Integrand Reduction 2/2

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TUM

We'll see now how to perform in practice the reduction to boxes, triangles, bubbles and tadpoles for an N-point integral of rank r

$$I_N^{(r)} = \int \frac{d^D l}{(2\pi)^D} \frac{\prod_{i=1}^{r} (u_i \cdot l)}{D_1 D_2 \dots D_N}$$

$$u_i^\mu = \underset{\Uparrow}{\text{external vectors}} = \{ \varepsilon_j^\mu ;\ \bar{u}\gamma^\mu u ;\ p_i^\mu \dots \}$$

remember, they live in $D=4$ space-time dimensions

STEPS

1] reduction tensor integrals $N \geq 5$ $\Rightarrow$ tensor ints with $N' < N$ + scalar N point

2] reduction scalar integrals with $N > 5$

3] reduction scalar integrals with $N = 5$

4] reduction tensor integrals $N = 4, 3, 2, 1$ $\Rightarrow$ scalar ints

TENSOR INTEGRALS $\quad N \geq 5$

if $N \geq 5$, we have at least 4 independent region momenta, we pick first four $q_1^\mu, q_2^\mu, q_3^\mu, q_4^\mu$ to define van Neerven - Vermaseren BASIS

$$v_1^\mu = \frac{\delta^{\mu q_2 q_3 q_4}_{q_1 q_2 q_3 q_4}}{\Delta_4} \quad \text{etc} \quad \text{plus } n_\varepsilon \text{ in } -2\varepsilon \text{ dim}$$

$$l^\mu = \sum_{i=1}^{4} (l \cdot q_i)\, v_i^\mu + (l \cdot n_\varepsilon)\, n_\varepsilon^\mu$$

- $u_i \cdot n_\varepsilon = 0$!

so numerator factors $u_i \cdot l$ become

$$u_{i\mu} \left[\sum_{j=1}^{4} (l \cdot q_j)\, v_j^\mu + (l \cdot n_\varepsilon)\, n_\varepsilon^\mu \right] = \sum_{j=1}^{4} (l \cdot q_j)(u_i \cdot v_j)$$

only dep. on loop momenta

external DATA

now remember

$$\ell \cdot q_j = \frac{1}{2}\left[D_j - D_N + q_j^2 + m_j^2 - m_N^2 \right]$$ such that

$$u_i \cdot \ell = \frac{1}{2}\sum_{j=1}^{4}\left[D_j - D_N\right](v_j \cdot u_i) \Rightarrow$$ cancel denominators & reduce $N \to N-1$ (possibly higher rank)

$$+ \frac{1}{2}\sum_{j=1}^{4}\left[m_j^2 - m_N^2 - q_j^2\right](v_j \cdot u_i) \Rightarrow$$ generate scalar integrals

repeating this for every factor of the numerator we are left either with SCALAR INTS with $\underline{N \text{ points}}$

or with TENSOR INTS with $\underline{N' < 4}$

SCALAR INTS $N \geq 4$

two different cases if $N > 5$ or $N = 5$!

- $N > 5$ consider integral $\int \frac{d^D l}{(2\pi)^D} \frac{1}{D_1 \cdots D_N}$

where $D_i = (l + q_i)^2 - m_i^2$

since $N > 5$, there are at least 6 denominators

BUT only 4 independent momenta + the (-2ε) part of the loop momentum = 5 objects

$\Rightarrow$ there must exist at least 1 relation among the D_j !

IN PRACTICE we use it as follows, consider linear comb:

$$\sum_{i=1}^{N} a_i D_i = \sum_{i=1}^{N} a_i \left(l^2 + 2 l \cdot q_i + q_i^2 - m_i^2 \right) =$$

$$= \ell^2 \sum_{i=1}^{N} \alpha_i + 2\ell_\mu \cdot \sum_{i=1}^{N} \alpha_i q_i^\mu + \sum_{i=1}^{N} \alpha_i (q_i^2 - m_i^2)$$

since $N > 5$, there exist at least one solution to

$$\begin{cases} \sum_{i=1}^{N} \alpha_i = 0 \\ \sum_{i=1}^{N-1} \alpha_i q_i^\mu = 0 \end{cases}$$

$\Rightarrow$ remove last one for example

(because $q_N^\mu = 0$ in our notation.)

so we can write

$$\Rightarrow \sum_{i=1}^{N} \alpha_i D_i = \sum_{i=1}^{N} \alpha_i (q_i^2 - m_i^2)$$

which implies

$$1 = \frac{\sum_{i=1}^{N} \alpha_i D_i}{\sum_{i=1}^{N} \alpha_i (q_i^2 - m_i^2)}$$

only dependence on loop momentum is here.

