4 - Little Group Scaling and Color Ordering 1/2

WS 2021

TUM

LITTLE GROUP SCALING

We started this discussion saying that we were searching for a way to represent scattering amplitudes, which makes their transformation properties under LITTLE GROUP manifest

Given a momentum p^μ, Little Group is set of transformations that leaves it invariant.

$$\cancel{p} = |p\rangle[p| + |p]\langle p| \quad \text{or} \quad p^{\dot{a}b} = |p\rangle^{\dot{a}}[p|^b \ \text{etc}$$

for p to be invariant, we have only one possibility

$$|p\rangle \to z\,|p\rangle \qquad |p] \to \frac{1}{z}\,|p]$$

massless left handed

Incoming Right-handed PART

Conventional

notice that FOR REAL MOMENTA z must be a phase! this is because

$$|p\rangle^* = [p| \implies z^* = \frac{1}{z}$$

$$\implies \boxed{|z|^2 = 1}$$

WHAT DOES THIS IMPLY FOR SPIN-1 OBJECTS?

$$\varepsilon_+^\mu = -\frac{[r\gamma^\mu p\rangle}{\sqrt{2}\,[rp]} \implies \begin{cases} LG(p) & \varepsilon_+^\mu \to z^2 \varepsilon_+^\mu \\ LG(r) & \varepsilon_+^\mu \to \varepsilon_+^\mu \quad \underline{\text{invariant}} \end{cases}$$

$$\varepsilon_-^\mu = \frac{\langle r\gamma^\mu p]}{\sqrt{2}\langle rp\rangle} \implies \begin{cases} LG(p) & \varepsilon_-^\mu \to z^{-2} \varepsilon_-^\mu \\ LG(r) & \varepsilon_-^\mu \to \varepsilon_-^\mu \quad \text{invariant} \end{cases}$$

this is another manifestation of the fact that τ^μ is just a <u>GAUGE MOMENTUM</u>

- Notice that all relations among spinor products must respect the correct scaling

$$[p|\gamma^\mu|q\rangle\,[e|\gamma_\mu|k\rangle = 2\,[pe]\langle kq\rangle$$

lhs $\longleftrightarrow$ rhs must scale the same way

- importantly, we know that scattering amplitudes must transform under <u>LITTLE GROUP</u> !

So now we see that this representation allows us to constraint form of (massless) scattering amplitude in terms of which $|p\rangle$ or $|p]$ are allowed to appear !

- Scattering Amplitude for <u>MASSLESS particles</u> will be function of momenta and polarizations
$\Longrightarrow$ it can be rewritten as $\langle\,\rangle$ $[\,]$ only!

- For each external particle, amplitude must scale properly under Little group

$$A(p_1, \dots, p_i, \dots p_N) \longrightarrow A(p_1, \dots W^{\mu}{}_{\nu} p_i{}^{\nu}, \dots p_N)$$

$$= z^{+2h_i} A(p_1, \dots, p_N)$$

h_i helicity of particle i !

if i fermion of helicity $-\frac{1}{2}$ $\Rightarrow$ $1/z$ LEFT

$+\frac{1}{2}$ $\Rightarrow$ z RIGHT$_4$

Notice that

$|p\rangle$ & $\langle p|$ scale same $\sim z$

u_R $\qquad$ $\bar{u}_L$

$|p]$ & $[p|$ scale same $\sim \frac{1}{z}$

u_L $\qquad$ $\bar{u}_R$

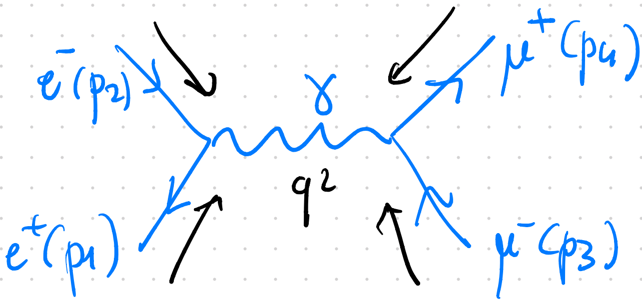
$\Rightarrow \langle 2 \gamma_\mu 1]$

Left-handed Fermion Line

EXAMPLE 1 $e^+e^- \to \mu^+\mu^-$

MASSLESS e! and μ!

let us compute the tree-level QED process



momenta all incoming!

if λ_i helicity of particle with momentum p_i

$$i\mathcal{M}_{\lambda_1\lambda_2\lambda_3\lambda_4} = \frac{-ie^2}{q^2}\,\bar{u}_{\lambda_1}(p_1)\,\gamma^\mu\, u_{\lambda_2}(p_2)\;\bar{u}_{\lambda_4}(p_4)\,\gamma_\mu\, u_{\lambda_3}(p_3)$$

with $q^2 = (p_1+p_2)^2 = (p_3+p_4)^2$!

