{ade} f^{bce}$

MOREOVER JACOBI IDENTITY

$$f^{abe} f^{ecd} + f^{ace} f^{edb} + f^{ade} f^{ebc} = 0$$

so actually only two are supposed to be independent!

2) WHY ARE COLOR-ORDERED AMPLITUDES USEFUL?

* "PARTIAL" ORTHOGONALITY ensures gauge invariance

$$\sum \mathrm{Tr}(12\ldots n)\left[\mathrm{Tr}(\sigma(12\ldots n))\right]^* = N^{(n-2)}(N^2-1)\left(\delta_{\sigma 1} + \mathcal{O}\left(\frac{1}{N^2}\right)\right)$$

↑ Leading color sub

gauge invariance must hold order by order in $1/N$

* ONLY COLOR ORDERED DIAGRAMS CONTRIBUTE

↳ Fewer terms to compute

* Overcompleteness manifests as further relations among these amplitudes!

PROPERTIES of n-gluon color ordered amplitudes.
(at tree-level)

1. Cyclicity $A(12\ldots n) = A(2\ldots n1)$

$\Rightarrow$ reason why they are $(n-1)!$

2. Reflection $A(12\ldots n) = (-1)^n A(n\ldots 21)$

$\Rightarrow$ can be proved using antisymmetry of color ordered Feynman rules.

| true at ℓ-LOOPS for gluons |

3. Photon Decoupling

$$A^{tree}_{ng}(1,2,3,\ldots,n) + A^{tree}_{ng}(2,1,3,\ldots n) + A^{tree}_{ng}(2,3,1,\ldots,n)$$

$$+ \cdots + A^{tree}_{ng}(2,3,\ldots,n-1,1,n) = 0$$

Consider tree-level decomposition

$$A = \sum_{\sigma(23\ldots n) = S_n/\mathbb{Z}_n} A(1\,\sigma(23\ldots n))\ \mathrm{Tr}(1\,\sigma(23\ldots n)) \quad (*)$$

remember, n gluon amplitudes @ tree level are equal if we compute them in $SU(N)$ or $U(N)$

but in $U(N)$ if we pick for one gluon the $U(1)$ photon, the amplitude must vanish, as we saw!

put $T^1 = \mathbb{1}$ in $(*)$ and enforce that it vanishes, this gives the photon decoupling

EXPLICITLY FOR 4 - GLUON CASE

$$A_1 \mathrm{Tr}(1234) + A_2 \mathrm{Tr}(1243) + A_3 \mathrm{Tr}(1324)$$

$$+ A_4 \mathrm{Tr}(1342) + A_5 \mathrm{Tr}(1423) + A_6 \mathrm{Tr}(1432)$$

if $1 = \mathbb{1}$

$$A_1 \mathrm{Tr}(234) + A_2 \mathrm{Tr}(243) + A_3 \mathrm{Tr}(324)$$

$$+ A_4 \mathrm{Tr}(342) + A_5 \mathrm{Tr}(423) + A_6 \mathrm{Tr}(432) = 0$$

$$\mathcal{M} = A(1234) + A(1342) + A(1423)$$

$$= A(1234) + A(2134) + A(2314)$$

using cyclicity

in general for n gluons, sum of $n-1$ amplitudes:

$$A(123\ldots n) + A(213\ldots n) + \cdots + A(23\ldots 1n) = 0$$

PHOTON DECOUPLING IDS.

Photon Decoupling is sometimes called DUAL WARD IDENTITY $\Rightarrow$ it can be derived in string theory, dual theory of some QFT through AdS / CFT Correspondence.

there are more identities, one can prove there are only $(n-3)!$ independent color ordered amplitudes

$\triangleright$ 4-gluon scattering only 1 amplitude!

- Kleiss-Kuijf relations $\rightarrow$ due to overcompleteness of color traces $(n-2)!$
(for 4 gluons $\equiv$ $U(1)$ decoupling)

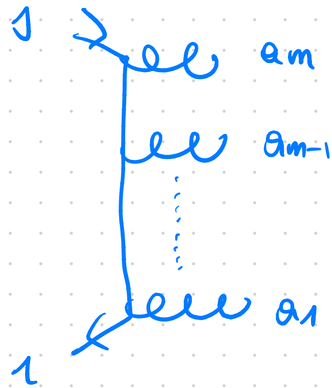
- BCJ relations $\Rightarrow$ we will see an example $(n-3)!$

COLOR DECOMPOSITION WITH FERMIONS

Remember, external quark lines start and end

with T^as with open fundamental indices

$$\Rightarrow \quad [T^{a_1} \cdots\cdots T^{a_m}]_{ij}$$



tree-level color decomposition for $q\bar{q}\underbrace{g \cdots g}_{n-2}$

$$A^{tree} = \sum_{\sigma \in S_{n-2}} \left(T^{a_{\sigma(3)}} \cdots T^{a_{\sigma(n)}}\right)_{ij} A_n^{tree}\left(1_{\bar{q}}, 2_q, \sigma(3) \ldots \sigma(n)\right)$$

Ex: do explicitly $q\bar{q}Q\bar{Q}g$, only color decomposition!

