SCATTERING AMPLITUDES

3 - Spinor-Helicity Formalism.

Massless bosons and fermions

WS 2021

TUM

SPINOR INDICES

When studying $(\frac{1}{2}, \frac{1}{2})$ repr of Lorentz group, we introduced the notation

$$p^{\dot{a}b} = \sigma^{\dot{a}b}_{\mu} \, p^{\mu} \quad \text{for a momentum } p^{\mu}$$

transforms in $(\frac{1}{2}, \frac{1}{2})$

4-vector

$SO^{+}(1,3) \longleftrightarrow SL(2,\mathbb{C})$

vector $\longleftrightarrow (\frac{1}{2}, \frac{1}{2})$ repres.

$\dot{a}$ right handed, b left-handed

Let us try to generalise this index notation for physical DIRAC SPINORS : $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\psi_L^{\#?} \qquad\qquad \psi_R^{\#?}$$

which indices should we use?

To be consistent with our previous definition

1.

$$p^{\dot{a}b} = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix}$$

note that :

$$(p^{\dot{a}b})^* = \begin{pmatrix} p^0 + p^3 & p^1 + ip^2 \\ p^1 - ip^2 & p^0 - p^3 \end{pmatrix} \equiv (p^{\dot{a}b})^T = p^{b\dot{a}}$$

Complex conjugation exchanges <u>dotted</u> / <u>undotted</u>

2. we have seen that to go from $L \to R$ or $R \to L$

complex conj and multiply by $i\sigma^2$

$$\boxed{\begin{cases} i\sigma^2 \cdot \psi_L^* \equiv \psi_R \\ -i\sigma^2 \psi_R^* = \psi_L \end{cases}}$$

3. $(i\sigma^2) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in matrix (spinor) notation is special !

Recall form of left-handed boost, rotations

$$R^L_x(\theta_x) = \begin{pmatrix} \cos\frac{\theta_x}{2} & i\sin\frac{\theta_x}{2} \\ i\sin\frac{\theta_x}{2} & \cos\frac{\theta_x}{2} \end{pmatrix} \qquad B^L_x(\phi_x) = \begin{bmatrix} \cosh\frac{\phi_x}{2} & \sinh\frac{\phi_x}{2} \\ \sinh\frac{\phi_x}{2} & \cosh\frac{\phi_x}{2} \end{bmatrix}$$

etc

it's easy to see that

$$R^L_j\,(i\sigma_2)\,(R^L_j)^T = B^L_j\,(i\sigma_2)\,(B^L_j)^T = (i\sigma_2)$$

$$R^R_j\; i\sigma_2\; (R^R_j)^T = B^R_j\,(i\sigma_2)\,(B^R_j)^T = (i\sigma_2)$$

$\forall j$!

matrix $i\sigma_2$ that goes from $L \Leftrightarrow R$

is invariant under L and R Lorentz transf.!

As for $g^{\mu\nu}$ (COMMENT) we can use $i\sigma^2$ to raise or lower L-R indices, for example

$$\begin{cases} (i\sigma_2)^{ab}\psi_b = \psi^a \\ (i\sigma_2)^{\dot{a}\dot{b}}\psi_{\dot{b}} = \psi^{\dot{a}} \end{cases}$$

but who is what?

- CONVENTION -

let us fix who we call Left

- $\psi_L = \psi_a$ then since complex conj must $a \to \dot{a}$ we must put

- $(\psi_L)^* = \psi_{\dot{a}}$ and we raise the dotted index with spinor metric

- $\psi_R = (i\sigma^2)(\psi_L)^* = (i\sigma_2)^{\dot{a}\dot{b}}\psi_{\dot{b}} = \psi^{\dot{a}}$

- right to left instead $\psi_L = (-i\sigma_2)\psi_R^* = \underbrace{(-i\sigma_2^{\dot{a}})_{ab}}_{\text{lower indices!}}\psi^b$