NOTATION :

$u_L = p]$ $\quad u_R = p\rangle$ $\quad \bar{u}_L = \langle p$ $\quad \bar{u}_R = [p$

So now: there are 4 possibilities (clear why?)

$$\cancel{i}\mathcal{M}_{LLLL} = -\frac{ie^2}{q^2}\langle 1\gamma^\mu 2]\langle 4\gamma_\mu 3]$$

① left handed incoming antiparticle = right-handed outgoing particle

$$= -\frac{\cancel{i}e^2}{q^2}\,2\langle 14\rangle[32]$$

$$\cancel{i}\mathcal{M}_{LLRR} = -\frac{ie^2}{q^2}\langle 1\gamma^\mu 2][4\gamma_\mu 3\rangle$$

$$= -\frac{\cancel{i}e^2}{q^2}\langle 1\gamma^\mu 2]\langle 3\gamma_\mu 4] = -\frac{ie^2}{q^2}\,2\langle 13\rangle[42]$$

$$\cancel{i}\mathcal{M}_{RRLL} = -\frac{ie^2}{q^2}[1\gamma^\mu 2\rangle\langle 4\gamma_\mu 3]$$

$$= -\frac{\cancel{i}e^2}{q^2}[1\gamma^\mu 2\rangle[3\gamma_\mu 4\rangle = -\frac{ie^2}{q^2}\,2[13]\langle 42\rangle$$

$$\cancel{i}\mathcal{M}_{RRRR} = -\frac{ie^2}{q^2}[1\gamma^\mu 2\rangle[4\gamma_\mu 3\rangle$$

$$= -\frac{\cancel{i}e^2}{q^2}\,2[14]\langle 32\rangle$$

$$\mathcal{M}_{LL,LL} = -\frac{2e^2}{q^2}\langle 14\rangle[32]$$

$$\mathcal{M}_{LL,RR} = -\frac{2e^2}{q^2}\langle 13\rangle[42]$$

$$\mathcal{M}_{RR,LL} = -\frac{2e^2}{q^2}[13]\langle 42\rangle$$

$$\mathcal{M}_{RR,RR} = -\frac{2e^2}{q^2}[14]\langle 32\rangle$$

Complex conj! PARITY

A parity transformation reverses all helicities and since QED is invariant under P we find that the "coefficient" is the same

scaling little group is correct:

$\mathcal{M}_{LL,LL}$

$$\begin{cases} 1 \to z \\ 4 \to z \\ 2 \to 1/z \\ 3 \to 1/z \end{cases}$$

Notice that, Amplitudes squared (and cross-sections) must be invariant under Little Group. Indeed

$$|\mathcal{M}_{LL,LL}|^2 = |\mathcal{M}_{RR,RR}|^2$$

$$= + \frac{4e^4}{q^4} \langle 14\rangle [32] [41] \langle 23\rangle$$

$$= + \frac{4e^4}{q^4} (2p_1 \cdot p_4)(2p_2 \cdot p_3)$$

Define usual mandelstam $(p_1+p_2)^2 = s$

$(p_1+p_3)^2 = t$

$(p_2+p_3)^2 = u$

$$= + \frac{4e^2}{s^2} u^2$$

similarly

$$|\mathcal{M}_{LL,RR}|^2 = |\mathcal{M}_{RR,LL}|^2 = +\frac{4e^4}{q^4} \langle 13\rangle[42][31]\langle 24\rangle$$

$$= +\frac{4e^4}{q^4} \, 2p_1\cdot p_3 \;\; 2p_2\cdot p_4$$

$$= +\frac{4e^2}{s^2} \, t^2$$

average

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{1}{4} 2\left(|\mathcal{M}_{LL,LL}|^2 + |\mathcal{M}_{LL,RR}|^2\right)$$

$$= 2e^2\left(\frac{t^2+u^2}{s^2}\right)$$

Using $t = -\frac{s}{2}(1-\cos\theta)$

$u = -\frac{s}{2}(1+\cos\theta)$

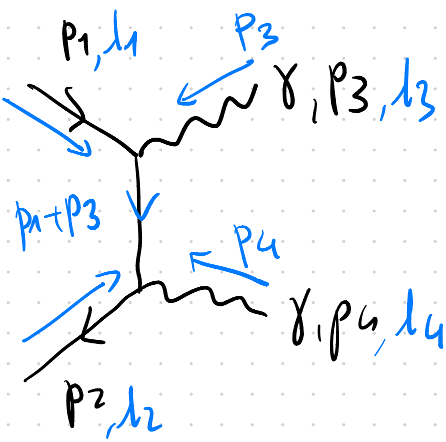
$$\Rightarrow \sim \boxed{e^2(1+\cos^2\theta)}$$