4- GLUONS @ tree level

- We have in principle $2^4 = 16$ helicity configurations and $3! = 6$ color ordered amplitudes. many related by BOSE SYMMETRY !

Pick 1 ordering $A(1234)$, the others related by crossing, in general, plus other relations !

* Helicities

- $A(1^+2^+3^+4^+) = A(1^-2^-3^-4^-)$ parity = 1

similarly $\Rightarrow$ 8 independent ones !

- $A(1^-2^+3^+4^+)$ 4 of these = 4

- $A(1^-2^-3^+4^+) = \frac{4\cdot 3}{2} = 6$ / 2 parity = 3

- all + (or all −) are always zero @ tree level

remember that if we choose the same gauge vector for 2 gluons $\Rightarrow$ $\boxed{\varepsilon^{+}(p_i, q)^{\mu}\, \varepsilon^{+}_{\mu}(p_j, q) = 0}$

if you think about it, @ tree level all gluon amplitudes have only stuff like

$$\varepsilon_1^{\mu_1} \dots \varepsilon_n^{\mu_n} \left[A_{\mu_1 \dots \mu_n} \right]$$

$$A_{\mu_1, \dots \mu_n} = \{ p_{1\mu_1} \dots p_{n\mu_n} ;\; g_{\mu_i \mu_j} \text{ etc} \}$$

but at tree level, only 3-gluon vertex gives p_μ !

there are always FEWER vertices than external lines $(n-2)$

so each term must have at least one $\varepsilon_i^{\pm} \cdot \varepsilon_j^{\pm} = 0$!

this proves easily that at tree-level all

n-gluon amplitudes with EQUAL HELICITIES

are zero!

WHAT ABOUT 1 negative hel ?

if 1 helicity is minus, we'll have (say ε_1^-)

$$\varepsilon_1^- \cdot \varepsilon_j^+ \; ; \qquad \varepsilon_i^+ \cdot \varepsilon_j^+ \qquad i,j = 2, \dots, n$$

· choose all $\varepsilon_j^+(p_1, p_j)$ for $j \neq 1$!

$\Rightarrow \quad \varepsilon_i^+ \cdot \varepsilon_j^+ = 0$ equal reference momentum

$$\varepsilon_1^-(p_1, q) \cdot \varepsilon_j^+(p_j, p_1) \propto \frac{[1 \gamma_\mu q\rangle [1 \gamma^\mu j\rangle}{\langle 1q\rangle \quad [j1]}$$

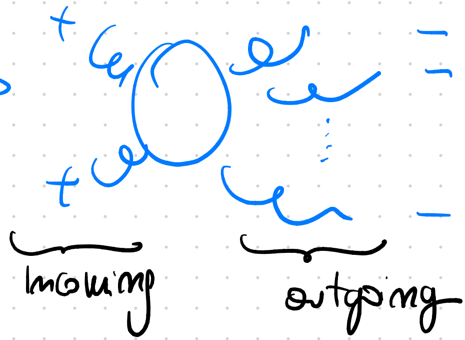
$$\propto [11] = 0 \quad !$$

so $A(1^- 2^+ \dots n^+) = 0$ tree level!

so we proved that ALL Equal hel
ALL with 1 Diff hel $= 0$

the first amplitudes that aren't zero, are the so-called MAXIMALLY HELICITY VIOLATING (MHV)

All + $\Rightarrow$



$\rightarrow$ All VIOLATED!

1 minus

A bit less VIOLATING

2 minuses

the "most" we can violate helicity without getting zero.

for 4-gluons then we are only left with 1 amplitude to compute

$A(1^- 2^- 3^+ 4^+)$ all the others can be obtained by PARITY,

get all types of $--++$ [and crossings of external legs.

how do I get $-+-+$?