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- Finally complex conjugation swaps dotted and undotted so, since $i\sigma^2$ REAL $= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$(i\sigma_2)^{ab} = (i\sigma_2)^{\dot{a}\dot{b}} = (-i\sigma_2)_{ab} = (-i\sigma_2)_{\dot{a}\dot{b}}$$

the spinor metric is often called

$$\boxed{\varepsilon^{ab} = \varepsilon^{\dot{a}\dot{b}} = -\varepsilon_{ab} = -\varepsilon_{\dot{a}\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}$$

- Define <u>SPINOR PRODUCT</u> $\psi^a \chi_a$, then

$$\psi^a \chi_a = \psi^a \chi^b \varepsilon_{ab} \to \text{Lorentz Transf}$$

$$\psi'^a \chi'^b \varepsilon_{ab} = \psi^c \chi^d \underbrace{\Lambda^a_c \Lambda^b_d \varepsilon_{ab}}_{\varepsilon_{cb}} = \psi^c \chi_c$$

Lorentz Invariant!

Similarly for Right-Handed ones!

- Also $\psi^a \chi_a = \psi^a \chi^b \varepsilon_{ab} = -\psi^a \chi^b \varepsilon_{ba}$

$$= -\chi^a \psi_a$$ Antisymmetric

$\psi^a \psi_a = 0$!

Notation

$$\left.\begin{aligned} \psi^a &= [\psi^a \\ \chi_a &= \chi]_a \end{aligned}\right\} \quad [\psi \chi]$$

[similarly for right-handed ones !] $\underline{\psi_R = \psi^{\dot{a}}}$

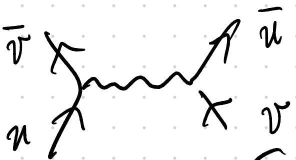
$$\left.\begin{aligned} \psi_{\dot{a}} &= \langle \psi_{\dot{a}} \\ \chi^{\dot{a}} &= |\chi\rangle^{\dot{a}} \end{aligned}\right\} \quad \langle \psi \chi \rangle$$

we will not use these spinors most of the time ⟹ DIRAC !

DIRAC SPINOR (in Chiral Repr)

Focus on wave functions which appear in scatt. Amplitudes
$\Rightarrow$ Feynman rules

$u(p)$, $v(p)$ particles, antiparticles of momentum p



$u \rightarrow$ incoming particle

$\bar{v} \rightarrow$ incoming antiparticle
= outgoing particle

satisfy

$$(\not{p} - m)\, u(p) = 0$$

$$(\not{p} + m)\, v(p) = 0$$

Dirac Equation

focus on massless case $\Rightarrow$ $\not{p}\, u(p) = \not{p}\, v(p) = 0$

in CHIRAL representation $\not{p} = p^\mu \gamma_\mu$ where

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \qquad \sigma^\mu = (\mathbb{1}, \vec{\sigma}) \qquad \bar{\sigma}^\mu = (\mathbb{1}, -\vec{\sigma})$$

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

let's connect this with L, R spinors !

Define $P_L = \frac{1-\gamma_5}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad P_R = \frac{1+\gamma_5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$u_{L,R}(p) = P_{L,R}\, u(p) \Rightarrow u(p) = \begin{pmatrix} u_L(p) \\ u_R(p) \end{pmatrix}$$

notice that Dirac Eq becomes Weyl Eq (2x2)

$$p^\mu \sigma_\mu u_L(p) = p^\mu \bar{\sigma}_\mu u_R(p) = 0$$

now it's easy to prove that

$$\tilde{u}(p) = (i\sigma_2)(u_L(p))^* \quad \text{fullfils}$$

ASSUMES REAL MOMENTA!

$$p^\mu \bar{\sigma}_\mu \tilde{u}(p) = 0 \,,$$ ie it's a right-handed $u_R(p)$!

