2 - The Lorentz Group

WS 2021

TUM

LORENTZ TRANSFORMATIONS

We work in Minkowski space-time $\mathbb{R}_{1,3}$, 4-dimensions (FOR NOW ☺)

$g^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$ metric

Lorentz Transf: $x^\mu \in \mathbb{R}_{1,3}$

$x'^\mu = \Lambda^\mu{}_\nu \, x^\nu$ such that

$x'^\mu x'_\mu = x^\mu x_\mu = x_0^2 - \vec{x}^{\,2}$

$$\Rightarrow \quad \Lambda^\mu{}_\nu \Lambda^\rho{}_\sigma \, g_{\rho\mu} \, x^\nu x^\sigma = x'^\mu x'_\mu$$

$$\Rightarrow \Lambda^\mu{}_\nu \, g_{\mu\rho} \, \Lambda^\rho{}_\sigma = g_{\nu\rho}$$

$$\boxed{\Lambda^T g \Lambda = g}$$

notation $O(1,3)$ = Lorentz $\det\Lambda = \pm 1$

$O^{+}(1,3)$

$\Lambda^0{}_0 \geq 1$

orthochronous

$SO(1,3)$

$\det\Lambda = +1$

$SO^{+}(1,3)$

proper, or restricted

Every element of $O(1,3)$ written as semidirect product as $SO^{+}(1,3)$ and $\{1, T, P, PT\}$

Discrete transformations

RELATION TO $SL(2,\mathbb{C}) \longleftrightarrow SO^{+}(1,3)$

$SL(2,\mathbb{C})$ group 2×2 complex matrices with $\det = 1$

it turns out to be the universal cover of $SO^{+}(1,3)$

$\forall\, x^\mu \in \mathbb{R}^{1,3}$ define $X = x^\mu \sigma_\mu$

$\sigma_\mu = (\mathbb{1}, \vec{\sigma})$ PAULI HAVE LOWER INDEX $\boxed{\sigma^\mu = \bar{\sigma}_\mu}$

$$\boxed{\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

then $X = \begin{bmatrix} x^0 + x^3 & x^1 - i x^2 \\ x^1 + i x^2 & x^0 - x^3 \end{bmatrix}$

this is the most general Hermitian matrix $\boxed{X^\dagger = X}$

so $\forall\, x^\mu \to X$ and $\forall\, X$ hermitian $\to x^\mu$

Notice that $\det X = (x^0)^2 - \vec{x}^{\,2} = x^\mu x_\mu$!

A Lorentz Transf must preserve the determinant!

now transform X' under $GL(2,\mathbb{C})$:

$$X' = A X A^\dagger \quad \text{with} \quad A \in GL(2,\mathbb{C})$$

- $(X')^T = X'$ so hermiticity is preserved
- $\det X' = (\det A)(\det A^\dagger) \det X$

$\Rightarrow$ norm preserved if $|\det A| = 1$

so this must be a Lorentz transformation

$$A\, X^\mu \sigma_\mu A^\dagger = (\Lambda^\mu{}_\nu(A) X^\nu)\, \sigma_\mu$$

. Now if A, $A' = e^{i\varphi} A$ then

$$A X A^\dagger = A' X (A')^\dagger \;!$$

use $e^{i\varphi}$ to fix $\det A = 1$

$SL(2,\mathbb{C})$

$SL(2,\mathbb{C})$ special linear group, 3 complex parameters

- if $A \in SL(2,\mathbb{C})$, also $-A$ is!

  $\det A = \det(-A)$! so $A, -A$ produce $1\ \Lambda$!

$$\Rightarrow SO(1,3)^{+} = SL(2,\mathbb{C})/\mathbb{Z}_2$$

$SL(2,\mathbb{C})$ is the universal covering group of $SO(1,3)^{+}$

GENERATORS AND ALGEBRA

$\mathfrak{so}^{+}(1,3)$

Rotations $\sim$ Generators

R $\qquad$ J

Boosts $\sim$ Generators

B $\qquad$ K

Action on $\mathbb{R}^{1,3}$ (space-time 4-vectors)

