4 - Unitarity 1/2

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TUM

In the previous lectures we have proven that any 1loop N-point Amplitude can be decomposed as

$$A_N^{1\ell} = \sum_{j=1}^{n_{box}} c_{4,j}\, I_{(4)}^{j} + \sum_{j=1}^{n_{tri}} c_{3,j}\, I_{(3)}^{j} + \sum_{j=1}^{n_{bub}} c_{2,j}\, I_{(2)}^{j} + \sum_{j=1}^{n_{TAD}} c_{1,j}\, I_{(1)}^{j} + R$$

at the INTEGRAL level.

the method of generalized Unitarity allows us to determine the coefficients $c_{4,j}$, $c_{3,j}$, $c_{2,j}$, $c_{1,j}$, and the rational part R, only resorting to the manipulation of ON-SHELL quantities.

$\Rightarrow$ products of on-shell tree-level amplitudes

GENERAL IDEA (why "unitarity")

We start from the S matrix Unitarity

$SS^\dagger = \mathbb{1}$ $\Longrightarrow$ implies conservation probability

Write $S = \mathbb{1} + iT$

↑ non-trivial scattering

then Unitarity Requirement becomes

$$(\mathbb{1} + iT)(\mathbb{1} - iT^\dagger) = \mathbb{1}$$

$$\mathbb{1} + i(T - T^\dagger) + TT^\dagger = \mathbb{1}$$

$$(*) \quad \boxed{-i(T - T^\dagger) = TT^\dagger}$$

Formal Statement OPTICAL THEOREM

Non perturbative (non-linear) statement

let's now expand T perturbatively, for example in QCD (coupling "g")

$$T = g^2 T^{(0)} + g^4 T^{(1)} + g^{(6)} T^{2} + \dots$$

non trivial interaction at tree-level

plug this into (*) to find order by order in g

ORDER g^2:

- $-i(T^{(0)} - T^{(0)\dagger}) = 0 \Rightarrow \underline{\underline{T^{(0)} = T^{(0)\dagger}}}$

$\Rightarrow$ tree-level amplitude has no <u>branch cut</u>

order g^4

- $-i(T^{(1)} - T^{(1)\dagger}) = T^{(0)} T^{(0)\dagger} = T^{(0)} T^{(0)}$

one-loop

"products" of tree-level

Consider equation evaluated sandwiching between incoming / outgoing asymptotic states :

$$\langle out | T^{\dagger} | in \rangle = \langle in | T | out \rangle^{*} = T_{io}^{*}$$

then 1-loop equation becomes

$$-i\left(T_{oi}^{(1)} - \underline{\underline{T_{io}^{(1)*}}}\right) = \langle out | T^{(0)} T^{(0)} | in \rangle$$

$$= \int d\mu \, \langle out | T^{(0)} | \mu \rangle \langle \mu | \, T^{(0)} | in \rangle$$

sum over all particle states (helicities bosons, fermions etc)

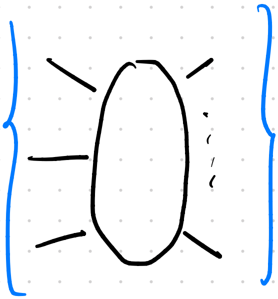
now, using time-reversal I can write $\underline{T_{io}^{(1)*} = T_{oi}^{(1)*}}$

$$2 \, \text{Im}\left(T_{oi}^{(1)}\right) = \int d\mu \, T_{o\mu}^{(0)} T_{\mu i}^{(0)}$$

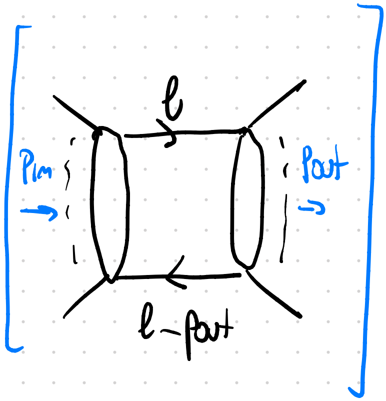
$$2\,\mathrm{Im}\left(T^{(1)}_{0i}\right) = \int d\mu \; T^{(0)}_{0\mu}\, T^{(0)}_{\mu i}$$

implies 1 loop amplitude has non-trivial analytic structure $\Rightarrow$ branch cuts when intermediate particles can go on-shell (see previous course!)

$$\Downarrow$$

$2\,\mathrm{Im}\Big\{$  $\Big\} =$

$$\int d^4\ell \; \delta^{(+)}(\ell^2)\, \delta^{(+)}\big((\ell - p_{out})^2\big)$$



remember energy > 0 $\quad \theta(p_{out} - \ell)$!