$\Rightarrow$ as for Lorentz spinors, indeed u_L, u_R are Left and Right handed ones!

indeed, writing Weyl Eq. in components:

MASSLESS PARTICLES

$$u_L(p) = -\frac{\bar{\sigma}\cdot\bar{p}}{p_0} u_L(p) \,; \quad u_R(p) = +\frac{\bar{\sigma}\cdot\bar{p}}{p_0} u_R(p)$$

incoming particle with spin opposite to momentum

incoming particle with spin along its momentum similarly, since massless particles

$u_L(p)$ = outgoing right-handed antiparticles

$u_R(p)$ = outgoing left-handed antiparticles

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outgoing PARTICLES are Dirac conjugate spinors $\overline{u}_{L,R}$

$\Rightarrow$ also incoming ANTIPARTICLES with L, R switched !

Explicit solution, exercise

$$u_L(p) = e^{i\alpha} \begin{pmatrix} -\sin\frac{\theta}{2}\, e^{-i\phi/2} \\ \cos\frac{\theta}{2}\, e^{i\phi/2} \end{pmatrix}$$

Little group !

we here also introduced the notation

$$u_L(p) = u_a \qquad u_R(p) = u^{\dot{a}}$$

Let us write now [SPINOR HELICITY]

$$U_L = |p] = N_p \begin{pmatrix} u_a \\ 0 \end{pmatrix} \quad u_a \to u]_a \qquad U_R = |p\rangle = N_p \begin{pmatrix} 0 \\ u^{\dot{a}} \end{pmatrix} = |u\rangle^{\dot{a}}$$

$$\bar{U}_L = \langle p| = N_p \begin{pmatrix} 0 & u_{\dot{a}} \end{pmatrix} \quad u_{\dot{a}} = \langle u_{\dot{a}} \qquad \bar{U}_R = [p| = N_p \begin{pmatrix} u^a & 0 \end{pmatrix} \quad u^a = [u^a$$

⟹ Bra-ket notation

so that

$$\langle p\, q] = [p\, q\rangle = 0 \,; \qquad \langle p\, p\rangle = [p\, p] = 0$$

antisymmetry!

$$\langle p\, p\rangle = N_p^2\, u_{\dot{a}} u^{\dot{a}} = -N_p^2\, u_{\dot{a}} u^{\dot{a}} \quad \text{antisym!}$$

$$= 0 \,!$$

what about $\langle p q \rangle$, $[p q]$?

to get there, first notice that

$$\langle p q \rangle^* = (\bar{U}_L U_R)^* = \bar{U}_R U_L = [q p] \quad !$$

<u>moreover:</u>

completeness rel.

$$\sum_\lambda u_\lambda(p)\, \bar{u}_\lambda(p) = \cancel{p} = |p\rangle[p| + |p]\langle p|$$

$$= u_L(p)\, \bar{u}_L(p) + u_R(p)\, \bar{u}_R(p)$$

operator rhs in bra-ket of QM

how compute (<u>real momenta</u>!)

$$\langle p q \rangle [q p] = -|\langle p q \rangle|^2 = -|[p q]|^2$$

$$= \langle p| \big(|q\rangle[q| + |q]\langle q| \big) |p] = \langle p| \cancel{q} |p]$$

$$\bar{u}_L(p)\ \not{q}\ u_L(p) = \bar{u}(p)\left(\frac{1+\gamma_5}{2}\right)\not{q}\left(\frac{1-\gamma_5}{2}\right)u(p)$$

$$= \bar{u}_a\left(\frac{1+\gamma_5}{2}\right)_{ab}\ \not{q}_{bc}\left(\frac{1-\gamma_5}{2}\right)_{cd}\ u_d$$

$$= \mathrm{Tr}\left[\not{p}\ \not{q}\left(\frac{1-\gamma_5}{2}\right)\right] = 2\,p\cdot q =$$ MANDELSTAM INVARIANT!

then using (real momenta!)

$$\langle pq\rangle[qp] = -|\langle pq\rangle|^2 = -|[pq]|^2$$

$$\Rightarrow \begin{cases} \langle pq\rangle = \sqrt{|2p\cdot q|}\ e^{i\phi} \\ [pq] = -\sqrt{|2p\cdot q|}\ e^{-i\phi} \end{cases}$$

physically important
square-root singularity
see what we need
PHYSICALLY in
collinear limits!

