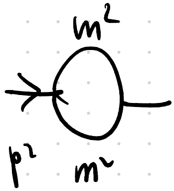
UNITARITY:

Cutkosky Rules & Dispersion Relations

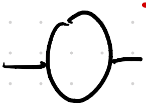
RECAP

We have played with one-loop bubble



1. Computed it for $p_M^2 < 0 \Rightarrow \underbrace{p^2 > 0}_{\text{Euclidean momentum}}$

$D \sim 2$



$$= (m^2)^{-\varepsilon}\,\Gamma(1+\varepsilon)\left[-\frac{2}{m^2}\,\frac{\xi}{1-\xi^2}\,\ln\xi + O(\varepsilon)\right]$$

introduced $p^2 = m^2\,\dfrac{(1-\xi)^2}{\xi}$ LANDAU VARIABLE

2. We have seen that in order to get Minkowskian result we need to continue such that

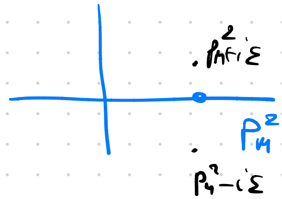
$$p^2 \to -\left(\underline{\underline{p_M^2 + i\varepsilon}}\right) \iff \xi \to -(x - i\delta)$$

$$0 < x < 1 \iff \underline{p_M^2 > 4m^2}$$

$$= (m^2)^{-\varepsilon}\,\Gamma(1+\varepsilon)\left[\frac{2}{m^2}\,\frac{x}{1-x^2}\,\ln(x) + \frac{2}{m^2}\left(\frac{x}{1-x^2}\right) i\pi + O(\varepsilon)\right]$$

such that, above threshold ($p^2 > 4m^2$) the bubble develops an imaginary part

$$\mathrm{Disc}\left(-\!\bigcirc\!-\right) = 2i\,\mathrm{Im}\left(-\!\bigcirc\!-\right)$$



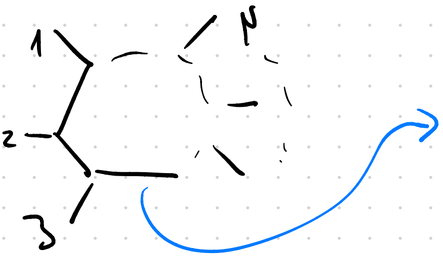
$$\mathrm{Im}\left(-\!\bigcirc\!-\right)^{D\sim 2} = +\frac{2}{m^2}\pi\,\frac{x}{1-x^2}\,\underbrace{\Theta\left(p_M^2 - 4m^2\right)}_{\Uparrow\ \text{only defined here}} + O(\varepsilon)$$

physically, imaginary part $\equiv$ discontinuity produced when $p^2 > 4m^2$ because there is enough energy to produce virtual particles

ON-SHELL

How to COMPUTE DISCONTINUITIES

Imaginary parts come from intermediate particles going on-shell.



$$\frac{1}{k^2 - m^2 + i\varepsilon}$$ Feynman propagator

We have

$$\mathrm{Im}\left(\frac{1}{k^2 - m^2 + i\varepsilon}\right) = \frac{1}{2i}\left(\frac{1}{k^2 - m^2 + i\varepsilon} - \frac{1}{k^2 - m^2 - i\varepsilon}\right)$$

$$= \frac{-\varepsilon}{(k^2 - m^2)^2 + \varepsilon^2}$$

And notice also

$$\lim_{\varepsilon \to 0}\left(-\varepsilon \int_0^{+\infty} \frac{1}{x^2 + \varepsilon^2} f(x)\, dx\right) = -\pi f(0)$$

such that $\begin{cases} \mathrm{Im}\left(\dfrac{1}{k^2-m^2+i\varepsilon}\right) = -\pi\,\delta(k^2-m^2) \\ \mathrm{Disc}\left(\dfrac{1}{k^2-m^2+i\varepsilon}\right) = -2\pi i\,\delta(k^2-m^2) \end{cases}$

$\Rightarrow$ propagators are real except when particles go on-shell !