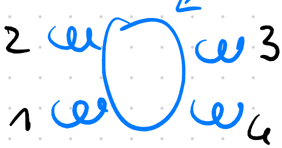
on-shell intermediate states

CUTKOSKY VELTMANN RULES

Unitarity can be used to "FIX" the coefficients of the expansion into master integrals without having to compute Feynman diagrams!

EXAMPLE: the best way to understand how this works is with an example. Let us consider 4-gluon scattering in **PURE YM**

only gluons allowed



each gluon can have 2 helicities

$h = \pm$

$A(1^+2^+3^+4^+)$ $\quad$ $A(1^+2^+3^+4^-)$ + permutations

$A(1^+2^+3^-4^-)$ + permutations

pure YM means, only gluons in the game

Feynman Rules:

↑ ghosts

Let's take one of the so-called MHV amplitudes

COLOR ORDERED (gluons only appear in a given permutation)

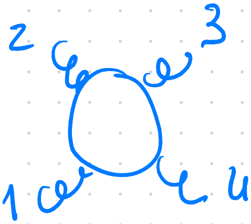
We know now that:

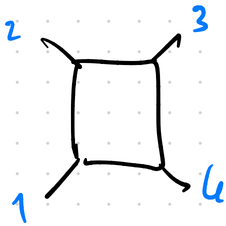
$$A^{(1\,loop)}_{4point}(1^+2^+3^-4^-) = c_4 I_4 + \sum_i c_{3,i} I_{3,i} + \sum_i c_{2,i} I_{2,i} + R$$

the idea is to take "cuts" or "discontinuities" of this equation (lhs and rhs)

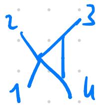
R and $c_{k,i}$ do not contribute (rational functions)

this is also a reason why amplitudes without rational part R are called CUT CONSTRUCTIBLE