REMARK ON NOTATION

we call $|p\rangle$ $|p]$ etc "Dirac Spinors"

$$U_L = |p] = N_p \begin{pmatrix} u_a \\ 0 \end{pmatrix} \qquad U_R = |p\rangle = N_p \begin{pmatrix} 0 \\ u^{\dot{a}} \end{pmatrix}$$

some references instead : $\begin{cases} u_a = |p]_a \\ u^{\dot{a}} = |p\rangle^{\dot{a}} \end{cases}$

in the end, completely equivalent !

but pay attention to how to interpret things like

$$\not{p} = \underbrace{|p\rangle[p|}_{p^{\dot{a}b}} + \underbrace{|p]\langle p|}_{p_{a\dot{b}}}$$

different spinors

One more remark

with our construction, we discovered that

we can write

$\not{p} = |p\rangle[p| + |p]\langle p|$ or equivalently

$p^{\dot{a}b} = |p\rangle^{\dot{a}}[p|^{b}$ $\qquad\qquad$ $p_{a\dot{b}} = |p]_{a}\langle p|_{\dot{b}}$

remember $p^{\dot{a}b} = p^{\mu}(\sigma_{\mu})^{\dot{a}b}$ = matrix $\underline{\underline{2\times 2}}$

such that $\det(p^{\dot{a}b}) = p^{2} = 0$!

so it's a matrix of rank 1 !

$p^{\dot{a}b} \equiv \tilde{\lambda}^{\dot{a}}\lambda^{b}$ $\Rightarrow$ indeed the spinor products above !

$\tilde{\lambda}^{\dot{a}} = |p\rangle^{\dot{a}}$ $\qquad$ $\lambda^{b} = |p]^{b}$

MORE PROPERTIES OF SPINOR PRODUCTS

we have seen that

- $\langle pq \rangle = -\langle qp \rangle$; $[pq] = -[qp]$
- $\langle pp \rangle = [pp] = 0$
- $\langle pq \rangle = [qp]^*$ (for real momenta)
- $\langle pq \rangle [qp] = 2p\cdot q$

NEW ONES

1. $\langle p| \gamma^\mu |p] = [p|\gamma^\mu|p\rangle = 2p^\mu$ [Gordon]
2. $[p|\gamma^\mu|q\rangle = \langle q|\gamma^\mu|p]$ [conjugation]
3. $[p|\gamma^\mu|q\rangle [k|\gamma_\mu|\ell\rangle = 2[pk]\langle \ell q\rangle$ [Fierz]

   $\langle p|\gamma^\mu|q]\langle k|\gamma_\mu|\ell] = 2\langle pk\rangle[\ell q]$

4. $\langle p|\gamma^{\mu}|q]\,\gamma_{\mu} = 2\left(|q]\langle p| + |p\rangle[q|\right)$ [Fierz 2] ?

5. $|p\rangle[p| = \frac{1+\gamma_5}{2}\,\not{p}$ [projectors]

$|p]\langle p| = \frac{1-\gamma_5}{2}\,\not{p}$

6. $\langle pq\rangle\langle kl\rangle = \langle pk\rangle\langle ql\rangle + \langle pl\rangle\langle kq\rangle$

[Schouten]

7. n-point $\sum_{i=1}^{n} p_i = 0$

$$\Rightarrow [p|\sum_{i=1}^{n}\not{p}_i|q\rangle = \sum_{i=1}^{n}[pi]\langle iq\rangle = 0$$

obvious

[momentum conservation]

8. $\langle p|\gamma^{\mu_1}\cdots\gamma^{\mu_{2n+1}}|q] = [q|\gamma^{2n+1}\cdots\gamma^{\mu_1}|p\rangle$ ODD

$\langle p|\gamma^{\mu_1}\cdots\gamma^{\mu_{2n}}|q\rangle = -\langle q|\gamma^{2n}\cdots\gamma^{\mu_1}|p\rangle$ EVEN