Now, let's consider the Feynman Propagator

NOTICE, WE DEFINE IT WITHOUT i !!

$$\Pi_F(k^2) = \frac{1}{k^2-m^2+i\varepsilon} = \frac{1}{2\omega_k}\left[\frac{1}{k_0-\omega_k+i\varepsilon} - \frac{1}{k_0+\omega_k-i\varepsilon}\right]$$

with $\omega_k = \sqrt{\vec{k}^2+m^2}$ and

$$= \frac{1}{2\omega_k}\left[\frac{1}{k_0-\omega_k-i\varepsilon} - \frac{1}{k_0+\omega_k-i\varepsilon}\right] + \frac{1}{2\omega_k}\left[\frac{1}{k_0-\omega_k+i\varepsilon} - \frac{1}{k_0-\omega_k-i\varepsilon}\right]$$

see next page

$$\frac{1}{k^2-m^2+i\varepsilon} = \frac{1}{2\omega_k}\left[\frac{1}{k_0-\omega_k+i\varepsilon} - \frac{1}{k_0+\omega_k-i\varepsilon}\right]$$

$$= \underbrace{\frac{1}{2\omega_k}\left[\frac{1}{k_0-\omega_k-i\varepsilon} - \frac{1}{k_0+\omega_k-i\varepsilon}\right]}_{\Pi_R(k)}$$

RETARDED PROPAGATOR

$$+ \frac{1}{2\omega_k}\underbrace{\left[\frac{1}{k_0-\omega_k+i\varepsilon} - \frac{1}{k_0-\omega_k-i\varepsilon}\right]}$$

$$\frac{k_0-\omega_k-i\varepsilon-k_0+\omega_k-i\varepsilon}{(k_0-\omega_k)^2+\varepsilon^2} = \frac{-2i\varepsilon}{(k_0-\omega_k)^2+\varepsilon^2}$$

$$= \Pi_R(k) - \frac{i}{\omega_k}\frac{\varepsilon}{(k_0-\omega_k)^2+\varepsilon^2}$$

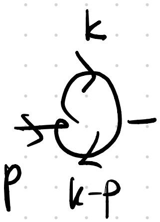
$$= \Pi_R(k) - \frac{i}{\omega_k}\pi\,\delta(k_0-\omega_k)$$

so we can rewrite Feynman Propagator as

$$\Pi_F(k) = \Pi_R(k) - i\,\frac{\pi}{\omega_k}\,\delta(k_0-\omega_k)$$

the retarded propagator has ONLY POLES above the real <u>AXIS</u>.

Let's take our bubble in $D \simeq 2$

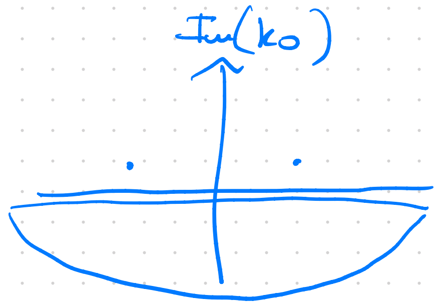


$$= \int \frac{d^D k}{i\pi^{D/2}} \underbrace{\frac{1}{(k^2-m^2+i\varepsilon)}}_{\Pi_F(k)} \underbrace{\frac{1}{(k-p)^2-m^2+i\varepsilon}}_{\Pi_F(k-p)}$$

$$= \int \frac{d^D k}{i\pi^{D/2}} \left[\Pi_R(k) - i\,\frac{\pi}{\omega_k}\,\delta(k_0-\omega_k)\right]\left[\Pi_R(k-p) - i\,\frac{\pi}{\omega_k}\,\delta(k_0-p_0-\omega_{k-p})\right]$$

$$\omega_k = \sqrt{\vec{k}^2+m^2} \qquad \omega_{k-p} = \sqrt{(\vec{k}-\vec{p})^2+m^2}$$

there are 4 terms



1. $$\int \frac{d^D k}{i\pi^{D/2}} \Pi_R(k)\,\Pi_R(k-p) = 0$$

2. $$\int \frac{d^D k}{i\pi^{D/2}} \frac{\pi^2}{\omega_k \omega_{k-p}} \delta(k_0-\omega_k)\,\underbrace{\delta(k_0-p_0-\omega_{k-p})}_{\text{no support}} = 0 \qquad \left(p^\mu = (M,\vec{0})\right)$$