So starting from integrand

$$I_N = \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{D_1 \cdots D_N} \cdot \left\{ 1 = \frac{\sum_{i=1}^{N} \alpha_i D_i}{\sum_{i=1}^{N} \alpha_i (q_1^2 - m_1^2)} \right\}$$

$$= \sum_{i=1}^{N} C_k \, I_{N-1,\,i}$$

scalar integral with propagator i missing

IMPORTANT this does NOT work for $N=5$
not enough α_i to satisfy both constraints in general!

WE PROVED

$$I_{N\geq 5}^{(r)} = C_5 I_5^{(1)} + \sum_{N=1}^{4} \sum_{r'=0}^{N} C_{N,r'} I_N^{(r')}$$

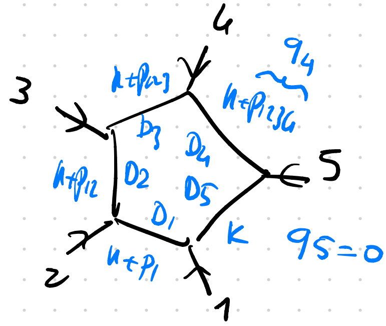
scalar pentagon

tensor ints $N' \leq 4$

REDUCTION OF SCALAR PENTAGONS

Consider

$$I_5^{(1)} = \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{D_1 \cdots D_5}$$



it depends on 4 independent external q_i^μ, so $d_P = 4$ out d_t has "only" -2ε dimensions

$$\ell^\mu = \sum_{i=1}^{4} v_i^\mu (\ell\cdot q_i) + \underset{\underset{(\ell\cdot n_\varepsilon)n_\varepsilon^\mu}{\parallel}}{\ell_\varepsilon^\mu}$$

loop momentum decomposition

$$= \frac{1}{2}\sum_{i=1}^{4} v_i^\mu \left(D_i - D_5 + m_i^2 - m_5^2 - q_i^2 \right) + \ell_\varepsilon^\mu$$

now let's square this using

$$\ell^2 = D_5^2 + m_5^2$$

$$D_5 + m_5^2 = \frac{1}{4}\sum_{i,j=1}^{4} v_i \cdot v_j \left(D_i - D_5 + m_i^2 - m_5^2 - q_i^2\right)\left(D_j - D_5 - m_j^2 - m_5^2 - q_j^2\right)$$

$$+ \, l_\varepsilon^2 \qquad \text{using} \quad \boxed{v_i \cdot l_\varepsilon = 0}$$

$\Rightarrow$

$$l_\varepsilon^2 = m_5^2 - \frac{1}{4}\sum_{i,j=1}^{4} v_i \cdot v_j \left(m_i^2 - m_5^2 - q_i^2\right)\left(m_j^2 - m_5^2 - q_j^2\right)$$

$$+ \, \mathcal{O}(D_i) + \mathcal{O}(D_i \cdot D_j)$$

i could $= j$

so we can write

$$l_\varepsilon^2 = \mathcal{N}(u_i, m_i) + \mathcal{O}(D_i) + \mathcal{O}(D_i D_j)$$

loop independent

$$\Rightarrow \quad \mathbb{1} = \frac{l_\varepsilon^2}{\mathcal{N}(u_i, m_i)} + \mathcal{O}(D_i) + \mathcal{O}(D_i \cdot D_j)$$

So inserting this "1" into the scalar pentagon we get

$$I_5^{(1)} = \frac{1}{\mathcal{N}} \int \frac{d^D\ell}{(2\pi)^D} \frac{\ell_\varepsilon^2}{D_1 D_2 D_3 D_4 D_5} + \underbrace{\sum_{N=1}^{4} \sum_{r=1}^{N} c_{N,r} I_N^{(r)}}_{N \leq 4 \quad \text{rank } r}$$

what about this ?