SOME

PROOFS : Some properties are obvious

2. $[p|\gamma^\mu|q\rangle = \overline{U}_R(p)\,\gamma^\mu\,U_R(q)$

$$= N_p N_q \begin{pmatrix} u^a & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} \begin{pmatrix} 0 \\ u^{\dot a} \end{pmatrix}$$

$$= N_p N_q\, u^a(p)\, (\sigma^\mu)_{a\dot a}\, u^{\dot a}(q)$$

$(\sigma_\mu)^{\dot a b}$

$\sigma_i^{\dot a b} = \text{PAULI}$

$$\langle q|\gamma^\mu|p] = N_p N_q \begin{pmatrix} 0 & u_{\dot a} \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} \begin{pmatrix} u_a \\ 0 \end{pmatrix}$$

$$= N_p N_q\, u_{\dot a}(q)\, (\bar\sigma^\mu)^{\dot a a}\, u_a(p)$$

$$= N_p N_q\, u^b(p)\, u^{\dot b}(q)\, \underbrace{\varepsilon_{ab}\,\varepsilon_{\dot a\dot b}\,(\bar\sigma^\mu)^{\dot a a}}$$

$$\underbrace{(-i\sigma^2)_{\dot a\dot b}}_{\text{TRANSPOSE}}\,(\bar\sigma^\mu)^{\dot a a}\,(-i\sigma^2)_{ab} = \left[\sigma^{\mu T}\right]_{\dot b b}$$

$$= (\sigma^\mu)_{b\dot b}$$

PROOF

$$\begin{bmatrix}0&1\\-1&0\end{bmatrix}^T \begin{bmatrix} \begin{pmatrix}1&0\\0&1\end{pmatrix} \\ -\begin{pmatrix}0&1\\1&0\end{pmatrix} \\ -\begin{pmatrix}0&-i\\i&0\end{pmatrix} \\ -\begin{pmatrix}1&0\\0&-1\end{pmatrix} \end{bmatrix} \begin{bmatrix}0&1\\-1&0\end{bmatrix}$$

$$\begin{bmatrix}0&1\\-1&0\end{bmatrix}^T = \begin{bmatrix}0&-1\\1&0\end{bmatrix}$$

$$\begin{bmatrix} \begin{pmatrix}0&-1\\1&0\end{pmatrix} \\ \begin{pmatrix}1&0\\0&-1\end{pmatrix} \\ \begin{pmatrix}i&0\\0&i\end{pmatrix} \\ \begin{pmatrix}0&-1\\-1&0\end{pmatrix} \end{bmatrix} \begin{bmatrix}0&1\\-1&0\end{bmatrix} = \begin{bmatrix} \begin{pmatrix}1&0\\0&1\end{pmatrix} \\ \begin{pmatrix}0&1\\1&0\end{pmatrix} \\ \begin{pmatrix}0&i\\-i&0\end{pmatrix} = (\sigma^2)^T \\ \begin{pmatrix}1&0\\0&-1\end{pmatrix} \end{bmatrix}$$

$$= (\sigma^{\mu T})_{\dot{b}b} \, !$$

1. $\langle p|\gamma^\mu|p] = \bar{v}_L(p)\,\gamma^\mu u_L(p)$

$$= \bar{u}(p)\left(\frac{1+\gamma_5}{2}\right)\gamma^\mu u(p)$$

$$= \sum_{\text{spins}} u_a \bar{u}_b \left(\frac{1+\gamma_5}{2}\right)_{bc} \gamma^\mu_{ca}$$

$$= \mathrm{Tr}\left[\cancel{p}\left(\frac{1+\gamma_5}{2}\right)\gamma^\mu\right] = \mathrm{Tr}\left(\frac{1-\gamma_5}{2}\,\cancel{p}\,\gamma^\mu\right)$$

$$= 2p^\mu$$

3. Fierz

$$[p|\gamma^\mu|q\rangle[k|\gamma_\mu|\ell\rangle = \overbrace{N_p N_q N_k N_\ell}^{N}$$