3. $$\int \frac{d^D k}{i\pi^{D/2}} \pi \left[\frac{1}{\omega_k}\delta(k_0-\omega_k)\,\Pi_R(k-p)\right]$$

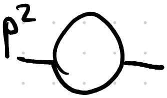
4. $$\int \frac{d^D k}{i\pi^{D/2}} \pi \left[\frac{1}{\omega_{k-p}}\delta(k_0-p_0-\omega_{k-p})\,\Pi_R(k)\right]$$

In these two terms we can re-use same relation:

$$\Pi_R(k) = \Pi_F(k) + i\frac{\pi}{\omega_k}\delta(k_0-\omega_k)$$

$$\Pi_R(k-p) = \Pi_F(k-p) + i\frac{\pi}{\omega_{k-p}}\delta(k_0-p_0-\omega_{k-p})$$

And throw away $\delta\delta$ again we get:



$$= -i \int \frac{d^D k}{i\pi^{D/2}} \left[\frac{\pi}{\omega_k} \delta(k_0-\omega_k)\, \Pi_F(k-p) + \frac{\pi}{\omega_{k-p}} \delta(k_0-p_0-\omega_{k-p})\, \Pi_F(k) \right]$$

Till now just simple "manipulations"

Let's take now the <u>discontinuity</u>

$\mathrm{Disc}\Big($  $\Big)$

$$= -i \int \frac{d^D k}{i\pi^{D/2}} \left[\frac{\pi}{\omega_k} \delta(k_0-\omega_k)\, \mathrm{Disc}\big(\Pi_F(k-p)\big) + \frac{\pi}{\omega_{k-p}} \delta(k_0-p_0-\omega_{k-p})\, \mathrm{Disc}\big(\Pi_F(k)\big) \right]$$

$$\mathrm{Disc}\big(\Pi_F(k)\big) = -2\pi i\, \delta(k^2-m^2)$$

$$\text{Disc}\left(\overset{p^2}{\to}\bigcirc-\right) = -i \int \frac{d^D k}{i\pi^{D/2}} \Bigg[\frac{\pi}{\omega_k}\,\delta(k_0-\omega_k)\,(-2\pi i)\,\delta\big((k-p)^2-m^2\big)$$

$$\frac{\pi}{\omega_{k-p}}\,\delta(k_0-p_0-\omega_{k-p})\,(-2\pi i)\,\delta(k^2-m^2)\Bigg]$$

notice also that

$$\delta(k^2-m^2) = \delta\big[(k_0-\omega_k)(k_0+\omega_k)\big]$$

$$= \frac{\delta(k_0-\omega_k)}{2\omega_k}\,\theta(k_0) + \frac{\delta(k_0+\omega_k)}{2\omega_k}\,\theta(-k_0)$$

$$\delta\big((k-p)^2-m^2\big) = \frac{\delta(k_0-p_0-\omega_{k-p})\,\theta(k_0-p_0)}{2\omega_{k-p}}$$

$$+ \frac{\delta\overset{(k_0-p_0+\omega_k)}{(p_0-k_0-\omega_{k-p})}}{2\omega_{k-p}}\,\theta(p_0-k_0)$$

so we obtain at once 4 products :