$$(R_x)^\mu{}_\nu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_x & \sin\theta_x \\ 0 & 0 & -\sin\theta_x & \cos\theta_x \end{bmatrix} \qquad (B_x)^\mu{}_\nu = \begin{bmatrix} \cosh\phi_x & -\sinh\phi_x & 0 & 0 \\ -\sinh\phi_x & \cosh\phi_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

similarly for R_y, R_z, B_y, B_z

Read out generators INFINITESIMAL TRANSFORMATIONS

$$(J_x)^\mu{}_\nu = \left[-i \left.\frac{dR_x(\theta_x)}{d\theta_x}\right|_{\theta_x=0}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & +i & 0 \end{bmatrix}$$

$$(J_y)^\mu{}_\nu = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix} \qquad (J_z)^\mu{}_\nu = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

FOR BOOSTS WE GET:

$$(K_x)^{\mu}{}_{\nu} = \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (K_y)^{\mu}{}_{\nu} = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(K_z)^{\mu}{}_{\nu} = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}$$

From generators we can work out the **Lorentz Algebra**

$$\left.\begin{aligned} [J_i, J_j] &= i\,\varepsilon_{ijk} J_k \\ [J_i, K_j] &= i\,\varepsilon_{ijk} K_k \\ [K_i, K_j] &= -i\,\varepsilon_{ijk} J_k \end{aligned}\right\} \quad \varepsilon_{123} = +1$$

a general Lorentz Transformation will be

$$\Lambda = e^{i\vec{J}\cdot\vec{\theta} + i\vec{K}\cdot\vec{\Phi}}$$

By introducing $N_i^{\pm} = \frac{1}{2}(J_i \pm i K_i)$

$$\Rightarrow \begin{cases} [N_i^+, N_j^+] = i\,\varepsilon_{ijk} N_k^+ \\ [N_i^-, N_j^-] = i\,\varepsilon_{ijk} N_k^- \\ [N_i^+, N_j^-] = 0 \end{cases}$$

$$\boxed{so^+(1,3) = su(2) \times su(2)}$$ Lie Algebras

any representation of $so^+(1,3)$ specified by doublet (j, j') $\Rightarrow$ $(2j+1)(2j'+1)$ degrees of freedom

- REMEMBER Full Lorentz group $O(3,1)$ not generated by $so(1,3)$ algebra as $e^{i t_i R_i}$
  only $SO^+(1,3)$, proper orthochronous group which is connected to the identity ($t_i = 0$, $\mathbb{1}$)

REGULAR "PHYSICAL" ROTATION GENERATORS ARE

$$\vec{J} = \vec{N}^+ + \vec{N}^-$$

their eigen. give the "spin" of the representation

↓

on repr (A, B) of Lorentz group, generates

$$J = A+B,\ A+B-1,\ \dots,\ |A-B|$$

representations SO(3) or SU(2)

	LORENTZ $SU(2) \otimes SU(2)$	ROTATION $SO(3)$	"Spin"
	$(0, 0)$	0	
	$(\frac{1}{2}, 0)$	$\frac{1}{2}$	
	$(0, \frac{1}{2})$	$\frac{1}{2}$	
irred	$(\frac{1}{2}, \frac{1}{2})$	$1 \oplus 0$	reducible
	$(1, 0)$	1	
irred	$(1, 1)$	$2 \oplus 1 \oplus 0$	reducible

REMEMBER: Exponentiating Lie Algebra, we get UNIVERSAL COVERING GROUP

$$\text{Exp}: su(2) \Rightarrow SU(2) \quad \text{U.C. of } SO(3)$$

$$\text{Exp}: su(2) \oplus su(2) \Rightarrow \underline{\underline{SL(2,\mathbb{C})}} \quad \text{UC} \quad \underline{\underline{SO^+(1,3)}}$$