$$\times\, u^a(p)\,(\sigma^\mu)_{a\dot{a}}\,u^{\dot{a}}(q)\;u^b(k)\,(\sigma_\mu)_{b\dot{b}}\,u^{\dot{b}}(\ell)$$

what is $(\sigma^\mu)_{a\dot{a}}(\sigma_\mu)_{b\dot{b}} = ?$

$$(\sigma^\mu)_{a\dot a}(\sigma_\mu)_{b\dot b} \overset{?}{=} 2\,(i\sigma_2)_{ab}\,(i\sigma_2)_{\dot a\dot b}$$ **Ex**

$$= (\mathbb{1})_{a\dot a}(\mathbb{1})_{b\dot b} - (\vec\sigma)_{a\dot a}\cdot(\vec\sigma)_{b\dot b}$$

$$= (\delta_{a\dot a})(\delta_{b\dot b}) - \left[A\,\delta_{a\dot a}\,\delta_{b\dot b} + B\,\delta_{a\dot b}\,\delta_{b\dot a}\right]$$

$$(\vec\sigma)_{a\dot a}\cdot(\vec\sigma)_{b\dot b} = A\,\delta_{a\dot a}\,\delta_{b\dot b} + B\,\delta_{a\dot b}\,\delta_{b\dot a}$$

$\boxed{\cdot}$ $$\delta_{a\dot a}(\vec\sigma_{a\dot a})\cdot(\vec\sigma)_{b\dot b} \overset{\vec\sigma \text{ Traceless!}}{=} 0 = 2A\,\delta_{b\dot b} + B\,\delta_{b\dot b}$$

$\boxed{\cdot}$ $$\delta_{b\dot a}(\vec\sigma_{a\dot a})\cdot(\vec\sigma)_{b\dot b} = 3\,\delta_{a\dot b} = A\,\delta_{a\dot b} + 2B\,\delta_{a\dot b}$$

$$\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix} = \begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix}$$ **Ex.**

$$\begin{matrix}\sigma_1 & \mathbb{1} & \sigma_1\\ \sigma_2 & \mathbb{1} & \sigma_2\\ \sigma_3 & \mathbb{1} & \sigma_3\end{matrix} \quad = \text{always } \underline{\mathbb{1}}$$

$$\Rightarrow \begin{cases} (2A+B)\,\delta_{b\dot{b}} = 0 \\ (A+2B)\,\delta_{a\dot{b}} = 3\,\delta_{a\dot{b}} \end{cases} \Rightarrow B = -2A$$

$$\Downarrow$$

$$-3A = 3$$

$$A = -1 \;//$$

$$B = +2$$

so we find $(\vec{\sigma}_{a\dot{a}})(\vec{\sigma})_{b\dot{b}} = 2\,\delta_{a\dot{b}}\delta_{b\dot{a}} - \delta_{a\dot{a}}\delta_{b\dot{b}}$

and finally

$$\underline{\underline{(\sigma^{\mu})_{a\dot{a}}(\sigma_{\mu})_{b\dot{b}}}} = \delta_{a\dot{a}}\delta_{b\dot{b}} + \delta_{a\dot{a}}\delta_{b\dot{b}} - 2\,\delta_{a\dot{b}}\delta_{b\dot{a}}$$

$$= 2\left[\delta_{a\dot{a}}\delta_{b\dot{b}} - \delta_{a\dot{b}}\delta_{b\dot{a}}\right]$$

$$= \underline{\underline{2\,\varepsilon_{ab}\,\varepsilon_{\dot{a}\dot{b}}}}$$

$$= 2\,(i\sigma_2)_{ab}\,(i\sigma_2)_{\dot{a}\dot{b}} \quad \checkmark$$

so going back to Fierz identity we have

$$[p|\gamma^\mu|q\rangle [k|\gamma_\mu|l\rangle = \overbrace{N_p N_q N_k N_l}^{N}$$

