Integrand Reduction 1/2

LORENZO TANCREDI
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TUM

A comment on integrals of type

$$\int \frac{d^D \vec{l}}{(2\pi)^D} \frac{(l \cdot l)^{m} \prod_{j=1}^{r-2m} (l \cdot u_j)}{D_1 \cdots D_N}$$

what about this ?

$$l \cdot l = \left[(l+q_i)^2 - m_i^2\right] - 2\, l \cdot q_i + m_i^2 - q_i^2$$

$$= \underbrace{D_i + m_i^2 - q_i^2}_{\text{repeat this for all available denominators}} - 2\, l \cdot q_i$$

same structure I have but lower rank

0

INTEGRAND REDUCTION

the computation of one-loop scattering amplitudes has received a strong boost once it was realised that any one-loop tensor integral can always be reduced to a **BASIS** of **master integrals** in $D = 4 - 2\varepsilon$ space-time dimensions

$$I_N = \int \frac{d^D l}{(2\pi)^D} \frac{\mathcal{N}(l^\mu, p^\mu, \gamma^\mu, \varepsilon^\mu \ldots)}{D_1 D_2 \ldots D_N}$$

$$= \sum_i c_{4,i} I_4^{(i)} + \sum_i c_{3,i} I_3^{(i)} + \sum_i c_{2,i} I_2^{(i)}$$

$$+ \sum_i c_{1,i} I_1^{(i)} + R + O(\varepsilon) \qquad (1)$$

these coefficients are ε independent

$I_N^{(i)}$ denote SCALAR (rank zero) N-point integrals $\Rightarrow$ in $D = 4 - 2\varepsilon$ we need at most scalar BOXES

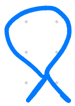
the index i is necessary, because in general there is more than one box, more than one triangle etc —

First to prove this were $\boxed{\text{PASSARINO, VELTMAN 1979}}$

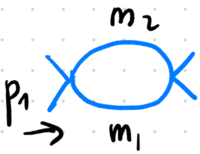
this result is, in a sense, independent and more "stringent" than what one can achieve with Integration by parts identities (IBPs), but it is also less general and applies "only" @ 1 loop

[We will say more about a comparison of this with IBPs in later part of the course]

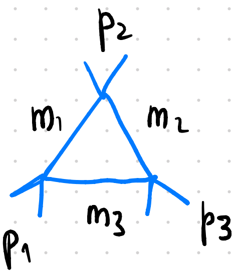
For the integral basis we have



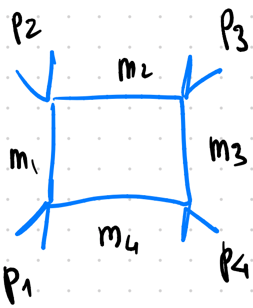
$$I_1(m_1^2) = \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{D_1}$$



$$I_2(p_1^2; m_1^2, m_2^2) = \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{D_1 D_2}$$



$$I_3(p_1^2, p_2^2, p_3^2; m_1^2, m_2^2, m_3^2) = \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{D_1 D_2 D_3}$$



$$I_4(p_1^2, p_2^2, p_3^2, p_4^2; s_{12}, s_{23}; m_1^2, m_2^2, m_3^2, m_4^2)$$

$$= \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{D_1 D_2 D_3 D_4}$$

$$D_i = (\ell + q_i)^2 - m_i^2 \qquad s_{ij} = (p_i + p_j)^2$$

All these integrals are known analytically and we will learn more about the functions required for their calculation in the next lectures.

First thing we want to do now, is to prove our main formula (1). We will prove 2 things

reduction of higher rank

1] $$\int \frac{d^D\vec{l}}{(2\pi)^D} \frac{l^{\mu_1} l^{\mu_2} \dots l^{\mu_r}}{D_1 \dots D_N} \longrightarrow \sum_{N' \leq N} \int \frac{d^D\vec{l}}{(2\pi)^D} \frac{1}{D_1 \dots D_{N'}} + R$$

reduction of higher point to lower point

2] $$\int \frac{d^D\vec{l}}{(2\pi)^D} \frac{1}{D_1 \dots D_N} \rightarrow \sum_{N' \leq 4} \int \frac{d^D\vec{l}}{(2\pi)^4} \frac{1}{D_1 \dots D_{N'}} + R$$

RATIONAL PART

In order to achieve this, we will expand the loop momentum in terms of the region momenta q_i^μ (or equivalently p_i^μ, but less convenient)

- if space-time dimensions $N \geq D+1$, then we have enough region momenta to span the full space

$$l^\mu = c_1 q_1^\mu + \dots + c_D q_D^\mu + l_\varepsilon^\mu$$

residual part in DIM reg

$D-2\varepsilon$ dim

! REASON WHY PENTAGON DISAPPEARS IN $D=4$, modulo a rational remainder !