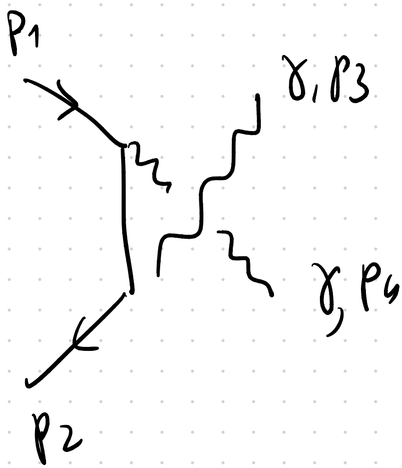
EXAMPLE 2 $\quad e^+e^- \to \gamma\gamma$ in QED

For now we remain in QED, abelian $[U(1)]$ gauge group. Compute an amplitude with spin 1 and 1/2

$$e^-(p_1) + e^+(p_2) + \gamma(p_3) + \gamma(p_4) \longrightarrow 0$$

All incoming





$$p_4^\mu = -(p_1^\mu + p_2^\mu + p_3^\mu)$$

$$i\mathcal{M}_{\lambda_1\lambda_2\lambda_3\lambda_4} = -ie^2\, \bar{u}_{\lambda_2}(p_2)\left[\ \Gamma_{\mu\nu}\ \right] u_{\lambda_1}(p_1)\, \varepsilon^{\mu}_{\lambda_3}\, \varepsilon^{\nu}_{\lambda_4}$$

$$\Gamma_{\mu\nu} = \frac{\gamma_\nu(\not{p}_2+\not{p}_4)\gamma_\mu}{(p_2+p_4)^2} + \frac{\gamma_\mu(\not{p}_2+\not{p}_3)\gamma_\nu}{(p_2+p_3)^2}$$

$$= \frac{\gamma_\nu(\not{p}_2+\not{p}_4)\gamma_\mu}{t} + \frac{\gamma_\mu(\not{p}_2+\not{p}_3)\gamma_\nu}{u}$$

As expected from general considerations, helicity along massless fermion line should be conserved

$\lambda_1 = \lambda_2 = L$ or $\lambda_1 = \lambda_2 = R$

we see it explicitly, as there are always three

γ matrices between the two spinors !

so we have two possibilities times $2\times 2 = 4$ for photons

$$i\mathcal{M}_{LL,\,\lambda_3\lambda_4} = -ie^2 \langle 2| \Gamma_{\mu\nu} |1] \, \varepsilon^{\mu}_{\lambda_3} \varepsilon^{\nu}_{\lambda_4}$$

$$i\mathcal{M}_{RR,\,\lambda_3\lambda_4} = -ie^2 \, [2| \Gamma_{\mu\nu} |1\rangle \, \varepsilon^{\mu}_{\lambda_3} \varepsilon^{\nu}_{\lambda_4}$$

$$\langle 2|\Gamma|1] = \langle \bar{u}_L(p_2)\, \Gamma \; u_L(p_1) \,]$$

1.) photons of equal helicities $\Rightarrow$ + case :

$$\varepsilon^{\mu}_{+}(p_3, q) = -\frac{[q_3\,\mu\,3\rangle}{\sqrt{2}\,[q_3 3]} \qquad \varepsilon^{\nu}_{+}(p_4, r) = -\frac{[q_4\,\nu\,4\rangle}{\sqrt{2}\,[q_4 4]}$$

photons are contracted with γ^μ, γ^ν !

$$\gamma^\mu \frac{[q_3\,\mu\,3\rangle}{\sqrt{2}\,[q_3 3]} = \;?\; = \frac{1}{\sqrt{2}\,[q_3 3]}\Big[A\,|q_3]\langle 3| + B\,|3\rangle[q_3| \Big]$$

right Little Group scaling

13

take matrix element with generic momenta

- $\langle k \left(\gamma^\mu \dfrac{[q_3 \mu 3\rangle}{\sqrt{2}[q_3 3]} \right) \ell] =$

$$\Rightarrow \langle k \mu \ell] \frac{[q_3 \mu 3\rangle}{\sqrt{2}[q_3 3]} = \frac{B \langle k 3\rangle [q_3 \ell]}{\sqrt{2}[q_3 3]}$$

