Dimensional Shift for Feynman Integrals

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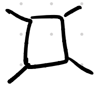
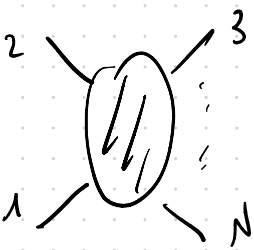
TUM

Recap

We have demonstrated the statement that, for one-loop scattering amplitudes, a universal decomposition exists:



$$= \sum_i C_{4;i} I_{4;i} + \sum_i C_{3;i} I_{3;i} + \sum_i C_{2;i} I_{2;i}$$

$$+ \sum_i C_{1;i} I_{1;i} + R$$

they come from integrals

$$\int \frac{d^D\ell}{(2\pi)^D} \frac{(\ell \cdot n_\varepsilon)^4}{D_1 \cdots D_4} \; ; \quad \int \frac{d^D\ell}{(2\pi)^D} \frac{(\ell \cdot n_\varepsilon)^2}{D_1 D_2 D_3} \; ; \quad \int \frac{d^D\ell}{(2\pi)^D} \frac{(\ell \cdot n_\varepsilon)^2}{D_1 D_2}$$

which have UV divergences!

Similarly $\displaystyle\int \frac{d^D\ell}{(2\pi)^D} \frac{(\ell \cdot n_\varepsilon)^2}{D_1 \cdots D_5} = O(\varepsilon)$

connected to

$$\int \frac{\Delta_5}{\Delta_4} \frac{1}{D_1 \cdots D_5} \frac{d^D\ell}{(2\pi)^D}$$

Dimensional Regularisation lets us consider Feynman integrals as continuous functions of the variable "D".

Many interesting consequences result from this, one of the most useful ones is the fact that Feynman integrals can be "shifted" of an even number of dimensions

$$I(D) \longleftrightarrow \begin{cases} I(D+2) \\ I(D-2) \end{cases}$$

there are different ways to see this, but one of the clearest involves deriving a new representation for the Feynman Integrals called <u>BAIKOV REPRESENTATION</u>

BAIKOV REPRESENTATION

Let us consider a generic, L-loop Feynman int

$$I = \int \prod_{j=1}^{L} \frac{d^D k_j}{\pi^{D/2}} \frac{1}{D_1 \dots D_N}$$

note euclidean int. measure.

I use k_j^μ instead of ℓ_j^μ to avoid confusion with # loops "L"

What we would like to do is to derive a new integral representation, as integral over the "scalar products" $\{ k_i \cdot p_j$ and $k_i \cdot k_j \}$

Let's introduce, only for this case, the notation

$$z_i = \underbrace{(k_1, \dots, k_L, p_1, \dots, p_E)}_{\text{total number } = M} \qquad z_i \cdot z_j = S_{ij}$$

these are NOT the standard Mandelstam

Now, let us (conventionally) split the loop momenta into a $\parallel$ and $\perp$ component, defined as follows:

- $\begin{cases} k_{1\parallel} \subset \{k_2, \ldots, k_L, p_1, \ldots p_E\} \\ k_{1\perp} \text{ orthogonal subspace} \end{cases}$

- $\begin{cases} k_{2\parallel} \subset \{k_3, \ldots, k_L, p_1 \ldots p_E\} \\ k_{2\perp} \text{ orthogonal subspace} \end{cases}$ etc

then

$$\prod_{j=1}^{L} d^D k_j = \left(d^{M-1} k_{1\parallel}\, d^{D-M+1} k_{1\perp}\right)\left(d^{M-2} k_{2\parallel}\, d^{D-M+2} k_{2\perp}\right) \cdots$$

$$\cdots \left(d^{M-L} k_{L\parallel}\, d^{D-M+L} k_{L\perp}\right)$$

the first step now is parametrizing the $/\!/$ spaces through the scalar products $s_{ij} = r_i \cdot r_j$

$$d^{M-1} k_{1/\!/} = \frac{ds_{12}\, ds_{13} \dots ds_{1M}}{\sqrt{\Delta(k_2, \dots, k_L, p_1 \dots p_E)}}$$

$$d^{M-2} k_{2/\!/} = \frac{ds_{23}\, ds_{24} \dots ds_{2M}}{\sqrt{\Delta(k_3, \dots, k_L; p_1 \dots p_E)}}$$