- if $N < D+1$ then we don't have enough momenta

$$l^\mu = c_1 q_1^\mu + \dots + c_{N-1} q_{N-1}^\mu + b_1 n_1^\mu + \dots + b_{D-N+1} n_{D-N+1} + l_\varepsilon^\mu$$

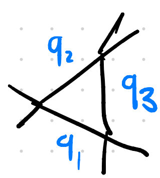
we would like to use these decompositions to prove formula (1). Still, we can do a bit better. The issue is that the q_i^μ are not orthogonal, so if we decompose

$$l^\mu = c_1 q_1^\mu + \dots + c_{N-1} q_{N-1}^\mu + \dots$$

these coefficients are rather complicated to compute since $q_i \cdot q_j \neq \delta_{ij}$

it is then convenient to use a different basis called VAN NEERVEN - VERMASEREN BASIS

to see how this works, let's imagine for simplicity that $D = 2$, and that we are dealing with a three-point function



In this case in $2-2\varepsilon$ dimensions:

$$l^\mu = c_1 q_1^\mu + c_2 q_2^\mu + l_\varepsilon^\mu$$

VAN NEERVEN - VERMASEREN BASIS is built as follow:

Consider Levi-Civita tensor in $D=2$ $\varepsilon^{\mu\nu}$

$$\varepsilon^{\mu\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$ STRICTLY in $D=2$!

with $\varepsilon^{\mu\nu}\varepsilon_{\rho\sigma} = \delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho$

[we keep here $g^{\mu\nu}$ minkowski, δ^μ_ν euclidean

so when we contract $\varepsilon^{\mu\nu}\varepsilon_{\rho\sigma}$ always one has lower

and one upper indices]

and build the two vectors

$$\bar{v}_1^{\,\mu} = \varepsilon^{\mu\nu} q_{2\nu} \qquad \bar{v}_2^{\,\mu} = \varepsilon^{\nu\mu} q_{1\nu}$$

notice order is different

$$\bar{v}_1^{\,\mu} = \varepsilon^{\mu q_2} \quad ; \quad \bar{v}_2^{\,\mu} = \varepsilon^{q_1\mu}$$

common notation

such that

- $\bar{v}_i \cdot \bar{v}_j \neq \delta_{ij}$ so it still not orthonormal

BUT

they are independent vectors and by construction

$$\begin{bmatrix} \bar{v}_1 \cdot q_1 = \varepsilon^{q_1 q_2} & \bar{v}_1 \cdot q_2 = 0 \\ \bar{v}_2 \cdot q_1 = 0 & \bar{v}_2 \cdot q_2 = \varepsilon^{q_1 q_2} \end{bmatrix}$$

orthogonal to the q_j!

$$\varepsilon^{q_1 q_2} = \varepsilon^{\mu\nu} q_{1\mu} q_{2\nu}$$

we'll see what this is in a second

let's define the "normalised" vectors

$$v_1^\mu = \frac{\vec{v}_1^{\,\mu}}{\varepsilon^{q_1 q_2}} = \frac{\varepsilon^{\mu q_2}}{\varepsilon^{q_1 q_2}} \qquad v_2^\mu = \frac{\varepsilon^{q_1 \mu}}{\varepsilon^{q_1 q_2}}$$

$$v_1 \cdot q_1 = 1 \qquad v_2 \cdot q_2 = 1 \qquad \underline{v_i \cdot q_j = 0 \quad i \neq j}$$

so if I want to decompose l^μ in $D = 2$ dimensions, I can write

$$l^\mu = (q_1 \cdot l)\, v_1^\mu + (q_2 \cdot l)\, v_2^\mu \qquad \left[\text{strictly in } D = 2!\right]$$

Different way to see this is from Schouten id

$$l^\mu \varepsilon^{\nu\rho} = l^\nu \varepsilon^{\mu\rho} + l^\rho \varepsilon^{\nu\mu}$$

which is a consequence of the fact that in $D=2$ you cannot have more than 2 independent vectors [prove it!]

$$\varepsilon^{\mu\nu\rho} p_{1\mu} p_{2\nu} p_{3\rho} = 0 \quad \text{in} \quad \underline{D=2}$$

$$l^\mu = (l \cdot q_1) v_1^\mu + (l \cdot q_2) v_2^\mu$$

- very convenient because as we know, $(q_i \cdot l)$ can always be written as linear comb of propagators of the graph!

$$l \cdot q_i = D_i - D_N - q_i^2 + m_i^2 - m_N^2 \quad \text{so}$$

$$l^\mu = \frac{1}{2} \sum_{i=1}^{2} \left[D_i - D_N - q_i^2 + m_i^2 - m_N^2 \right] v_i^\mu$$

- Still, this decomposition is valid STRICTLY in $D=2$, where we can define $\varepsilon^{\mu\nu}$!