it turns out $O(\varepsilon)$,

it can be "neglected" in $D = 4 - 2\varepsilon$

if only interested in $O(\varepsilon^0)$, finite piece.

if all propagators are massive, no IR divergences,

moreover, integral is clearly UV-finite close $D \sim 4$

$\Rightarrow$ it is easy to convince yourself that therefore as

$\ell_\varepsilon^2 \to 0$ $(\varepsilon \to 0)$, the integral "must" go to zero

moreover, $(\ell \cdot n_\varepsilon)^2 \to 0$ if $\ell^\mu \to 0$! cures IR divergences too!

$\ell^\mu \to c\, p_1^\mu$

In order to do this properly, it is useful to try to provide an explicit representation for the extra momentum ℓ_ε^μ

$\Rightarrow$ Let's go back to

$$\ell^\mu = \sum_{j=1}^{4} (\ell \cdot q_j)\, v_j^\mu + \ell_\varepsilon^\mu$$

We need

$$\left\{ \begin{array}{l} \ell_\varepsilon \cdot q_j = 0 \\ \ell_\varepsilon \cdot v_j = 0 \end{array} \right\}$$

We can generalise Van Neerven - Vermaseren construction and define

$$w^\mu = \frac{\delta^{q_1 \dots q_4 \mu}_{q_1 \dots q_4 \ell}}{\mathcal{N}}$$

normalisation

Clearly, $\left\{ \begin{array}{l} w^\mu \cdot q_{j\mu} = 0 \\ w^\mu \cdot v_{j\mu} = 0 \end{array} \right\}$ which also implies

$w^\mu(\ell)$ depends on the loop momentum, and it spans exactly the remaining space, orthogonal to $q_1^\mu \cdots q_4^\mu$!

Notice that

$$w \cdot w = \frac{\Delta_{q_1 \cdots q_4} \, \Delta_{q_1 \cdots q_4 \ell}}{N^2} \equiv \frac{\Delta_{q_1 \cdots q_4}}{\Delta_{q_1 \cdots q_4 \ell}}$$

$$w \cdot \ell = \frac{\Delta_{q_1 \cdots q_4 \ell}}{N} \equiv 1$$

pick $N = \Delta_{q_1 \cdots q_4 \ell}$ then $w \cdot \ell = 1$

$$l^\mu = \sum_{j=1}^{4} (l \cdot q_j)\, v_j^\mu + c\, w^\mu(e)$$

multiply by w_μ

$$\underbrace{l \cdot w}_{=1} = c\, \underbrace{w \cdot w}\, \frac{\Delta_{q_1 \cdots q_4}}{\Delta_{q_1 \cdots q_4 l}}$$

so $$c = \left(\frac{\Delta_{q_1 \cdots q_4 l}}{\Delta_{q_1 \cdots q_4}} \right)$$

$$l^\mu = \sum_{j=1}^{4} (l \cdot q_j)\, v_j^\mu + \underbrace{\left(\frac{\Delta_{q_1 \cdots q_4 l}}{\Delta_{q_1 \cdots q_4}} \right) w^\mu}_{\text{explicit representation for } l_\epsilon^\mu}$$

remember, integral we had to do was

$$\int \frac{d^D\ell}{(2\pi)^D} \frac{\ell_\varepsilon^2}{D_1 \cdots D_5} =$$

$$= \int \frac{d^D\ell}{(2\pi)^D} \frac{\Delta_5^2}{\Delta_4^2} \left(w \cdot w = \frac{\Delta_4}{\Delta_5} \right)$$

$$= \frac{1}{\Delta_4} \int \frac{d^D\ell}{(2\pi)^D} \frac{\Delta_{q_1 \cdots q_4 \ell}}{D_1 \cdots D_5}$$

this object is interesting now, it is the grace determinant of 5 momenta, it can be $\neq 0$