$\Big\downarrow$ s_{23}- channel $= C_4$ 

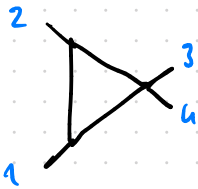
$\longrightarrow$ s_{12}-channel

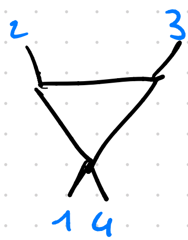
$\Big\{ + \tilde{C}_{3,1}$ 

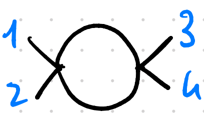
$\Big\}$ $\Big\{ + \tilde{C}_{3,2}$ 

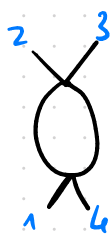
$\Big\}$

// same // same by symmetry

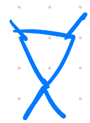
$+ \; C_{3,1}$ 

$+ \; C_{3,2}$ 

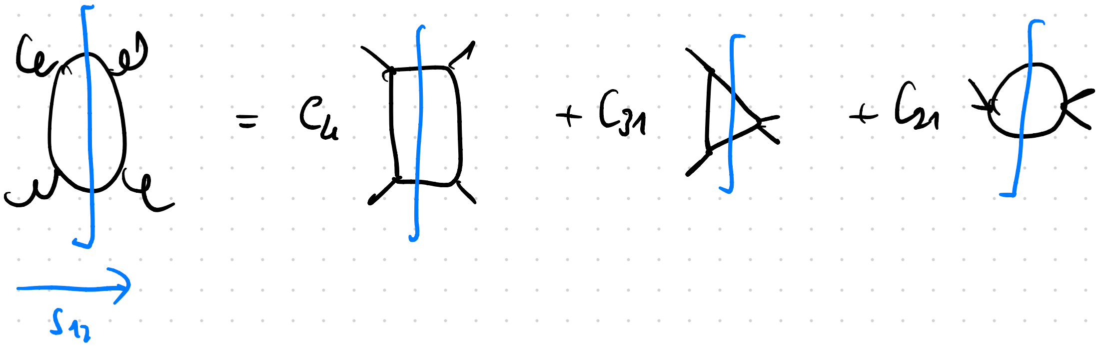
$+ \; C_{2,1}$ 

$+ \; C_{2,2}$ 

$+ \; R$

- S12- CHANNEL, cut kills R,  , 

so we get :



or more precisely:

$$\sum_{h_1, h_2} \int d\mu \; A_4^{(\text{tree})}(1^+ 2^+ \ell_1^{h_1} \ell_2^{h_2}) \, A_4^{(\text{tree})}(3^+, 4^+, -\ell_2^{-h_2}, -\ell_1^{-h_1})$$

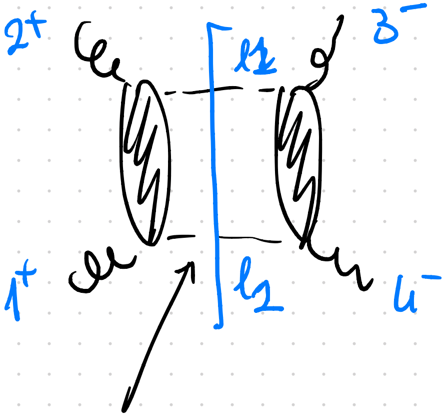
$$= C_4 \, \text{Cut}_{s_{12}}(I_4) + C_{3,1} \, \text{Cut}_{s_{12}}(I_{3,1}) + C_{2,1} \, \text{Cut}_{s_{12}}(I_{2,1})$$

where

$$d\mu = d^4\ell_1 \, d^4\ell_2 \; \underbrace{\delta^{(+)}(\ell_1^2) \, \delta^{+}(\ell_2^2) \; \delta^{(4)}(\ell_1 + \ell_2 + p_1 + p_2)}$$

"Double cut", puts the two internal particles on-shell.

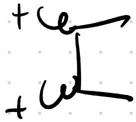
Left-Hand-Side



these ones must be gluons with $\underline{\underline{h = -}}$

in fact: all + tree-level $= 0$

all +, one - tree-level $= 0$

also 

$= 0$

or even with scalars turn out to be <u>zero</u>!

more general than pure <u>YM</u>!

so left-hand-side becomes

$$\int d\mu \; A_4^{(\text{tree})}(1^+ 2^+ \ell_1^- \ell_2^-)\; A_4^{(\text{tree})}(3^- 4^- \, -\ell_2^+ \, -\ell_1^+)$$

$$\ell_1 + \ell_2 = p_3 + p_4 = -(p_1 + p_2)$$

it's very easy to write the integrand, remembering Parke-Taylor formula @ tree level

$$A_n^{(\text{tree})}(1^- 2^- \dots i^+ \dots j^+ \dots n^-) = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle \dots \langle n1\rangle}$$

so we get

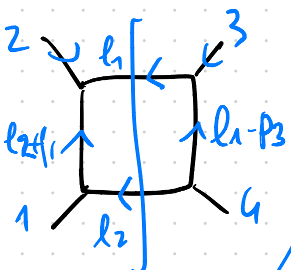
$$\int d\mu \; \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 2\ell_1\rangle\langle \ell_1\ell_2\rangle\langle \ell_2 1\rangle} \; \frac{\langle \ell_1\ell_2\rangle^4}{\langle \ell_1 3\rangle\langle 34\rangle\langle 4\ell_2\rangle\langle \ell_2\ell_1\rangle}$$

$$= -\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \int d\mu \; \frac{\langle 23\rangle\langle 41\rangle\langle \ell_1\ell_2\rangle^2}{\langle 2\ell_1\rangle\langle \ell_2 1\rangle\langle \ell_1 3\rangle\langle 4\ell_2\rangle}$$

we could try to perform the integral and then compare to cuts of "master integrals" this is in general very complicated, and also not necessary! the real power of unitarity is that we can make it work at the INTEGRAND LEVEL

· Cut the master integrals, noting that "$d\mu$" is the same.