Fierz

$$\frac{2\langle k 3\rangle [q_3 \ell]}{\sqrt{2}[q_3 3]} = \frac{B \langle k 3\rangle [q_3 \ell]}{\sqrt{2}[q_3 3]} \Rightarrow B = 2$$

- $[k \left(\gamma^\mu \dfrac{[q_3 \mu 3\rangle}{\sqrt{2}[q_3 3]} \right) \ell\rangle =$

$$[k \mu \ell\rangle [q_3 \mu 3\rangle = A [k q_3] \langle 3 \ell\rangle$$

$$\Rightarrow A = 2$$

so we find

- $\gamma_\mu \, \varepsilon_+^\mu(p_3, q_3) = -\dfrac{\sqrt{2}}{[q_3 3]} \Big(|q_3]\langle 3| + |3\rangle [q_3| \Big)$

  $= \not{\varepsilon}_+(p_3, q)$

- $\gamma_\nu \, \varepsilon_+^\nu(p_4, q_4) = -\dfrac{\sqrt{2}}{[q_4 4]} \Big(|q_4]\langle 4| + |4\rangle [q_4| \Big)$

  $= \not{\varepsilon}_+(p_4, r)$

in $\mathcal{M}_{44++}$ only two pieces survive :

- $\langle 2 \, \not{\varepsilon}_+(p_4, q_4) \, (\not{p}_2 + \not{p}_4) \, \not{\varepsilon}_+(p_3 q_3) \, | 1]$

$\propto \langle 24\rangle [q_4 \dots 3\rangle [q_3 1] \quad \xrightarrow{q_3 = p_1} \quad \underline{\underline{\text{ZERO}}}$

- $\langle 2 \, \not{\varepsilon}_+(p_3, q_3) \, (\not{p}_2 + \not{p}_3) \, \varepsilon_+(p_4, q_4) \, | 1]$

$\propto \langle 23\rangle [q_3 \dots 4\rangle [q_4 1] \quad \xrightarrow{q_4 = 1} \quad \text{ZERO}$

similarly for $\mathcal{M}_{RR,++}$

- $[2q_4]\langle 4 \ldots q_3]\langle 31\rangle$ $\quad q_4 = p_2$ $= \text{ZERO}$

- $[2q_3]\langle 3 \ldots q_4]\langle 41\rangle$ $\quad q_3 = p_2$ $= \text{ZERO}$

we proved that for equal photon helicity

$$\mathcal{M}_{LL++} = \mathcal{M}_{RR++} = 0$$

$$\mathcal{M}_{LL--} = \mathcal{M}_{RR--} = 0 \quad \text{by PARITY}$$

all tree-level amplitudes with photons with equal helicities are <u>zero</u>!

only independent ones

$\mathcal{M}_{LL+-}$, $\mathcal{M}_{LL-+}$, $\mathcal{M}_{RR+-}$, $\mathcal{M}_{RR-+}$

only two truly independent, the other two come from parity

we need corresponding relations for

$$\cdot\ \gamma_\mu \, \varepsilon_-^\mu(p_3, q_3) = \frac{\sqrt{2}}{\langle q_3 3\rangle}\Big(|3]\langle q_3| + |q_3\rangle[3|\Big)$$

$$\cdot\ \gamma_\nu \, \varepsilon_-^\nu(p_4, q_4) = \frac{\sqrt{2}}{\langle q_4 4\rangle}\Big(|4]\langle q_4| + |q_4\rangle[4|\Big)$$

$$\mathcal{M}_{LL-+} = -e^2 \langle 2| \underbrace{\left\{ \frac{\not{\epsilon}_{4+} (\not{p}_2 + \not{p}_4) \not{\epsilon}_{3-}}{t} + \frac{\not{\epsilon}_{3-} (\not{p}_2 + \not{p}_3) \not{\epsilon}_{4+}}{u} \right\}}_{} |1]$$

① ②

choose reference momenta to simplify structure !

$$\not{\epsilon}_{3-} = \frac{\sqrt{2}}{\langle q_3 3\rangle} \left(|3]\langle q_3| + |q_3\rangle[3| \right) \qquad q_3 = p_2 \quad \langle 22\rangle = 0$$

$$\not{\epsilon}_{4+} = -\frac{\sqrt{2}}{[q_4 4]} \left(|q_4]\langle 4| + |4\rangle[q_4| \right) \qquad q_4 = p_1 \quad [11] = 0$$