$\vdots$

$$d^{M-L} k_{L/\!/} = \frac{ds_{LL+1} \dots ds_{LM}}{\sqrt{\Delta(p_1, \dots, p_E)}}$$

$s_{12} = (k_1 + k_2)^2$

$= (k_{1/\!/} + k_{2/\!/})^2$

$+ (k_{1\perp} + k_{2\perp})^2$

$$\Delta(q_1 \dots q_n) = \det(q_i \cdot q_j)$$

GRAM DETERMINANT

$\Rightarrow$ volume of parallelogram !

What about the orthogonal components?

Denominators only depend on $k_\perp$ through $k_\perp^2$

since $k_\perp \cdot p_i = 0$; $k_\perp \cdot k_{//} = 0$ etc

($k_{i\perp} \cdot k_{j\perp} = 0$ for $i \neq j$!)

$$d^n k_{i\perp} = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} k_{i\perp}^{n-1} \, d k_{i\perp}$$

$y_i = k_{i\perp}^2$

$dy = 2 k_{i\perp} \, d k_{i\perp}$

$$= \frac{1}{2} \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} y_i^{(n-2)/2} \, dy_i$$

now using $k_1^2 = k_{1//}^2 + k_{1\perp}^2 = S_{11}$ =>

$$dk_{1\perp}^2 = dy_i = dS_{11}$$

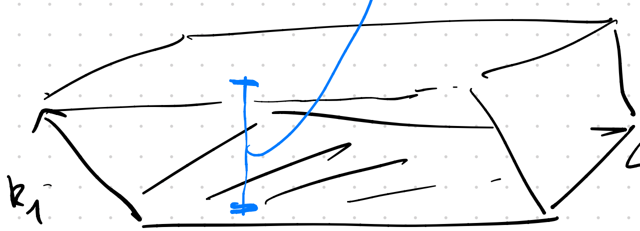
change of variables involves another sets of GRAMS

$$d^{d-M+1} k_{1\perp} = \frac{1}{2}\Omega_{d-M+1}\left[\frac{\Delta(k_1,\dots,k_L,p_1,\dots,p_E)}{\Delta(k_2,\dots,k_L,p_1,\dots p_E)}\right]^{\frac{d-M-1}{2}} dS_{11}$$

$$\vdots$$

$$d^{d-M+L} k_{L\perp} = \frac{1}{2}\Omega_{d-M+L}\left[\frac{\Delta(k_L,p_1,\dots p_E)}{\Delta(p_1\dots p_E)}\right]^{\frac{d-M+L-2}{2}} dS_{LL}$$

this because $|k_{i\perp}|$ is height of the parallelogram with base formed by $(k_{i+1},\dots,k_L,p_1,\dots p_E)$ plus vector k_i



Now plugging everything together

$$I = \frac{\pi^{-L(L-1)/4 - LE/2}}{\prod_{i=1}^{L} \Gamma\left(\frac{D-M+i}{2}\right)} \Delta(p_1 \cdots p_E)^{(E+1-D)/2}$$

$$\times \int \prod_{i=1}^{L} \prod_{j=i}^{M} ds_{ij} \frac{G(k_1, \dots, k_L, p_1, \dots, p_E)^{\frac{D-M-1}{2}}}{D_1 \cdots D_N}$$

these are now functions of the s_{ij}

Each $D_i = \sum_{j=1}^{L} \sum_{k=j}^{M} a_{jk}^{(i)} s_{jk}$

Linear by construction

So we can rotate once more and use $D_i = z_i$ as integration variables!

$$I = \frac{\pi^{-L(L-1)/4 \; -LE/2}}{\prod_{i=1}^{L} \Gamma\left(\frac{D-M+i}{2}\right)} \frac{\left[\det a_{jk}^{(i)}\right]}{\Delta(p_1 \ldots p_E)^{(D-E-1)/2}}$$

$$\oint_{\mathcal{C}} \frac{dz_1 \ldots dz_N}{z_1 \ldots z_N} \left(B(z_1, \ldots, z_N)\right)^{\frac{D-M-1}{2}}$$

integration contour is particularly messy ...