BOX



$$= \int d\mu \, \frac{1}{(\ell_2+p_1)^2(\ell_1-p_3)^2}$$

which can be written as:

[remember $\ell_i^2 = 0$ (on-shell)

$\ell_1+\ell_2 = -p_1-p_2$ etc]

$$= -\int d\mu \, \frac{1}{\langle \ell_2 1\rangle [1 \ell_2] \langle \ell_1 3\rangle [3 \ell_1]}$$

Now let's compare left and right hand side of the INTEGRAND LEVEL.

$$A_4^{\text{tree}}(1^+2^+3^-4^-)\cdot \boxed{-\frac{\langle 23\rangle\langle 41\rangle\langle \ell_1\ell_2\rangle^2}{\langle 2\ell_1\rangle\langle \ell_2 1\rangle\langle \ell_1 3\rangle\langle 4\ell_2\rangle}}$$

$$\overset{?}{\sim} + \frac{c_4}{(\ell_2+p_1)^2(\ell_1-p_3)^2} + \dots$$

to be fixed

other contributions from 

multiply up and down by $[1\ell_2][3\ell_1]$

$$+\frac{\langle 23\rangle\langle 41\rangle \overbrace{[3\ell_1]\langle\ell_1\ell_2\rangle}^{[34]\cancel{\langle 4\ell_2\rangle}} \overbrace{[1\ell_2]\langle\ell_2\ell_1\rangle}^{-[12]\cancel{\langle 2\ell_1\rangle}}}{\cancel{\langle 2\ell_1\rangle}\cancel{\langle 4\ell_2\rangle}(\ell_2+p_1)^2(\ell_1-p_3)^2}$$

using $\ell_1+\ell_2 = p_3+p_4 = -p_1-p_2$

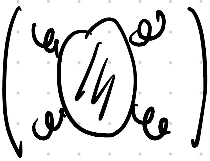
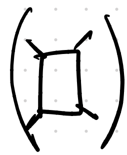
$$= - \frac{\langle 23\rangle \langle 41\rangle [34][12]}{(\ell_2+p_1)^2(\ell_1-p_3)^2}$$

$\langle 41\rangle[34] \to -\langle 43\rangle[32]$ using momentum conservation

$$= \frac{s_{23}\, s_{12}}{(\ell_2+p_1)^2(\ell-p_3)^2}$$

(in $D=4$ dimensions!)

So we have found, at the INTEGRAND LEVEL:

$\mathrm{Cut}_{s_{12}}\Big($  $\Big) = \underbrace{s_{12}\, s_{23}\, A_4^{(\text{tree})}(1^+2^+3^-4^-)}_{\text{which fixes } c_4 = s_{12} s_{23} A_4^{\text{tree}}}\ \mathrm{Cut}_{s_{12}}\Big($  $\Big)$

the box alone captures everything in the s_{12}-channel

there is no triangle / bubble contribution to $A(1^+2^+3^-4^-)$ in the s_{12}-channel $\Rightarrow$ true in ANY GAUGE THEORY!

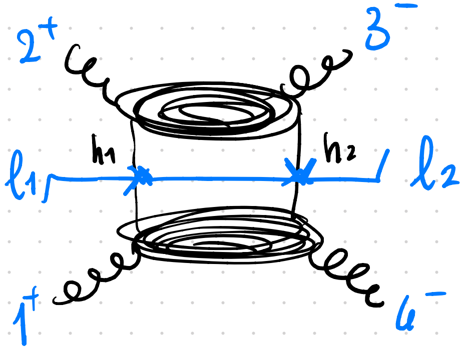
. just looking at one "unitarity cut", we have fixed the coefficient of the BOX.

$$A_4^{(1loop)}(1^+2^+3^-4^-) = s_{12}\, s_{23}\, A_4^{(tree)}(1^+2^+3^-4^-)\, I_4$$

$$+ \underbrace{C_{3,2}\, I_{3,2} + C_{2,2}\, I_{2,2}}_{} + R$$

only with cuts in s_{23}!
no contributions to s_{12} cut!