<u>ONLY</u> if $D \geq 5$

On top of that, $\Delta_{q_1 \cdots q_4 \ell} \to 0$ $\begin{cases} \text{if } \ell^\mu \to 0 & \text{(SOFT DIV)} \\ \text{if } \ell^\mu // q_j^\mu & \text{(COLLINEAR)} \end{cases}$

so actually, even if propagators are massless and pentagon could develop IR divergences, the gram determinant cures them all.

$$\underbrace{\int \frac{d^D l}{(2\pi)^D} \frac{\Delta_{q_1 \dots q_4 l}}{D_1 \dots D_5}} \quad \left\{ \begin{array}{l} \text{UV finite} \\ \text{IR finite} \\ \Delta_{q \dots q_4 l} \sim \mathcal{O}(\varepsilon) \to 0 \end{array} \right.$$

this integral can be neglected in $D=4$!

it actually turns out that there is even a more interesting interpretation of this $\Rightarrow$

$$\underbrace{\frac{1}{\Delta_{q_1 \dots q_4}} \int \frac{d^D l}{(2\pi)^D} \frac{\Delta_{q_1 \dots q_4 l}}{D_1 \dots D_5} = (D-4) \int \frac{d^{D+2} l}{(2\pi)^{D+2}} \frac{1}{D_1 \dots D_5}}_{\text{Dimensional shift}}$$

$\Downarrow$ 6-dimensional PENTAGON

REDUCTION OF $N=4$ Tensor Integrals

Consider

$$I_4^{(r)} = \int \frac{d^D \vec{l}}{(2\pi)^D} \frac{\prod_{i=1}^{r} (u_i \cdot l)}{D_1 D_2 D_3 D_4} \qquad \underline{\underline{r \leq 4}}$$

- 3 region momenta q_i (or p_i) span $d_p = 3$
- transverse space $d_t = 1 \underbrace{- 2\varepsilon}_{\text{in } d \text{ dimensions}}$

so we write

$$l^\mu = \sum_{i=1}^{3} (l \cdot q_i) v_i^\mu + (l \cdot n_4) n_4^\mu + (l \cdot n_\varepsilon) n_\varepsilon^\mu \qquad (*)$$

and clearly

$$u_i \cdot l = \sum_{i=1}^{3} (l \cdot q_i)(u_i \cdot v_i) + (l \cdot n_4)(n_4 \cdot u_i)$$

since $\underline{n_\varepsilon \cdot u_i = 0}$

now remember $\quad l\cdot q_i = \frac{1}{2}\left(D_i - D_N + m_i^2 - m_N^2 - q_1^2\right)$

$$= \frac{1}{2}\left(D_i - D_N\right) + \underset{\text{in } D_i}{\text{constant}}$$

We insert this, and keep only 4-point part

$$\frac{\prod_{i=1}^{7}(n_i\cdot l)}{D_1D_2D_3D_4} = \sum_{i=1}^{t} d_i\,\frac{(l\cdot n_4)^i}{D_1D_2D_3D_4} + \frac{d_0}{D_1D_2D_3D_4} + \text{lower point}$$

– explicit form doesn't matter – & $t \leq 4$!

Let's look at the integrals

$$\int \frac{d^D\ell}{(2\pi)^D} \frac{(\ell\cdot n_4)^i}{D_1\dots D_4} = n_4^{\mu_1}\cdots n_4^{\mu_i} \int \frac{d^D\ell}{(2\pi)^D} \frac{\ell_{\perp\mu_1}\cdots\ell_{\perp\mu_i}}{D_1\cdots D_4}$$

tensor integrals but all confined in transverse space!

Now notice that $D_i = (\ell+q_i)^2 - m_i^2 = \ell_\perp^2 + \underset{\substack{\Uparrow \\ \text{only in physical space}}}{\Delta_i}$

So all denominators always depend on $\ell_\perp^\mu$ through $\ell_\perp^2$ only $\Rightarrow$ rotational invariant

$$\int \frac{d^D\ell}{(2\pi)^D} \frac{1}{D(\ell_\perp^2, \ell_\parallel^\mu)} \left\{ \begin{matrix} 1 \\ \cancel{\ell_\perp^{\mu_1}} \to = 0 \\ \ell_\perp^{\mu_1}\ell_\perp^{\mu_2} \\ \cancel{\ell_\perp^{\mu_1}\ell_\perp^{\mu_2}\ell_\perp^{\mu_3}} \to 0 \\ \ell_\perp^{\mu_1}\ell_\perp^{\mu_2}\ell_\perp^{\mu_3}\ell_\perp^{\mu_4} \end{matrix} \right\}$$

sending $\ell_\perp \to -\ell_\perp$!