We can generalise this to $D = 2-2\varepsilon$ by using

$$v_1^\mu = \frac{\varepsilon_{q_1 q_2} \varepsilon^{\mu q_2}}{\varepsilon_{q_1 q_2} \varepsilon^{q_1 q_2}} \qquad v_2^\mu = \frac{\varepsilon_{q_1 q_2} \varepsilon^{q_1 \mu}}{\varepsilon_{q_1 q_2} \varepsilon^{q_1 q_2}}$$

and contracting

$$\varepsilon^{\mu\nu} \varepsilon_{\rho\sigma} = \delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho \overset{!}{\equiv} \delta^{\mu\nu}_{\rho\sigma} \quad \text{definition}$$

such that

$$\varepsilon^{q_1 q_2} \varepsilon_{q_1 q_2} = \delta^{q_1 q_2}_{q_1 q_2} = \det \begin{bmatrix} q_1^2 & q_1 \cdot q_2 \\ q_1 \cdot q_2 & q_2^2 \end{bmatrix}$$

$$= \Delta_2 = q_1^2 q_2^2 - (q_1 \cdot q_2)^2$$

GRAM DETERMINANT OF TWO VECTORS

So we write

$$v_1^\mu = \frac{\delta^{\mu q_2}_{q_1 q_2}}{\Delta_2} \qquad v_2^\mu = \frac{\delta^{q_1 \mu}_{q_1 q_2}}{\Delta_2}$$

valid in $D = 2 - 2\varepsilon$!

this is called the Van Neerven - Vermaseren basis

in $D = 2$

$$l^\mu = (l \cdot q_1)\, v_1^\mu + (l \cdot q_2)\, v_2^\mu + l_\varepsilon^\mu$$

extra component

in -2ε dimensions.

GENERAL CASE

Earlier we considered case $N=3$ $D=2$ where space-time can be spanned entirely by the two region momenta.

Consider now more general case

$$D = d_p + d_t$$

transverse space (all that remains!)

physical dim

$\neq$ depends on graph $= \underline{\underline{(N-1)}}$

considering physical case $D=4$ then

$$d_p = \min(N-1, 4)$$

if $N-1 > 4$, then still only 4 of the external momenta can be independent!

$\Rightarrow$ Important : even if we work in dim reg, extend momenta remain in $\underline{\underline{D=4}}$!

Now we construct the VAN NEERVEN - VERMASEREN basis in the d_p physical dimensions spanned by $q_1^\mu, \dots, q_{d_p}^\mu$ as:

$$v_i^\mu = \frac{\delta^{q_1 \dots q_{i-1}\, \mu\, q_{i+1} \dots q_{d_p}}_{q_1 \dots q_i \dots q_{d_p}}}{\Delta_{d_p}(q_1 \dots q_{d_p})} \qquad i = 1, \dots, d_p$$

$\Delta_{d_p} = \det(q_i \cdot q_j)$ gram determinant of region momenta.

$$\delta^{\mu_1 \dots \mu_{d_p}}_{\nu_1 \dots \nu_{d_p}} = \det \begin{bmatrix} \delta^{\mu_1}_{\nu_1} & \delta^{\mu_1}_{\nu_2} & \dots & \delta^{\mu_1}_{\nu_{d_p}} \\ \vdots & & & \\ \delta^{\mu_{d_p}}_{\nu_1} & \delta^{\mu_{d_p}}_{\nu_2} & \dots & \delta^{\mu_{d_p}}_{\nu_{d_p}} \end{bmatrix}$$

completely Antisymm in upper and lower indices !

$v_i \cdot q_j = \delta_{ij}$ orthogonal to the q_j !

how do we span the transverse space ?

• EUCLIDEAN space, spanned by n_j^μ $\quad j=1,\dots,d_t$

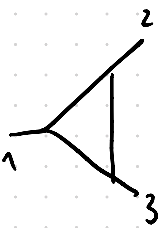
$$n_i\cdot n_j=\delta_{ij}\ ; \quad v_i\cdot n_j=0 \ \ \& \ \ q_i\cdot n_j=0$$

similarly also for the (-2ε) extra dimension n_ε^μ

so we write

$$l^\mu=\sum_{i=1}^{d_p}(l\cdot q_i)\,v_i^\mu+\sum_{i=1}^{d_t}(l\cdot n_i)\,n_i^\mu+(l\cdot n_\varepsilon)\,n_\varepsilon^\mu$$

so, for example, for a 3-point function



$$l^\mu=\sum_{i=1}^{2}(l\cdot q_i)\,v_i^\mu+\sum_{i=1}^{2}(l\cdot n_i)\,n_i^\mu+(l\cdot n_\varepsilon)\,n_\varepsilon^\mu$$

$d_p=2$ $\qquad$ $d_t=2+1$ extra to span -2ε dimensions