s_{23}- CHANNEL



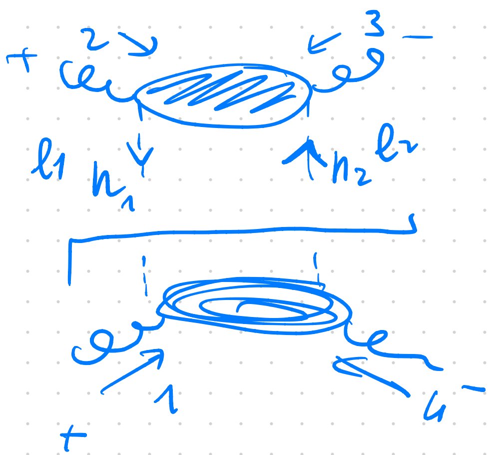
LHS

$$= \sum_{h_1,h_2} \int d\tilde{\mu}\; A_4^{(tree)}(-\ell_1^{-h_1}, 2^+, 3^-, \ell_2^{h_2}) \times A_4^{(tree)}(-\ell_2^{-h_2}, 4^-, 1^+, \ell_1^{h_1})$$

RHS

$$= C_4\, \mathrm{Cut}_{s_{23}}(I_4) + C_{3,2}\, \mathrm{Cut}_{s_{23}}(I_{3,2}) + C_{2,2}\, \mathrm{Cut}_{s_{23}}(I_{2,2})$$

Now situation for on-shell intermediate particles is different:



$$h_{1,2} = \{ \underbrace{-1, 1}_{\text{pure YM}}, \underbrace{-\tfrac{1}{2}, \tfrac{1}{2}, 0}_{\text{anything else}} \}$$

scalar particle

$$d\tilde{\mu} = d^4\ell_1\, d^4\ell_2\, \delta^{(+)}(\ell_1^2)\, \delta^{(+)}(\ell_2^2)\, \delta^{(4)}(-\ell_1+\ell_2+p_2+p_3)$$

We need then 4-point amplitudes with gluons of different helicities, exchanging other gluons

fermions

? scalars (N=4 SYM)

From little-group invariance arguments it's easy to see

$$A_4^{(tree)}\left(-\ell_1^{-h_1}, 2^+, 3^-, \ell_2^{h_2}\right) = \delta_{h_1 h_2} \frac{\langle -\ell_1 2\rangle^{2+2h_1} \langle \ell_2 2\rangle^{2-2h_1}}{\langle -\ell_1 2\rangle\langle 23\rangle\langle 3\ell_2\rangle\langle \ell_2(-\ell_1)\rangle}$$

$$A_4^{(tree)}\left(-\ell_2^{-h_2}, 4^-, 1^+, \ell_1^{h_1}\right) =$$

$$(-1)^{2h_1} \delta_{h_1 h_2} \frac{\langle -\ell_2 1\rangle^{2+2h_1} \langle \ell_1 1\rangle^{2-2h_1}}{\langle -\ell_2 4\rangle\langle 41\rangle\langle 1\ell_1\rangle\langle \ell_1(-\ell_2)\rangle}$$

so all helicities (and therefore field content) are allowed to propagate. Let's assume, in general

n_f = number fermions $\begin{cases} n_f = 1 \text{ in QCD (one quark species!)} \\ n_f = 4 \text{ in } N=4 \text{ SYM} \end{cases}$

n_s = number of scalars $\begin{cases} n_s = 0 \text{ in QCD} \\ n_s = 6 \text{ in SYM} \end{cases}$

Let's use them in order to evaluate the left-hand-side

$$\sum_h A_4^{(\text{tree})}(-\ell_1^h, 2^+, 3^-, \ell_2^h)\, A_4^{(\text{tree})}(-\ell_2^{-h}, 4^-, 1^+, \ell_1^h)$$

$$= \sum_{h=\{-1,-\frac{1}{2},0,\frac{1}{2},1\}} (-1)^{2h} \frac{n_{|h|}\left(\langle \ell_1 2\rangle\langle \ell_2 1\rangle\right)^{2+2h}\left(\langle \ell_2 2\rangle\langle \ell_1 1\rangle\right)^{2-2h}}{\langle \ell_1 2\rangle\langle 23\rangle\langle 3\ell_2\rangle\langle \ell_2\ell_1\rangle\langle \ell_2 4\rangle\langle 41\rangle\langle 1\ell_1\rangle\langle \ell_1\ell_2\rangle}$$

notice different sign!