$A(1^- 2^+ 3^- 4^+)$ can be obtained from photon decoupling as

$$A(1234) + A(2134) + A(2314) = 0$$

cyclicity

$$\hookrightarrow A(\overset{-+-+}{1234}) + A(\overset{--++}{1342}) + A(\overset{-++-}{1423}) = 0$$

• moreover, we know that (REFLECTION IDS.)

$$A(1234) = A(4321) \to A(1432)$$
$$A(1243) = A(3421) \to A(1342)$$
$$A(1324) = A(4231) \to A(1423)$$

$\leadsto$

$$A(1^-2^+3^-4^+) = -A(1 3^- 2^+ 4^+) - A(1 3^- 4^+ 2^+)$$

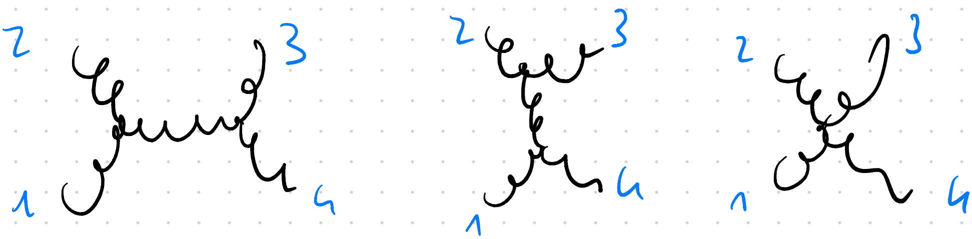
so if I know $--++$

I can get $\underline{\underline{-+-+}}$

plus photon decoupling and BCJ $\Rightarrow$ I need only one.

PARKE–TAYLOR

In your exercise, you'll verify that putting together the color ordered Feynman diagrams



you can get (all incoming)

$$A(1^+2^+3^-4^-) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

reasonable $\Rightarrow$

$$\left.\begin{aligned} 1 &\Rightarrow z^2 \\ 2 &\Rightarrow z^2 \\ 3 &\Rightarrow 1/z^2 \\ 4 &\Rightarrow 1/z^2 \end{aligned}\right\}$$

Correct Little Group scaling

similarly FOR ALL OUTGOING GLUONS

$$A(1^-2^-3^+4^+)_{OUT} = A(1^+2^+3^-4^-)_{IN}$$

$\boxed{\text{some books like all outgoing!}}$

notice that (stay OUTGOING FOR NOW)

$$A(1^-2^+3^-4^+) = -A(1^-3^-2^+4^+) - A(1^-3^-4^+2^+)$$

using

$$A(1^-3^-2^+4^+) = \frac{\langle 13\rangle^4}{\langle 13\rangle\langle 32\rangle\langle 24\rangle\langle 41\rangle}$$

exchanging $2 \Leftrightarrow 3$

$$A(1^-3^-4^+2^+) = \frac{\langle 13\rangle^4}{\langle 13\rangle\langle 34\rangle\langle 42\rangle\langle 21\rangle}$$

summing them

$$= \frac{\langle 13\rangle^4}{\langle 13\rangle\langle 24\rangle}\left[\frac{1}{\underline{\langle 32\rangle}\underline{\langle 41\rangle}} + \frac{1}{\underline{\langle 34\rangle}\underline{\langle 12\rangle}}\right]$$

$$= -\frac{\langle 13\rangle^4\left[\langle 34\rangle\langle 12\rangle + \langle 32\rangle\langle 41\rangle\right]}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle\ \langle 13\rangle\langle 24\rangle}$$

$$= +\frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\underbrace{\left[+\frac{\langle 21\rangle\langle 34\rangle + \langle 23\rangle\langle 41\rangle}{\langle 13\rangle\langle 24\rangle[42][31]}\right][42][31]}$$

$$+[43]\langle 31\rangle \qquad\qquad +[41]\langle 13\rangle$$

$$\frac{-\underbrace{[42]\langle 21\rangle}[13]\langle 34\rangle - \underbrace{[42]\langle 23\rangle}[31]\langle 14\rangle}{s_{13}\ s_{24}}$$

$$\parallel$$

$$\frac{s_{34}\cancel{s_{13}} + s_{14}\cancel{s_{13}}}{\cancel{s_{13}}\ s_{24}} = \frac{s_{12}+s_{23}}{s_{13}}$$

$$= -1 \ \checkmark$$

25

so we find that

$$A(1^- 2^+ 3^- 4^+) = \frac{\langle 13 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}$$

in general, one finds that for n-gluon scattering

$$A(1^+ \dots i^- \dots j^- \dots n^+) = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle}$$

always!

All MHV amplitudes, with as many gluons as you might like

$n=5 \Rightarrow$ sum 10 diagrams

$n=6 \Rightarrow$ sum 38 diagrams $\quad n=7$, 157 diagrams impressive simplicity hidden in hundreds or thousands of different terms ... something must be going on here!

$\Rightarrow$ we'll prove all Parke-Taylor formulas shortly, using ON-SHELL RECURSION techniques

BCFW

before getting there, we need to talk about general amplitudes with complex momenta