$$\times\, u^a(p) \underbrace{(\sigma^\mu)_{a\dot a}} u^{\dot a}(q)\; u^b(k) \underbrace{(\sigma_\mu)_{b\dot b}} u^{\dot b}(l)$$

$$= N\; u^a(p)\, u^{\dot a}(q)\, u^b(k)\, u^{\dot b}(l)\;\; 2\,(i\sigma^2)_{ab}\,(i\sigma^2)_{\dot a\dot b}$$

$$= 2\, N_p N_q N_k N_l \;\; \underset{[p}{u^a(p)}\, \underset{k]}{u_a(k)}\;\; \underset{q\rangle}{u^{\dot a}(q)}\;\; \underset{\langle l|}{u_{\dot a}(l)}$$

$$= 2\, [pk]\langle lq\rangle$$

similarly

$$\langle p|\gamma^\mu|q] \langle k|\gamma_\mu|l] = 2\langle pk\rangle [lq]$$

5. Projectors

$$|p\rangle[p| + |p]\langle p| = \not{p}$$

$$u_R \overline{u}_R + u_L \overline{u}_L$$

$$\frac{1+\gamma_5}{2} u_R = u_R \qquad \frac{1+\gamma_5}{2} u_L = 0 \quad \text{etc}$$

$$\left(\frac{1+\gamma_5}{2}\right)\not{p} = |p\rangle[p|$$

$$\left(\frac{1-\gamma_5}{2}\right)\not{p} = |p]\langle p|$$

6. Schouten Identity

$$\langle pq\rangle\langle k\ell\rangle = \langle pk\rangle\langle q\ell\rangle + \langle p\ell\rangle\langle kq\rangle$$

4 spinors p, q, k, ℓ

Spinors are 2 dimensional, so (choice)

$$|q\rangle = A\,|k\rangle + B\,|\ell\rangle$$

$$\langle kq\rangle = B\langle k\ell\rangle \qquad B = \frac{\langle kq\rangle}{\langle k\ell\rangle}$$

$$\langle \ell q\rangle = A\langle \ell k\rangle \qquad A = \frac{\langle \ell q\rangle}{\langle \ell k\rangle} = -\frac{\langle \ell q\rangle}{\langle k\ell\rangle}$$

$$\langle k\ell\rangle|q\rangle = \frac{\langle kq\rangle}{\cancel{\langle k\ell\rangle}}|\ell\rangle - \frac{\langle \ell q\rangle}{\cancel{\langle k\ell\rangle}}|k\rangle$$

$$\langle p|q\rangle\langle k\ell\rangle = \langle p|\ell\rangle\langle kq\rangle - \langle p|k\rangle\langle \ell q\rangle$$

schouten!

SPIN 1 particles [all incoming - convention]

how do we represent a gluon in spinor-hel?

- physical pol: $\varepsilon_\mu (\varepsilon^\mu)^* = -1$ normal.

  $p_\mu \varepsilon^\mu = 0$ transvers.

- gauge redundancy $r^\mu \Rightarrow r^\mu \varepsilon_\mu = 0$

  gauge choice

  $r^\mu A_\mu = 0$

- $$\sum_{\lambda\ \mathrm{pol}} \varepsilon^\mu_\lambda \varepsilon^{*\nu}_\lambda = -g^{\mu\nu} + \frac{k^\mu r^\nu + k^\nu r^\mu}{k \cdot r}$$

we start from: $\eta_1^\mu = [r \gamma^\mu p\rangle$; $\eta_2^\mu = \langle r \gamma^\mu p]$

$\eta^\mu_{1,2}\, r_\mu = \eta^\mu_{1,2}\, p_\mu = 0$ so good starting point

$$(\eta_1^\mu)^* = [r \gamma^\mu p\rangle^* = \langle r \gamma^\mu p] = \eta_2^\mu$$

then notice

$$\langle r\, \gamma^\mu\, p] \langle r\, \gamma_\mu\, p] \sim (\eta_1^\mu)^* \eta_{2\mu} = 0$$