where I used

$$|-p\rangle = -|p\rangle \qquad \Rightarrow \qquad \not{p} = |p\rangle[p| + |p]\langle p|$$

$$|-p] = +|p] \qquad \rightarrow -\not{p} \text{ by analytic continuation}$$

[alternative $|-p\rangle = i|p\rangle$
$|-p] = i|p]$]

$n_1 = n_b = 1$

$n_{1/2} = n_f$ (depends on theory)

$n_0 = n_s$ (depends on theory)

Let's look at the numerator, we have

$$\sum_h (-1)^{2h}\, n_h \left(\langle l_1 2\rangle\langle l_2 1\rangle\right)^{2+2h} \left(\langle l_2 2\rangle\langle l_1 1\rangle\right)^{2-2h}$$

$$= \left(\langle l_1 2\rangle\langle l_2 1\rangle\right)^4 + \left(\langle l_2 2\rangle\langle l_1 1\rangle\right)^4$$

$$- n_f \langle l_1 2\rangle^3\langle l_2 1\rangle^3 \langle l_2 2\rangle\langle l_1 1\rangle$$

$$- n_f \langle l_1 2\rangle\langle l_2 1\rangle\langle l_2 2\rangle^3 \langle l_1 1\rangle^3$$

$$+ n_s \langle l_1 2\rangle^2\langle l_2 1\rangle^2 \langle l_2 2\rangle^2 \langle l_1 1\rangle^2$$

use

$(a-b)^4 =$

$a^4 + b^4 - 4a^3b - 4ab^3$

$- 6a^2b^2$

$$= \left(\langle l_1 2\rangle\langle l_2 1\rangle - \langle l_2 2\rangle\langle l_1 1\rangle\right)^4$$

$$-(n_f - 4)\left(\langle l_1 2\rangle^2\langle l_2 1\rangle^2 + \langle l_2 2\rangle^2\langle l_1 1\rangle^2\right)\langle l_1 2\rangle\langle l_2 2\rangle\langle l_2 1\rangle\langle l_1 1\rangle$$

$$+ (n_s - 6)\langle l_1 2\rangle^2\langle l_2 1\rangle^2\langle l_2 2\rangle^2\langle l_1 1\rangle^2 \leftarrow$$

Notice also, for first term

$$\langle \ell_1 2\rangle\langle \ell_2 1\rangle - \underbrace{\langle \ell_2 2\rangle\langle \ell_1 1\rangle}_{-\langle \ell_2 \ell_1\rangle\langle 12\rangle - \langle \ell_2 1\rangle\langle 2 \ell_1\rangle} =$$

$$= \cancel{\langle \ell_1 2\rangle\langle \ell_2 1\rangle} + \langle 12\rangle\langle \ell_2 \ell_1\rangle - \cancel{\langle \ell_2 1\rangle\langle \ell_1 2\rangle}$$

So all in all, the left-hand-side becomes

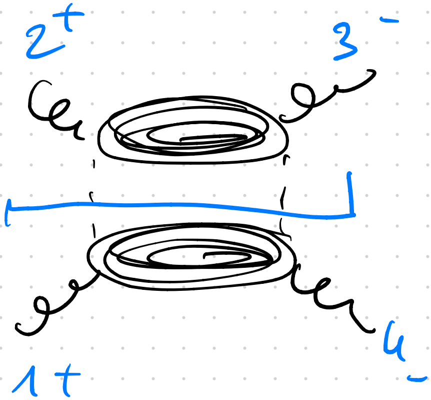
$$\sum_h A_4^{tree} A_4^{tree} = \Big\{ \langle 12\rangle^4 \langle \ell_1 \ell_2\rangle^4$$

$$- (n_f - 4)\left(\langle \ell_1 2\rangle^2\langle \ell_2 1\rangle^2 + \langle \ell_2 2\rangle^2\langle \ell_1 1\rangle^2\right)\langle \ell_1 2\rangle\langle \ell_2 2\rangle\langle \ell_2 1\rangle\langle \ell_1 1\rangle$$