BAIKOV POLYNOMIAL

(Gram det of all momenta, evaluated for z_i)

but we don't really care, we rarely use this representation to perform the integrals, only to study their structure

(we'll see more examples in next lectures)

In particular, let's take a generic integral and multiply its numerator by $\Delta(k_1,..,k_L,p_1..p_E)$

$$I(D) = \int \prod_{j=1}^{L} \frac{d^D k_j}{\pi^{D/2}} \, \frac{\Delta(k_1,..k_L,p_1..p_E)}{D_1 \cdots D_N}$$

and go now to Baikov's representation

$$= \frac{\pi^{-L(L-1)/4 \, - LE/2}}{\prod_{i=1}^{L} \Gamma\left(\frac{D-M+i}{2}\right)} \, \frac{\left[\det G_{jk}^{(ij)}\right]}{\Delta(p_1 \ldots p_E)^{(D-E-1)/2}}$$

$$\times \oint_{\mathcal{C}} \frac{dz_1 \ldots dz_N}{z_1 \ldots z_N} \, B(z_1,\ldots,z_N)^{\underbrace{\frac{D-M-1}{2}+1}}$$

up to prefactor, same int in $D+2$ dimensions!

$$I(D+2) = \frac{\pi^{-L(L-1)/4 - LE/2} \left[\det a_{jk}^{(ij)}\right]}{\prod_{i=1}^{L} \Gamma\left(\frac{D-M+i}{2}+1\right) \; \Delta(p_1 \dots p_E)^{(D-E-1)/2+1}}$$

$$\times \oint_{\mathcal{C}} \frac{dz_1 \dots dz_N}{z_1 \dots z_N} \; B(z_1, \dots, z_N)^{\frac{D-M-1}{2}+1}$$

once $(M = L+E)$

such that

$$= \frac{2^L}{(D-E-L+1)_L}$$

$$I(D+2) = \boxed{\frac{\prod_{i=1}^{L} \Gamma\left(\frac{D-M+i}{2}\right)}{\prod_{i=1}^{L} \Gamma\left(\frac{D-M+i}{2}+1\right)}} \; \frac{1}{\Delta(p_1 \dots p_E)}$$

$$\int \prod_{j=1}^{L} \frac{d^D k_j}{\pi^{D/2}} \; \frac{\Delta(k_1 \dots k_L;\, p_1 \dots p_E)}{D_1 \dots D_N}$$

if you take also Minkowsky metric, you get on top of that a factor $\underline{\underline{(-1)^L}}$

so for example, in the case of the 1 loop pentagon, we had

$$\int \frac{d^D\vec{\ell}}{(2\pi)^D}\frac{(\overrightarrow{\ell\cdot n_E})^2}{D_1\cdots D_5} = \frac{1}{\Delta_4}\int\frac{d^D\vec{\ell}}{(2\pi)^D}\frac{\Delta_{q_1\cdots q_4\ell}}{D_1\cdots D_5}$$

$$= \#\,\frac{(D-E-L+1)_L}{2^L}\int\frac{d^{D+2}\ell}{(2\pi)^{D+2}}\frac{1}{D_1\cdots D_5}$$

prefactor

Falling FACTORIAL

$x_n = x(x-1)\cdots(x-n+1)$

$$\begin{cases} L=1 \\ E=4 \end{cases}$$

$$\propto \frac{(D-4)}{2}\underbrace{I_5(D+2)}_{\text{6-dimensional pentagon}}$$

when I expand for $D \sim 4$ I get

$$\int \frac{d^D l}{(2\pi)^D} \frac{(l \cdot n_\varepsilon)^2}{D_1 \dots D_5} = c_1 \cdot (D-4)\, I_5^{(0)}(6) + \mathcal{O}((D-4)^2)$$

6-dimensional pentagon is $\underline{\text{FINITE}}$, no UV no IR

while integral with $\displaystyle\int \frac{(l \cdot n_\varepsilon)^2}{D_1 \dots D_5} \to 0$ as $\underline{\mathcal{O}(D-4)}$