so at the end we are only left with

$$\int \frac{d^D\ell}{(2\pi)^D} \frac{\prod_{j=1}^{7}(\ell\cdot n_j)}{D_1 D_2 D_3 D_4} = d_0 \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{D_1 D_2 D_3 D_4}$$

$$+ d_2 \int \frac{d^D\ell}{(2\pi)^D} \frac{(\ell\cdot n_4)^2}{D_1 D_2 D_3 D_4} + d_4 \int \frac{d^D\ell}{(2\pi)^D} \frac{(\ell\cdot n_4)^4}{D_1 D_2 D_3 D_4}$$

+ lower point integrals

what are these integrals ?

Now use same trick as earlier, square eq (*)

$$\ell^2 = D_N + m_N^2 \equiv (\ell\cdot n_4)^2 + (\ell\cdot n_\varepsilon)^2 + \text{const} + \mathcal{O}(D_i)$$

at least one D_i!

up to 3

so inverting it, I can write

$$(\ell\cdot n_4)^2 = -(\ell\cdot n_\varepsilon)^2 + \text{const} + \underset{\text{at least one}}{\mathcal{O}(D_i)}$$

which implies that all double appearances of $(\ell\cdot n_4)$ can be removed from

coeffs change!

$$\int \frac{d^D\ell}{(2\pi)^D} \frac{\prod_{j=1}^{7}(\ell\cdot n_j)}{D_1D_2D_3D_4} = \tilde{d}_0 \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{D_1D_2D_3D_4}$$

$$+ \tilde{d}_2 \int \frac{d^D\ell}{(2\pi)^D} \frac{(\ell\cdot n_\varepsilon)^2}{D_1D_2D_3D_4} + \tilde{d}_4 \int \frac{d^D\ell}{(2\pi)^D} \frac{(\ell\cdot n_\varepsilon)^4}{D_1D_2D_3D_4}$$

$$+ \text{ lower point integrals}$$

where now, once more, we expect $(\ell\cdot n_\varepsilon) \to 0$ as $\varepsilon \to 0$, and therefore if there are no divergences these integrals should drop!

If there are no IR divergences, power counting ensures that

$$\frac{(l\cdot n_\varepsilon)^2}{D_1\cdots D_4} \longrightarrow 0$$

while

$$\frac{(l\cdot n_\varepsilon)^4}{D_1\cdots D_4} \longrightarrow \frac{l^4}{l^8}\cdot d^4 l \to \begin{array}{l}\text{UV}\\ \text{Log div}\end{array}$$

only this one can produce a rational term

to prove that $\int \frac{(l\cdot n_\varepsilon)^2}{D_1\cdots D_4} \to 0$ in general, one can do a similar construction as for the pentagon

$$w^\mu = \frac{\delta^{q_1 q_2 q_3 n_4 \mu}_{q_1 q_2 q_3 n_4 l}}{\Delta 5} \qquad n_\varepsilon = \frac{\Delta_5}{\Delta_{q_1 q_2 q_3 n_4}} w^\mu$$

$$\int \frac{d^D l}{(2\pi)^D} \frac{(l\cdot n_\varepsilon)^2}{D_1\cdots D_4} = \frac{1}{\Delta 4} \int \frac{d^D l}{(2\pi)^D} \frac{\Delta 5}{D_1\cdots D_4}$$

UV, IR finite

$\to 0$

as $D \to 4$

One can also observe that $(\ell \cdot n_\varepsilon)^m \to 0$

both is soft limit (since $\ell \to 0$)

collinear limit (since $\ell \to c p_i$; $p_i \cdot n_\varepsilon = 0$!)