η_1 and η_2 are orthogonal

$$\eta_1^* \cdot \eta_1 = \langle r\, \gamma^\mu\, p]\, [r\, \gamma_\mu\, p\rangle = \langle r\, \gamma^\mu p] \langle p\, \gamma_\mu\, r]$$

$$= 2\langle rp\rangle[rp] \quad \text{so :}$$

define

FOR TWO POLAR.

$$\begin{cases} \varepsilon_1^\mu(p,r) = -\dfrac{[r\,\gamma^\mu\, p\rangle}{\sqrt{2}\,[rp]} = (\varepsilon_-^\mu)^* = \varepsilon_+^\mu \\[2ex] \varepsilon_2^\mu(p,r) = \dfrac{\langle r\,\gamma^\mu\, p]}{\sqrt{2}\,\langle rp\rangle} = (\varepsilon_+^\mu)^* = \varepsilon_-^\mu \end{cases}$$

$$\varepsilon_{1,2}^\mu p_\mu = \varepsilon_{1,2}^\mu r_\mu = 0 \qquad \xrightarrow{\text{DIRAC EQ}}$$

signs and names conventional $\Rightarrow$ all incoming

$$(\varepsilon_+^\mu)^* (\varepsilon_{+\mu}) = \frac{\langle r\, \gamma^\mu p]}{2\langle pr\rangle} \frac{\langle p\, \gamma_\mu r]}{[rp]} = \frac{2\langle rp\rangle[rp]}{2\langle pr\rangle[rp]} = -1$$

right norm.

$$(\varepsilon_+^\mu)^* (\varepsilon_{+\mu}) = -1$$

$$(\varepsilon_+^\mu)^* \varepsilon_{-\mu} = (\varepsilon_-^\mu)^* \varepsilon_+ = 0$$

$\varepsilon^\mu_{+,-}$ produce states with helicity $+,-$

↳ have a look later at transf under little group.

IMPORTANT reference vector r^μ has to do with gauge Freedom!

In fact, take 2 $\varepsilon^-_\mu(p,r)$ and $\varepsilon^-_\mu(p,q)$

$\underbrace{r \quad \quad q}_{\text{different vectors}}$

$$\varepsilon^-_\mu(p,r) - \varepsilon^-_\mu(p,q) = \frac{1}{\sqrt{2}}\left[\frac{\langle r\gamma^\mu p]}{\langle rp\rangle} - \frac{\langle q\gamma^\mu p]}{\langle qp\rangle}\right]$$

$$= \frac{1}{\sqrt{2}} \frac{\langle r\gamma^\mu p]\langle qp\rangle - \langle q\gamma^\mu p]\langle rp\rangle}{\langle rp\rangle\langle qp\rangle}$$

$$= -\frac{1}{\sqrt{2}} \frac{\langle r\gamma^\mu \not{p}\, q\rangle - \langle q\gamma^\mu \not{p}\, r\rangle}{\langle rp\rangle\langle qp\rangle}$$

reverse second string

$$= -\frac{1}{\sqrt{2}} \frac{\langle r(\gamma^\mu \not{p} + \not{p}\gamma^\mu)q\rangle}{\langle rp\rangle\langle qp\rangle}$$

$$\gamma^\mu \not{p} + \not{p}\gamma^\mu = p_\nu \{\gamma^\mu \gamma^\nu\} = 2 p_\nu g^{\mu\nu}$$
$$= 2p^\mu \quad !$$

so we get

$$\varepsilon^+_\mu(p,r) - \varepsilon^+_\mu(p,q) = -\frac{1}{\sqrt{2}} \frac{2\langle rq\rangle p^\mu}{\langle rp\rangle\langle qp\rangle} \propto p^\mu$$

now remember that ε_μ opens an

$\varepsilon_\mu M^\mu$ and WARD ID $p_\mu M^\mu = 0$!

so changing reference mom does not change physics due to ward identity $\Rightarrow$ GAUGE INVARIANCE

APPLIES INDEP. FOR EVERY BOSON !