$$+ (n_s - 6)\,\langle \ell_1 2\rangle^2\langle \ell_2 2\rangle^2\langle \ell_2 1\rangle^2\langle \ell_1 1\rangle^2 \Big\} \frac{1}{\mathrm{Den}}$$

$$\mathrm{Den} = -\langle \ell_1 2\rangle\langle 23\rangle\langle 3 \ell_2\rangle\langle \ell_2 4\rangle\langle 41\rangle\langle 1 \ell_1\rangle\langle \ell_1 \ell_2\rangle^2$$

results simplifies extremely if $\begin{Bmatrix} n_f = 4 \\ n_s = 6 \end{Bmatrix}$



$$= -\frac{\langle 12\rangle^4 \langle \ell_1 \ell_2\rangle^{\cancel{4}\,2}}{\langle \ell_1 2\rangle\langle 23\rangle\langle 3\ell_2\rangle\cancel{\langle \ell_1\ell_2\rangle^2}\langle \ell_2 4\rangle\langle 41\rangle\langle 1\ell_1\rangle}$$

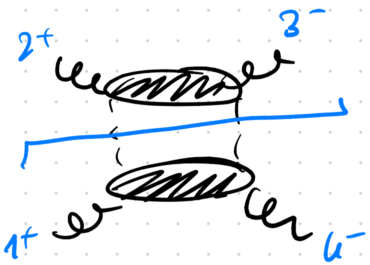
$$= \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\cdot\left[-\frac{\langle 12\rangle\langle 34\rangle\langle \ell_1\ell_2\rangle^2}{\langle \ell_1 2\rangle\langle 3\ell_2\rangle\langle \ell_2 4\rangle\langle 1\ell_1\rangle}\right]$$

$$= A_4^{(tree)}(1^+2^+3^-4^-)\left\{-\underbrace{\frac{\langle 12\rangle\langle 34\rangle\langle \ell_1\ell_2\rangle^2}{\langle 1\ell_1\rangle\langle \ell_2 4\rangle\langle \ell_1 2\rangle\langle 3\ell_2\rangle}}_{\substack{\text{very similar to s-channel} \\ \text{modulo} \\ 1\to 2,\ 2\to 3,\ 3\to 4,\ 4\to 1}}\right\}$$

and in fact it becomes, with same trick as before

$$\frac{s_{12}\, s_{23}}{(\ell_2+p_1)^2(\ell_1-p_3)^2} \Rightarrow \frac{s_{23}\, s_{34}}{(\ell_2+p_2)^2(\ell_1-p_4)^2}$$

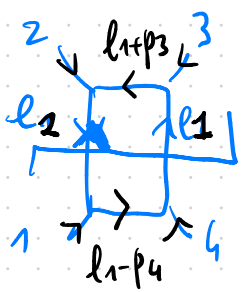
so at the integrand level



$$= A_4^{(\text{tree})} \frac{s_{23}\, s_{34}}{(\ell_2+p_2)^2(\ell_1-p_4)^2}$$

$$= C_4 \, \mathrm{Cut}_{s_{23}}\left(I_4\right) + C_{3,2}\, \mathrm{Cut}_{s_{23}} I_{3,2} + C_{2,2}\, \mathrm{Cut}_{s_{23}}(I_{2,2})$$

$$\Downarrow$$



$$= \begin{cases} \ell_1+p_3+p_2 = -\ell_2 \\ (\ell_1+p_3)^2 = (-\ell_2-p_2)^2 = (\ell_2+p_2)^2 \end{cases}$$

exactly same box !

So finally we have that, if $n_s = 6$, $n_f = 4$

$$A^{1\text{loop}}(1^+2^+3^-4^-) = \underbrace{s_{12}\, s_{23}\, A_4^{(\text{tree})}(1^+2^+3^-4^-)\, I_4(s_{12}, s_{23})}$$

no triangles
no bubbles !

this field content is special, it's the one of

N=4 SYM (supersymmetric version of QCD)

with maximum degree of supersymmetry