So, in conclusion, we are left with

$$\int \frac{d^D\ell}{(2\pi)^D} \frac{\prod_{j=1}^{?}(\ell \cdot u_j)}{D_1 \cdots D_4} = \tilde{d}_0 \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{D_1 \cdots D_4}$$

$$+ \boxed{\tilde{d}_4 \int \frac{d^D\ell}{(2\pi)^D} \frac{(\ell \cdot n_\varepsilon)^4}{D_1 \cdots D_4}} + \text{lower point} + \mathcal{O}(\varepsilon)$$

this here contributes only
to rational part

REDUCTION N=3 TENSOR INTEGRALS

We have seen, we only need to consider $r \leq 3$

$$\int \frac{d^D l}{(2\pi)^D} \frac{\prod_{j=1}^{2} (l \cdot u_j)}{D_1 D_2 D_3}$$

where now

$$l^\mu = \sum_{i=1}^{2} (l \cdot q_i)\, v_i^\mu + (l \cdot n_3)\, n_\varepsilon^\mu + (l \cdot n_4)\, n_4^\mu + l_\varepsilon^\mu$$

Analogously as for the boxes, I only need to consider

$$\int \frac{d^D l}{(2\pi)^D} \frac{\prod_{\substack{i,j=1 \\ |i+j| \leq 3}}^{3} (l \cdot n_3)^i (l \cdot n_4)^j}{D_1 D_2 D_3}$$

since

$$\boxed{l \cdot n_\varepsilon = 0}$$

and

$$l \cdot q_i = \frac{1}{2} \Big(\underbrace{D_1 - D_N}_{\text{give lower point}} - q_1^2 + \underbrace{q_N^2 - m_1^2 + m_N^2}_{\text{lower rank}} \Big)$$

max power of loop momentum is l^3 since $r \leq 3$

as earlier, integrating over transverse space we see

$$\int \frac{d^D l}{(2\pi)^D} \frac{1}{D_1 D_2 D_3} \begin{Bmatrix} l_\perp^{\mu_1} l_\perp^{\mu_2} l_\perp^{\mu_3} \rightarrow 0 \\ l_\perp^{\mu_1} l_\perp^{\mu_2} \\ l_\perp^{\mu_1} \rightarrow 0 \\ 1 \end{Bmatrix}$$

since $D_i^2 = l_\perp^2 + \Delta_i (l_\parallel)$

and we only need to consider

$$\int \frac{d^D l}{(2\pi)^D} \frac{\prod_{j=1}^{r} (u_j \cdot l)}{D_1 D_2 D_3} = c_0 \int \frac{d^D l}{(2\pi)^D} \frac{1}{D_1 D_2 D_3}$$

$$+ \int \frac{d^D l}{(2\pi)^D} \frac{1}{D_1 D_2 D_3} \left\{ c_3 (l \cdot n_3)^2 + c_4 (l \cdot n_4)^2 + c_{34} (l \cdot n_3)(l \cdot n_4) \right\}$$

+ lower point integrals !

or, alternatively, by rotation in coeffs c_i

$$= \int \frac{d^D \ell}{(2\pi)^D} \left\{ \tilde{c}_3 \left[(\ell\cdot n_3)^2 + (\ell\cdot n_4)^2 \right] + \tilde{c}_4 \left[(\ell\cdot n_3)^2 - (\ell\cdot n_4)^2 \right] + c_{34} (\ell\cdot n_3)(\ell\cdot n_4) \right\} + \text{rest}$$

this is nice because

. Squaring $\ell^\mu = \sum_{j=1}^{2} (\ell\cdot q_j) v_j^\mu + \sum_{j=3}^{4} (\ell\cdot n_j) n_j^\mu + \overbrace{\ell_\varepsilon^\mu}^{(\ell\cdot n_\varepsilon) n_\varepsilon^\mu}$

$$\ell^2 = \boxed{D_3^2 + m_3^2 = \sum_{i,j=1}^{2} (\ell\cdot q_i)(\ell\cdot q_j) v_i\cdot v_j} \Leftarrow \text{lower points or scalar int}$$

$$+ (\ell\cdot n_3)^2 + (\ell\cdot n_4)^2 + \ell_\varepsilon^2$$

so

$$(\ell\cdot n_3)^2 + (\ell\cdot n_4)^2 = -\ell_\varepsilon^2 + \text{lower points } \mathcal{O}(D_i)$$

so we can substitute coeff of $\tilde{c}_3$ with ℓ_ε^2

what about $\tilde{c}_4 \left[(\ell \cdot n_3)^2 - (\ell \cdot n_4)^2 \right]$?