② falls out only 1 piece survives in ①

$$\mathcal{M}_{LL-+} = +e^2 \frac{2}{\langle 23\rangle[14]} \frac{\langle 24\rangle[1(2+4)2\rangle[31]}{t}$$

$$\mathcal{M}_{LL-+} = + e^2 \frac{2}{\langle 23\rangle [14]} \frac{\langle 24\rangle [1(\cancel{2}+4)2\rangle [31]}{t}$$

$$= \frac{2e^2}{\cancel{[14]}\langle 23\rangle} \frac{\langle 24\rangle \cancel{[14]}\langle 42\rangle [31]}{t} \qquad t = \langle 13\rangle [31]\ !$$

$$= -2e^2 \frac{\langle 24\rangle^2 \cancel{[31]}}{\langle 23\rangle \langle 13\rangle \cancel{[31]}}$$

$$= -2e^2 \frac{\langle 24\rangle^2}{\langle 13\rangle\langle 23\rangle}$$

similarly, choosing $q_3 = p_1$; $q_4 = p_2$

$$\mathcal{M}_{LL+-} = -2e^2 \frac{\langle 23\rangle^2}{\langle 14\rangle\langle 24\rangle}$$

($\mathcal{M}_{LL-+}$ with $3 \leftrightarrow 4$ exchanged!)

19

Let's do this explicitly

$$\mathcal{M}_{LL+-} = -e^2 \langle 2| \underbrace{\qquad\qquad\qquad}_{\left\{ \frac{\not{\epsilon}_{4-} (\not{p}_2 + \not{p}_4) \not{\epsilon}_{3+}}{t} + \frac{\not{\epsilon}_{3+} (\not{p}_2 + \not{p}_3) \not{\epsilon}_{4-}}{u} \right\}} |1]$$

1 2

$$\not{\epsilon}_{3+} = -\frac{\sqrt{2}}{[q_3 3]} \Big(|q_3] \langle 3| + |3\rangle [q_3| \Big) \qquad q_3 = p_1$$

$$\not{\epsilon}_{4-} = +\frac{\sqrt{2}}{\langle q_4 4\rangle} \Big(|4] \langle q_4| + |q_4\rangle [4| \Big) \qquad q_4 = p_2$$

1 falls, only one term of 2 survives

$$\mathcal{M}_{LL+-} = +e^2 \frac{2}{[13]\langle 24\rangle} \frac{\langle 23\rangle [1 (2+3) 2\rangle [41]}{u}$$

$$\mathcal{M}_{LL+-} = +e^2 \frac{2}{[13]\langle 24\rangle} \frac{\langle 23\rangle[1(\cancel{2}+3)2\rangle[41]}{u = \langle 14\rangle[41]}$$

$$= \frac{2e^2}{\cancel{[13]}\langle 24\rangle} \frac{\langle 23\rangle\cancel{[13]}\langle 32\rangle\cancel{[41]}}{\langle 14\rangle\cancel{[41]}}$$

$$= -2e^2 \frac{\langle 23\rangle^2}{\langle 24\rangle\langle 14\rangle}$$

$$|\mathcal{M}|^2 = 2\left(|\mathcal{M}_{LL+-}|^2 + |\mathcal{M}_{LL-+}|^2\right) =$$

$$= 8e^4 \left\{ \frac{(\langle 23\rangle[32])^2}{\langle 24\rangle[42]\langle 14\rangle[41]} + \frac{(\langle 24\rangle[42])^2}{\langle 23\rangle[32]\langle 13\rangle[31]} \right\}$$