1. $\int \delta(k_0-\omega_k)\, \delta(k_0-p_0-\omega_k)\, \theta(k_0-p_0)$

2. $\int \delta(k_0-\omega_k)\, \delta(p_0-k_0-\omega_k)\, \theta(p_0-k_0)$

3. $\int \delta(k_0-p_0-\omega_k)\, \delta(k_0-\omega_k)\, \theta(k_0)$

4. $\int \delta(k_0-p_0-\omega_k)\, \delta(k_0+\omega_k)\, \theta(-k_0)$

go to frame $p^\mu = (M, \vec{0})$ and find

1. $\int \delta(k_0-\omega_k)\, \delta(-M)\, \theta(k_0-M) = 0$

$$\boxed{2.\ \int \delta(k_0-\omega_k)\, \delta(M-2\omega_k)\, \theta(p_0-k_0) \neq 0}$$

3. $\int \delta(-M)\, \delta(k_0-\omega_k)\, \theta(k_0) = 0$

4. $\int \delta(\underbrace{-M-2\omega_k}_{})\, \delta(k_0+\omega_k)\, \theta(-k_0) = 0$

$\omega_k > 0$ always! $M > 0$ too! 9

so we are left with

$$\mathrm{Disc}_{p^2}\left(\rightarrow\!\bigcirc\!- \right) = -i \int \frac{d^D k}{i\pi^{D/2}} \frac{\pi}{\omega_k} \delta(k_0 - \omega_k) \frac{(-2\pi i)}{2\,\omega_{k-p}} \delta(p_0 - k_0 - \omega_{k-p})\, \theta(p_0 - k_0)$$

Finally let's make it Lorentz invariant

by using $\theta(k_0)\,\delta(k^2 - m^2) = \frac{1}{2\omega_k} \delta(k_0 - \omega_k)$

$$\theta(p_0 - k_0)\, \delta((p-k)^2 - m^2) = \frac{1}{2\omega_{k-p}} \delta(p_0 - k_0 - \omega_{k-p})$$

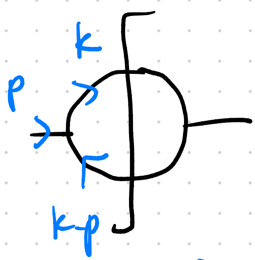
putting all together we get : $\left(\begin{array}{c}\text{putting } -i \\ \text{inside}\end{array}\right)$

$$\mathrm{Disc}_{p^2}\left(\rightarrow\!\bigcirc\!- \right) = \int \frac{d^D k}{i\pi^{D/2}} \left[(-2\pi i)\, \delta(k^2 - m^2)\, \theta(k_0) \right] \left[(-2\pi i)\, \delta((p-k)^2 - m^2)\, \theta(p_0 - k_0) \right]$$

we obtain discontinuity by "cutting" the suitable set of propagators through "rule"

$$\frac{1}{k^2-m^2+i\varepsilon} \to -2\pi i\,\delta(k^2-m^2)\,\theta(k_0)$$

paying attention to "DIRECTIONALITY" $\Longrightarrow$ positive energy $\theta(E_0)$ is the one that flows from LEFT to RIGHT

$$\mathrm{Im}\left(\text{—○—} \right) =$$ 

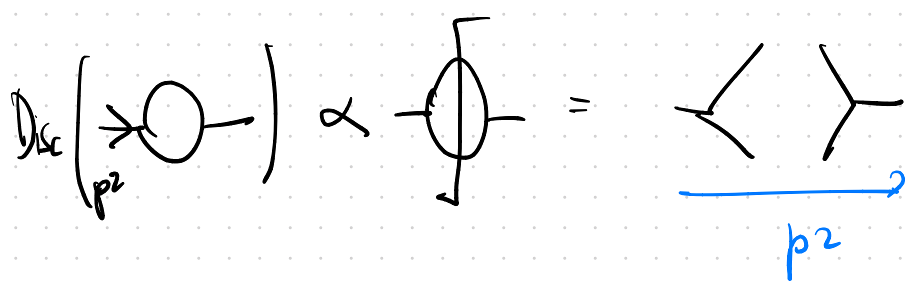
$p_0 - k_0$ not $k_0 - p_0$!!