EXAMPLES REPRESENTATIONS

1. $(0,0)$ $\underline{\underline{\text{scalar}}}$, trivial representation

2. $(\frac{1}{2}, 0)$ $N_i^- = 0$ so $\boxed{J_i = i K_i}$

N_i^+ must be $\frac{1}{2}$ rep $SU(2)$ $\quad N_i^+ = \frac{1}{2}\sigma_i$

$$N_i^+ = \frac{1}{2}(J_i + iK_i) = iK_i = \frac{1}{2}\sigma_i$$

$$\Rightarrow \quad J_i = \frac{1}{2}\sigma_i \qquad K_i = -\frac{i}{2}\sigma_i$$

$$\vec{R}(\vec{\theta}) = e^{i\vec{\theta}\cdot\vec{J}} = e^{i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}}$$

$$\vec{B}(\vec{\phi}) = e^{i\vec{\phi}\cdot\vec{K}} = e^{\vec{\phi}\cdot\frac{\vec{\sigma}}{2}}$$

so explicitly :

$$R_x(\theta_x) = \begin{pmatrix} \cos\frac{\theta_x}{2} & i\sin\frac{\theta_x}{2} \\ i\sin\frac{\theta_x}{2} & \cos\frac{\theta_x}{2} \end{pmatrix} ; \quad R_y(\theta_y) = \begin{bmatrix} \cos\frac{\theta_y}{2} & \sin\frac{\theta_y}{2} \\ -\sin\frac{\theta_y}{2} & \cos\frac{\theta_y}{2} \end{bmatrix}$$

$$R_z(\theta_z) = \begin{bmatrix} e^{i\theta_z/2} & 0 \\ 0 & e^{-i\theta_z/2} \end{bmatrix}$$

$$B_x(\phi_x) = \begin{bmatrix} \cosh\frac{\phi_x}{2} & \sinh\frac{\phi_x}{2} \\ \sinh\frac{\phi_x}{2} & \cosh\frac{\phi_x}{2} \end{bmatrix} \quad B_y(\phi_y) = \begin{bmatrix} \cosh\frac{\phi_y}{2} & -i\sinh\frac{\phi_y}{2} \\ i\sinh\frac{\phi_y}{2} & \cosh\frac{\phi_y}{2} \end{bmatrix}$$

$$B_z(\phi_z) = \begin{bmatrix} e^{\phi_z/2} & 0 \\ 0 & e^{-\phi_z/2} \end{bmatrix}$$

Left-Handed Spinor Representation of $SO^+(1,3)$

3. $\left(0, \frac{1}{2}\right)$ $\quad N_i^+ = 0 \Rightarrow J_i = -i K_i$

so $\underline{\underline{N_i^-}} = \frac{1}{2}(J_i - iK_i) = -iK_i = \underline{\underline{J_i = \frac{1}{2}\sigma_i}}$

$J_i = \frac{1}{2}\sigma_i$ so for $\left(\frac{1}{2}, 0\right)$

$K_i = \frac{i}{2}\sigma_i$ <u>different</u> from $\left(\frac{1}{2}, 0\right)$

$$\vec{R}(\vec{\theta}) = e^{i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}} \qquad \vec{B}(\vec{\phi}) = e^{-\vec{\phi}\cdot\frac{\vec{\sigma}}{2}}$$

$$\Downarrow$$

for example $B_z(\phi_z) = \begin{bmatrix} e^{-\phi_z/2} & 0 \\ 0 & e^{\phi_z/2} \end{bmatrix}$

Right-Handed Spinor representation

differs from Left-Handed on <u>BOOSTS</u>

FROM LEFT TO RIGHT :

$\Rightarrow$ ψ_L left-handed spinor means

- rotation $\quad \psi_L' = e^{i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}}\,\psi_L$
- boost $\quad \psi_L' = e^{\vec{\phi}\cdot\frac{\vec{\sigma}}{2}}\,\psi_L$

From here, consider $\quad \overline{\psi}_L = i\sigma_2\,\psi_L^*$

how does it transform ?

- rotation $\quad \overline{\psi}_L' = i\sigma_2(\psi_L')^* = i\sigma_2\left(e^{i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}}\,\psi_L\right)^*$

$$= i\sigma_2\, e^{-i\vec{\theta}\cdot\frac{\vec{\sigma}^*}{2}}\,\psi_L^*$$

$\Uparrow$ insert $[\mathbb{1} = (-i\sigma^2)(i\sigma^2)]$

$i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$= i\sigma_2\, e^{-i\vec{\theta}\cdot\frac{\vec{\sigma}^*}{2}}(-i\sigma_2)\,\underbrace{i\sigma_2\,\psi_L^*}_{\overline{\psi}_L}$$