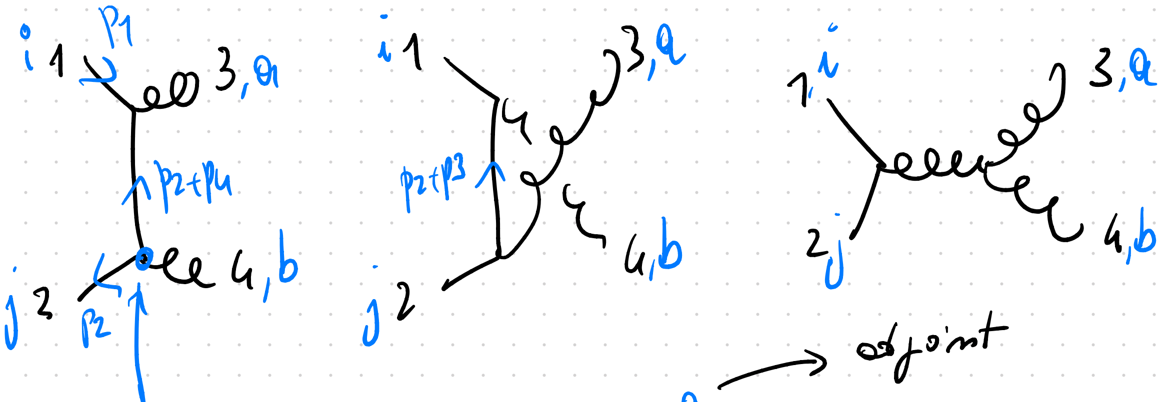
$$= 8e^4 \left\{ \frac{u^2}{tu} + \frac{t^2}{ut} \right\} = 8e^4 \left\{ \frac{u}{t} + \frac{t}{u} \right\}$$

(Tree-level) amplitudes with bosons with equal helicities tend to be zero... we'll see this more in general later on

WHAT CHANGES in QCD ?

$q\bar{q} \to gg$?

$$q(p_1) + \bar{q}(p_2) + g(p_3) + g(4) \longrightarrow 0$$



q-g coupling $\quad -ig^2\gamma^\mu t^a_{ij}$

$a \to$ adjoint

$ij \leftarrow$ fundamental

t^a_{ij} generators of $SU(3)$ (or in general $SU(N_c)$)

$$\mathrm{Tr}[t^a t^b] = \frac{\delta^{ab}}{2} \; ; \qquad [t^a, t^b]_{ij} = i f^{abc} t^c_{ij}$$

the first two diagrams can be "copied" from $e^+e^- \to \gamma\gamma$

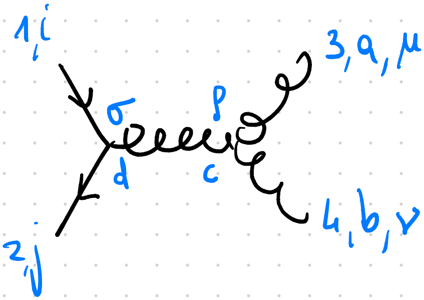
$\longrightarrow$ ABELIAN PART ($e \Leftrightarrow g_s$)

$$i\mathcal{M}^{[A]}_{\lambda_1\lambda_2\lambda_3\lambda_4} = -i g_s^2 \, \bar{u}_{\lambda_2}(p_2)\left[\Gamma_{\mu\nu} \right] u_{\lambda_1}(p_1)\, \varepsilon^{\mu}_{\lambda_3} \varepsilon^{\nu}_{\lambda_4}$$

$$\Gamma_{\mu\nu} = \frac{\gamma_\nu(\not{p}_2+\not{p}_4)\gamma_\mu}{t} t^b_{jk} t^a_{ki} + \frac{\gamma_\mu(\not{p}_2+\not{p}_3)\gamma_\nu}{u} t^a_{jk} t^b_{ki}$$

$\mathcal{M}^{[A]} \neq$ only for $\begin{bmatrix} \mathcal{M}_{LL+-} \; ; \; \mathcal{M}_{RR+-} \\ \mathcal{M}_{LL-+} \; ; \; \mathcal{M}_{RR-+} \end{bmatrix}$

NON ABELIAN PART



$$= -i\mathcal{M}^{[NA]}_{\lambda_1\lambda_2\lambda_3\lambda_4}$$

$$i\mathcal{M}^{[NA]}_{\lambda_1\lambda_2\lambda_3\lambda_4} = +i g_s^2\, \bar{u}_{\lambda_2}(p_2)\gamma^\sigma u_{\lambda_1}(p_1)\left[-i\,\frac{\delta_{cd}\, g_{\rho\sigma}}{(p_1+p_2)^2}\right]$$

$$T^d_{ji} f^{abc}\left[g^{\mu\nu}(p_3^\rho - p_4^\rho) + g^{\nu\rho}(p_4^\mu - p_{12}^\mu) + g^{\mu\rho}(p_{12}^\nu - p_3^\nu)\right]\varepsilon_{3\mu}\varepsilon_{4\nu}$$

3-gluon (and 4-gluon vertices) come from

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{ig}{\sqrt{2}}[A_\mu, A_\nu]$$