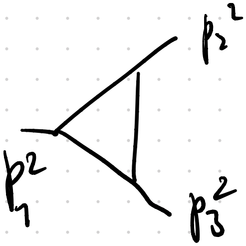
CUTKOSKY RULES can be proven by LARGEST-TIME EQ. (Veltman)

Cutkosky rules provide a way to compute "directly" the discontinuity of a graph by substituting propagators with δ-functions, ie putting intermediate particles on-shell.

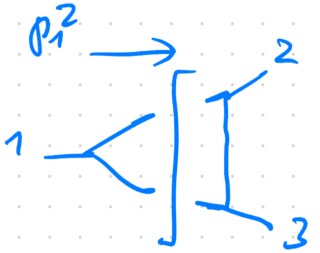
INDEED in general we must cut the diagram so to divide it into two separate pieces with the variable we are interested in flowing from LEFT to RIGHT

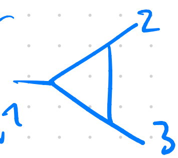
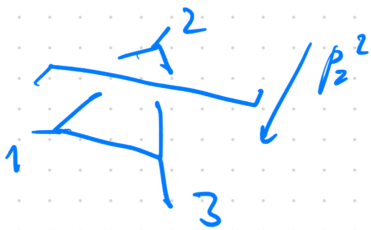
EXAMPLES

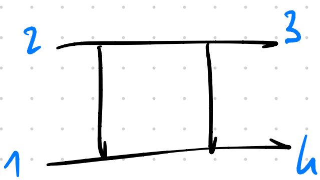


2. 

here there are in general three variables p_i^2

$\mathrm{Disc}_{p_1^2}\Big($  $\Big) \propto$ 

$\mathrm{Disc}_{p_2^2}\Big($  $\Big) \propto$  etc

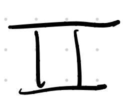
3. 

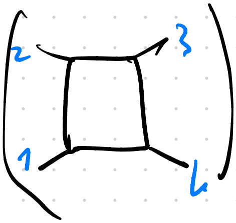
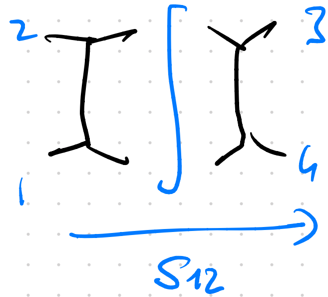
here in total 6 $\quad p_i^2 \quad i = 1, \dots, 4$

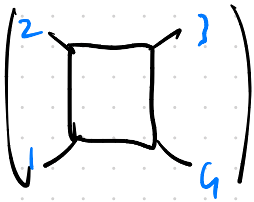
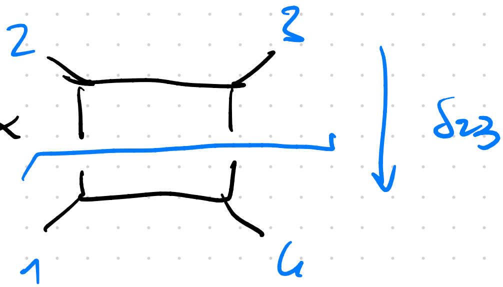
plus $\quad s_{12} = (p_1+p_2)^2$

$s_{23} = (p_2+p_3)^2$

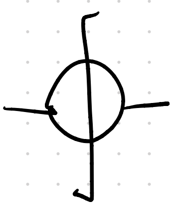
with $\quad \underline{s_{12} + s_{23} + s_{13} = \sum_i p_i^2}$

$\mathrm{Disc}_{p_1^2}\Big($  $\Big) \propto$  etc

$\mathrm{Disc}_{s_{12}}$  $\propto$ 

$\mathrm{Disc}_{s_{23}}$  $\propto$ 

Explicitly for one-loop <u>bubble</u> :