$$\int \frac{d^D \ell}{(2\pi)^D} \frac{(\ell \cdot n_3)^2 - (\ell \cdot n_4)^2}{D_1 D_2 D_3} =$$

$$= \left(n_3^\mu n_3^\nu - n_4^\mu n_4^\nu \right) \underbrace{\int \frac{d^D \ell}{(2\pi)^D} \frac{\ell_\mu \ell_\nu}{D_1 D_2 D_3}}_{I_{\mu\nu}}$$

$$I_{\mu\nu} = A\, g_{\mu\nu} + \underbrace{\sum_{ij=1}^{2} B_{ij}\, q_{i\mu}\, q_{j\nu}}$$

drops when multiplied by $n_{3\mu}$ or $n_{4\mu}$!

$$= \left(n_3^\mu n_3^\nu - n_4^\mu n_4^\nu \right) A\, g_{\mu\nu} = \left(n_3^2 - n_4^2 \right) A = 0 \, !$$

Similarly $\dfrac{(\ell \cdot n_3)(\ell \cdot n_4)}{D_1 D_2 D_3} \sim n_3^\mu n_4^\nu \cdot A \cdot g_{\mu\nu} = 0$

$\underline{\underline{n_3 \cdot n_4 = 0}}$

so in conclusion, for triangles we get

$$\int \frac{d^D \ell}{(2\pi)^D} \frac{\prod_{j=1}^{2} (\ell \cdot u_j)}{D_1 D_2 D_3} = \tilde{c}_0 \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{D_1 D_2 D_3}$$

$$+ \underbrace{\tilde{c}_3 \int \frac{d^D \ell}{(2\pi)^D} \frac{(\ell \cdot n_\varepsilon)^2}{D_1 D_2 D_3}}$$ + lower points

this guy here is UV divergent in $D=4$

so it can produce a RATIONAL PART,

since $(\ell \cdot n_\varepsilon)^2 \to 0$ as $\varepsilon \to 0$

REDUCTION TENSOR INTS $N=2$

for 2-point integrals, physical space is only 1-dimensional

$$l^\mu = (l \cdot q_1) v_1^\mu + \sum_{j=2}^{4} (l \cdot n_j) n_j^\mu + l_\varepsilon^\mu$$

A tensor integral is of the form

$$\int \frac{d^D l}{(2\pi)^D} \frac{\prod_{j=1}^{r} (l \cdot n_j)}{D_1 D_2} = \quad \text{with } r \leq 2$$

so plugging in the loop momentum, we get

$$= \int \frac{d^D l}{(2\pi)^D} \left\{ \frac{b_0 + b_j (l \cdot n_j) + b_{ij} (l \cdot n_i)(l \cdot n_j)}{D_1 D_2} \right\}$$

Again rotational symmetry guarantees $\displaystyle\int \frac{d^D l}{(2\pi)^D} \frac{l \cdot n_j}{D_1 D_2} = 0$

Similarly, if $i \neq j$

$$\int \frac{d^D\vec{\ell}}{(2\pi)^D} \frac{(\ell\cdot n_i)(\ell\cdot n_j)}{D_1 D_2} = n_i^\mu n_j^\nu \int \frac{d^D\vec{\ell}}{(2\pi)^D} \frac{\ell_\mu \ell_\nu}{D_1 D_2} \propto n_i\cdot n_j = 0$$

$$\text{so} \int \frac{d^D\vec{\ell}}{(2\pi)^D} \frac{\prod_{j=1}^{2} (\ell\cdot u_j)}{D_1 D_2} = b_0 \int \frac{d^D\vec{\ell}}{(2\pi)^D} \frac{1}{D_1 D_2} + b_{ii} \int \frac{d^D\vec{\ell}}{(2\pi)^D} \frac{(\ell\cdot n_i)^2}{D_1 D_2}$$