FIELD STRENGTH

$A_\mu = A_\mu^a t_a$

gluon field

$$= g_s^2\, \bar{u}_{s_2} \gamma_\rho u_{s_1} \;\; f_S^{abd} \;\; t^d_{ji} \longrightarrow$$ contributes to both Abelian diagrams

$$\times \Big[\varepsilon_3 \cdot \varepsilon_4 \left[p_3 - p_4\right]^\rho + 2\varepsilon_4^\rho \left(\varepsilon_3 \cdot p_4 - \cancel{p_{12} \cdot \varepsilon_3} \right) + 2\varepsilon_3^\rho \left(\cancel{\varepsilon_4 \cdot p_{12}} - \varepsilon_4 \cdot p_3 \right) \Big]$$

using momentum conservation :

$$p_{12} = p_1 + p_2 = -p_3 - p_4$$

$$\varepsilon_3 \cdot p_{12} = -\varepsilon_3 \cdot p_4 \; ; \; \varepsilon_4 \cdot p_{12} = -\varepsilon_4 \cdot p_3 \; !$$

MANIPULATE COLOR FACTOR:

$$\mathrm{Tr}[t^a t^b] = \frac{\delta^{ab}}{2} \; ; \qquad [t^a, t^b]_{ji} = i f^{abc} t^c_{ji}$$

$$\Rightarrow f^{abd} t^d_{ji} = -i\left[t^a_{jk} t^b_{ki} - t^b_{jk} t^a_{ki}\right]$$

$$= \boxed{i\, t^b_{jk} t^a_{ki}} - \boxed{i\, t^a_{jk} t^b_{ki}}$$

same color of diagram 1   same color of diagram 2

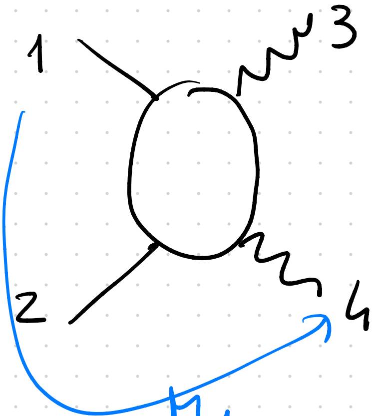
$\Rightarrow$

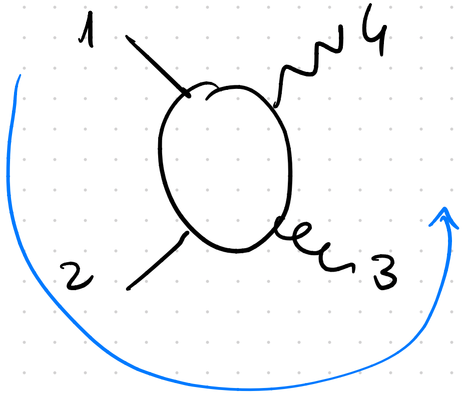
$$\mathcal{M}^{[A+NA]} = \mathcal{M}_1 (t^b t^a)_{ji} + \mathcal{M}_2 (t^a t^b)_{ji}$$

called "color ordered"

only 1 ordering contributes, NA contributes to both!

IMPORTANT POINT : M_1 and M_2 are independently gauge invariant and can be computed more simply!





M_1 $\Downarrow$ $M(1243)$

M_2 $M(1234)$

Color ordered amplitudes

To compute each of the two I need only 1 of the abelian diagrams plus a piece of the non abelian one $M(1243) = M(1234)\big|_{3\leftrightarrow 4}$

Let's compute one of the two

$$M(1234) = +g_s^2 \Bigg\{ \bar{u}_{\lambda_2} \, \not{\varepsilon}_3^{\lambda_3} \frac{(\not{p}_2+\not{p}_3)}{s_{23}} \not{\varepsilon}_4^{\lambda_4} \, u_{\lambda_1}$$

$$+ \frac{\bar{u}_{\lambda_2} \gamma^\mu u_{\lambda_1}}{s_{12}} \Big[\varepsilon_3^{\lambda_3} \cdot \varepsilon_4^{\lambda_4} (p_3 - p_4)_\mu + 2\, \varepsilon_3^{\lambda_3} \cdot p_4 \, \varepsilon_{4\mu}^{\lambda_4} - 2\, \varepsilon_4^{\lambda_4} \cdot p_3 \, \varepsilon_{3\mu}^{\lambda_3} \Big] \Bigg\}$$

the other one would be $(3 \Leftrightarrow 4)$

$$M(1234) = -g_s^2 \Bigg\{ \bar{u}_{\lambda_2} \, \not{\varepsilon}_4^{\lambda_4} \frac{(\not{p}_1+\not{p}_3)}{s_{13}} \not{\varepsilon}_3^{\lambda_3} \, u_{\lambda_1}$$

$$+ \frac{\bar{u}_{\lambda_2} \gamma^\mu u_{\lambda_1}}{s_{12}} \Big[\varepsilon_3^{\lambda_3} \cdot \varepsilon_4^{\lambda_4} (p_3 - p_4)_\mu + 2\, \varepsilon_3^{\lambda_3} \cdot p_4 \, \varepsilon_{4\mu}^{\lambda_4} - 2\, \varepsilon_4^{\lambda_4} \cdot p_3 \, \varepsilon_{3\mu}^{\lambda_3} \Big] \Bigg\}$$