 $= \displaystyle\int \frac{d^D k}{i\pi^{D/2}} \left[-2\pi i\,\delta(k^2-m^2)\,\theta(k_0)\right]$
$$\left[-2\pi i\,\delta((p-k)^2-m^2)\,\theta(p_0-k_0)\right]$$

go to reference frame $p^\mu = (M, \vec{0})$

$$2\,\mathrm{Im} = -4\pi^2 \int \frac{d^{D-1}k}{i\pi^{D/2}} \int_0^\infty dk_0 \; \delta(k_0^2-\omega_k^2)\,\theta(k_0)\; \delta\big((M-k_0)^2-\omega_k^2\big)\,\theta(M-k_0)$$

$$= -4\pi^2 \int \frac{d^{D-1}k}{i\pi^{D/2}} \int_0^M dk_0 \; \frac{\delta(k_0-\omega_k)}{2\omega_k}\; \delta(M^2+\cancel{\omega_k^2}-2M\omega_k-\cancel{\omega_k^2})$$

$$= -4\pi^2 \int \frac{d^{D-1}k}{i\pi^{D/2}} \; \frac{1}{2\omega_k}\; \delta(M^2-2M\omega_k)$$

$$= +i\, \frac{4\pi^2\, \Omega(D-1)}{2M\, \pi^{D/2}} \int_0^\infty dk\, |\bar{k}|^{D-2}\; \frac{1}{2\omega_k}\; \delta\Big(\omega_k-\frac{M}{2}\Big)$$

$$\omega_k = \sqrt{\vec{k}^2+m^2} \qquad d\omega_k = \frac{1}{\cancel{2}}\frac{\cancel{2}|\bar{k}|}{\omega_k}\, d|\bar{k}| = \frac{\sqrt{\omega_k^2-m^2}}{\omega_k}\, dk$$

$$= +i\, \frac{\cancel{2}\pi^{2-\frac{D}{2}}}{\cancel{2}M}\; \frac{2\pi^{\frac{D-1}{2}}}{\Gamma\big(\frac{D-1}{2}\big)} \int_{m^2}^\infty d\omega_k \left(\sqrt{\omega_k^2-m^2}\right)^{D-3} \delta\Big(\omega_k-\frac{M}{2}\Big)$$

$$= +i \, \frac{2\pi^{3/2}}{\Gamma\left(\frac{D-1}{2}\right)} \, \frac{\left(\sqrt{\frac{M^2}{4} - m^2}\right)^{D-3}}{M} \qquad M = \sqrt{p_M^2}$$

$$= +i \, \frac{2\pi^{3/2}}{\Gamma\left(\frac{D-1}{2}\right)} \, \frac{\left(\sqrt{\frac{p_M^2}{4} - m^2}\right)^{D-3}}{\sqrt{p_M^2}} \qquad \text{for } D=2$$

$$= +i \, \frac{2\pi^{3/2}}{\Gamma\left(\frac{1}{2}\right)} \, \frac{2}{\sqrt{p_M^2(p_M^2 - 4m^2)}} = -\frac{2\pi^{3/2}}{\sqrt{\pi}} \, \frac{1}{\sqrt{p_M^2(p_M^2 - 4m^2)}}$$

$$p_M^2 = m^2 \frac{(1+x)^2}{x} \qquad \sqrt{p_M^2(p_M^2 - 4m^2)} = m^2 \left(\frac{1-x^2}{x}\right)$$

$$= +i \, 4\pi \frac{1}{m^2} \left(\frac{x}{1-x^2}\right) = \underline{2i \left[2\pi \, \frac{1}{m^2} \, \frac{x}{1-x^2} \right]}$$

$\Uparrow$

the imaginary part that we found by <u>analytic</u> <u>continuation</u>

Now, the important point is that we can use this imaginary part, which comes from CUTKOSKY RULES, and is easier to compute, in order to reconstruct the full <u>result</u> $\Rightarrow$