+ lower point

then, squaring the loop momentum we get

$$\ell^2 = \underbrace{D_2 + m_2^2}_{\hookrightarrow} = \underbrace{(\ell\cdot q_1)^2\, v_1\cdot v_1}_{\text{lower points}} + (\ell\cdot n_j)^2 + \ell_\varepsilon^2$$

lower points
or scalar 2-point

$$(\ell\cdot n_2)^2 + (\ell\cdot n_3)^2 + (\ell\cdot n_4)^2 = -\ell_\varepsilon^2 + \mathcal{O}(D_i) + \mathcal{O}(D_i D_j)$$

Use it to remove some of the rank-2

$$\int \frac{d^D l}{(2\pi)^D} \frac{\prod_{j=1}^{2} (l \cdot n_j)}{D_1 D_2} = \tilde{b}_0 \int \frac{d^D l}{(2\pi)^D} \frac{1}{D_1 D_2}$$

$$+ \tilde{b}_2 \int \frac{d^D l}{(2\pi)^D} \frac{(l \cdot n_\varepsilon)^2}{D_1 D_2} + \tilde{b}_3 \int \frac{d^D l}{(2\pi)^D} \frac{(l \cdot n_2)^2 - (l \cdot n_4)^2}{D_1 D_2}$$

$$+ \tilde{b}_4 \int \frac{d^D l}{(2\pi)^D} \frac{(l \cdot n_3)^2 - (l \cdot n_4)^2}{D_1 D_2} + \text{lower point}$$

using a trivial "change of basis"

$$\begin{matrix} (l \cdot n_2)^2 \\ (l \cdot n_3)^2 \\ (l \cdot n_4)^2 \end{matrix} \longrightarrow \begin{cases} (l \cdot n_2)^2 - (l \cdot n_4)^2 = a \\ (l \cdot n_3)^2 - (l \cdot n_4)^2 = b \\ (l \cdot n_2)^2 + (l \cdot n_3)^2 + (l \cdot n_4)^2 = c \end{cases}$$

$$\left[\begin{aligned} +(l \cdot n_4)^2 &= -(a + b - c) = -a - b + c \\ (l \cdot n_2)^2 &= a - a - b + c = c - b \\ (l \cdot n_3)^2 &= b - a - b + c = c - a \end{aligned} \right]$$

Now, as for triangles

$$\int \frac{d^D \ell}{(2\pi)^D} \frac{(\ell \cdot n_i)^2 - (\ell \cdot n_j)^2}{D_1 D_2} \sim (n_i^2 - n_j^2) A \equiv 0$$

so we are left with

$$\int \frac{d^D \ell}{(2\pi)^D} \frac{\prod_{j=1}^{2} (u_j \cdot \ell)}{D_1 D_2} = \tilde{b}_0 \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{D_1 D_2}$$

$$+ \tilde{b}_2 \underbrace{\int \frac{d^D \ell}{(2\pi)^D} \frac{(\ell \cdot n_\varepsilon)^2}{D_1 D_2}}_{\text{UV divergent!}} + \text{lower points}$$

FINALLY for $N = 1$ there is nothing to do

$$\int \frac{d^D \ell}{(2\pi)^D} \frac{\ell \cdot u}{D_1} = a_0 \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{D_1} + a_i \underbrace{\int \frac{d^D \ell}{(2\pi)^D} \frac{\ell \cdot n_i}{D_1}}_{= 0 \text{ symmetry!}}$$

SCALAR

So finally putting all steps together, we have proven that

$$I_N^{(n)} = \tilde{d}_0^i I_4^i + c_0^i I_3^i + b_0^i I_2^i + a_0^i I_1^i + R$$

$$+ O(\varepsilon)$$

different boxes, triangles bubbles and tadpoles contribute

FINAL REMARKS

1] Coefficients $\tilde{d}_0^i$, $\tilde{c}_0^i$, $\tilde{b}_0^i$, $\tilde{a}_0^i$ can be determined using procedure above => Integrand Reduction PASSARINO VELTMAN

2] We are only interested in the fact that this decomposition exists, we'll use UNITARITY to compute coefficients