Again it turns out that $\left.\begin{matrix} LL-- & RR-- \\ LL++ & RR++ \end{matrix}\right\} = 0$

<u>Exercise</u>.

the independent ones are spin $\begin{matrix} LL-+ & RR-+ \\ LL+- & RR+- \end{matrix}$

by PARITY

Now let's compute $LL-+$ for $M(1234)$

$$M(1234)_{LL-+} = g_s^2 \left[\langle 2| \not{\varepsilon}_3^- \left(\frac{\not{p}_2+\not{p}_3}{s_{23}} \right) \not{\varepsilon}_4^+ |1] \right.$$

$$\left. + \frac{\langle 2\, \gamma^\mu\, 1]}{s_{21}} \left(\varepsilon_3^- \cdot \varepsilon_4^+ (p_3-p_4)^\mu + \cancel{2\, \varepsilon_3^- \cdot p_4\, \varepsilon_{4\mu}^+} - \cancel{2\, \varepsilon_4^+ \cdot p_3\, \varepsilon_{3\mu}^-} \right) \right]$$

see that if $q_3 = p_4$; $q_4 = p_3 \Rightarrow$ $\boxed{\varepsilon_3 \cdot p_4 = \varepsilon_4 \cdot p_3 = 0 \,!}$

Also with this choice

$$\varepsilon_3^- \cdot \varepsilon_4^+ \sim \langle 4 \gamma^\mu 3] [3 \gamma_\mu 4\rangle$$

$$= \langle 4 \gamma^\mu 3] \langle 4 \gamma_\mu 3] = 0 \; !$$

so non-abelian part disappears !

$$M(1234)_{LL-+} = g_s^2 \langle 2| \not{\varepsilon}_3^- \left(\frac{\not{p}_2 + \not{p}_3}{s_{23}} \right) \not{\varepsilon}_4^+ |1]$$

$$\not{\varepsilon}_4^+ = -\frac{\sqrt{2}}{[34]} \Big(|4\rangle[3| + |3]\langle 4| \Big)$$

$$\not{\varepsilon}_3^- = +\frac{\sqrt{2}}{\langle 43\rangle} \Big(|3]\langle 4| + |4\rangle[3| \Big)$$

$$M(1234)_{LL-+} = -\frac{2 g_s^2}{[34]\langle 43\rangle} \frac{1}{s_{23}} \langle 24\rangle [3| \not{p}_2 + \not{p}_3 |4\rangle [31]$$

$$M(1234)_{4-+} = -2g_s^2 \frac{\langle 24\rangle[32]\langle 24\rangle[31]}{\langle 43\rangle[34]\; s_{23}}$$

$$= -2g_s^2 \frac{\langle 24\rangle^2 \cancel{[32]}[31]}{\langle 43\rangle[34]\langle 23\rangle\cancel{[32]}}$$

to make it nicer

$$= +2g_s^2 \frac{\langle 24\rangle^2}{\langle 23\rangle\langle 34\rangle} \frac{[31]}{[34]} \frac{\langle 12\rangle\langle 41\rangle\langle 24\rangle}{\langle 12\rangle\langle 41\rangle\langle 24\rangle}$$

$$= 2g_s^2 \frac{\langle 24\rangle^3\langle 41\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \cancel{\frac{[31]\langle 12\rangle}{[34]\langle 24\rangle}}$$

in fact $[34]\langle 24\rangle = -[34]\langle 42\rangle = +[31]\langle 12\rangle$

so we found

$$M(1234)_{LL-+} = 2g_s^2 \frac{\langle 24\rangle^3 \langle 41\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

notice scaling

$$1 \to z^{1-1-1} = 1/z = 1]$$

$$2 \to z^{3-1-1} = z \Rightarrow |2\rangle$$

$$3 \to z^{-2} \Longrightarrow \varepsilon_3^-$$

$$4 \to z^{1+3-2} = z^2 \Rightarrow \varepsilon_4^+$$

Correct !