$\Uparrow$

$(i\sigma_2)\,\vec{\sigma}^*\,(-i\sigma_2) = -\vec{\sigma}$

$$\rightarrow e^{i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}}\,\overline{\psi}_L \qquad \text{same transf.}$$

• boost

$$\bar{\psi}_L' = i\sigma_2 (\psi_L')^*$$

$$= i\sigma_2 \left(e^{\frac{1}{2}\vec{\phi}\cdot\vec{\sigma}}\psi_L\right)^*$$

$$= \underline{\underline{e^{-\frac{1}{2}\vec{\phi}\cdot\vec{\sigma}}\,\bar{\psi}_L}}$$

<u>RIGHT-HANDED SPINOR!</u>

$$\left\{\begin{array}{ll} i\sigma_2 & \text{Left to Right} \\ -i\sigma_2 & \text{Right to Left} \end{array}\right\} \text{ conventional}$$

$$\boxed{\begin{array}{l} \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \sigma_2\sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}}$$

$$-i\sigma_2 (\bar{\psi}_L)^* = -i\sigma_2 \left[i\sigma_2 (\psi_L)^*\right]^*$$

$$= -i\sigma_2 (i\sigma_2)\psi_L = \psi_L$$

back to left-handed

complex conjugation + multiplying by $i\sigma_2$ switches from <u>LEFT</u> to <u>RIGHT</u>, <u>remember this</u>

we'll get back to this soon!

notice that to go from <u>RIGHT</u> to <u>LEFT</u>

$$(-i\sigma_2)(\psi_R)^* = \tilde{\psi}_L$$ transforms as a left handed <u>one</u>

need a minus sign to be consistent

in fact: I know $i\sigma_2 \psi_L^* = \tilde{\psi}_R$

$$-i\sigma_2 (\tilde{\psi}_R)^* = -i\sigma_2 \left[i\sigma_2 \psi_L^* \right]^*$$

$$= -i\sigma_2 \left(\left[i\sigma_2 \psi_L \right]^* \right)^*$$

$$= (-i\sigma_2)(i\sigma_2)\psi_L = \underline{\underline{\psi_L}}$$

<u>left handed again</u>

Conventional who gets + and - !

• $L\left(\frac{1}{2}, \frac{1}{2}\right) \Rightarrow \vec{J} \quad 1 \oplus 0$

Contains spin 1

Indeed, this turns out to be vector representation through usual identification

$$X^{\dot{a}b} = X^\mu (\sigma_\mu)^{\dot{a}b}$$

Conventional names for two indices, $\dot{a}$ in R

b in L

We will come back to this in exercises and Next lecture

WHAT DO WE NEED FOR PHYSICS ?

these are representations of $SO^+(1,3)$

Physics requires [for QED, QCD for example]

Invariance also under PARITY $\Rightarrow$ photon has two helicities!

For what concerns Lorentz irreps, it means that for SPIN $\frac{1}{2}$ parity swaps $(\frac{1}{2},0) / (0,\frac{1}{2})$:

boost $B_{L,R}(\phi) = e^{\pm\frac{1}{2}\vec{\phi}\cdot\vec{\sigma}}$ $\qquad \phi \Leftrightarrow -\phi$

but $\beta_i = \tanh \phi_i = \frac{v_i}{c}$ swapped by parity!

$$\left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right) = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

DIRAC SPINOR

in chiral Representation

PARITY AND HANDNESS

Boost on left-handed $(\frac{1}{2}, 0)$ $\quad B_L(\bar{\phi}) \sim e^{\frac{1}{2}\bar{\phi}\cdot\bar{\sigma}}$

right-handed $(0, \frac{1}{2})$ $\quad B_R(\bar{\phi}) \sim e^{-\frac{1}{2}\bar{\phi}\cdot\bar{\sigma}}$

from Left $\to$ Right $\quad \vec{\phi} \to -\vec{\phi}$

but $\phi_j \sim \beta_i$ so $\quad \beta_i = \tanh \phi_i = \dfrac{v_i}{c}$

PARITY SWAPS $\beta \to -\beta \Rightarrow \phi \to -\phi$

ie it swaps Left - Right boosts

A Lorentz Invariant theory must also take PARITY into account, which ultimately is the reason why we need DIRAC SPINORS

$$\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right) = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

in chiral Repr.