* Dispersion Relations

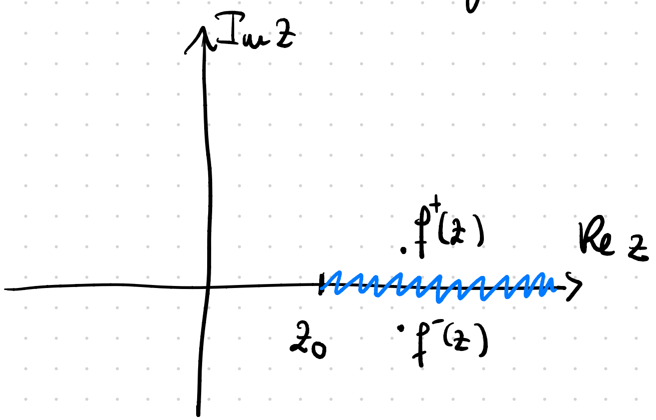
* Generalisation $\rightarrow$ GENERALISED UNITARITY (<u>next semester</u>)

At tree-level, knowledge of analytic structure of scattering amplitude (POLES, NO BRANCH CUTS) allowed us to use Complex Analysis to derive BCFW recursion

$\Rightarrow$ Disp Rel are first generalisation of this @ L-loops

DISPERSION RELATIONS

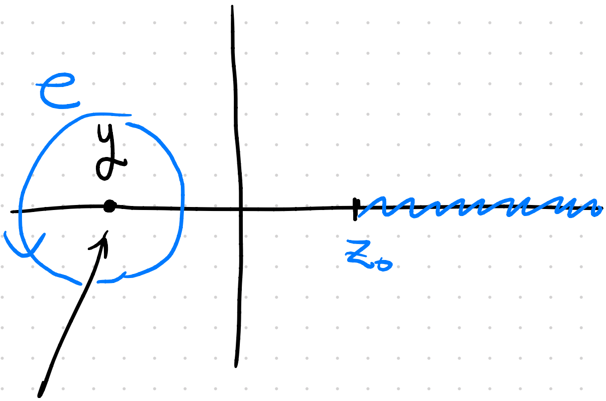
Consider a (complex) function $f(z)$ such that for $z > z_0$ $f(z)$ has a branch cut and develops therefore a discontinuity



$$\boxed{\begin{aligned}\text{Disc}\, f(z) &= f^+(z) - f^-(z)\\ &= f(z+i\varepsilon) - f(z-i\varepsilon)\end{aligned}}$$

Discontinuity across branch cut

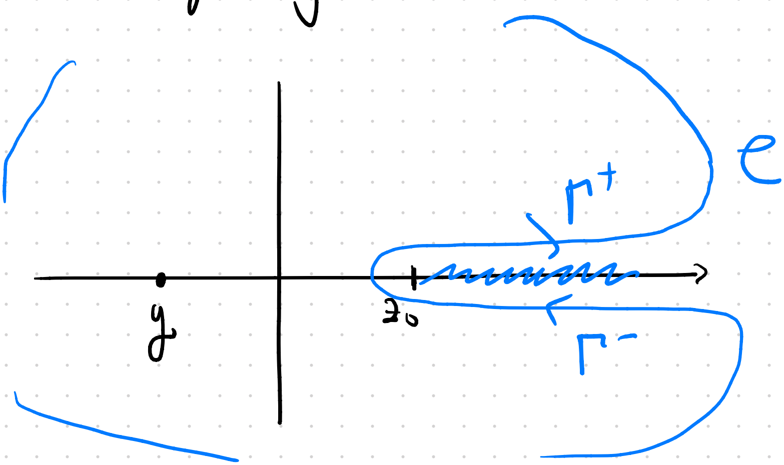
Suppose that someone gives us the discontinuity, dispersion relations tell us how to reconstruct full function



Consider a point y far from the branch cut and consider

$$\oint_{\mathcal{C}} \frac{f(z)}{z-y} = 2\pi i \, f(y) \quad \text{from Cauchy theorem}$$

But deforming contour $\mathcal{C}$ to infinity we get



now if $\frac{f(z)}{z-y} \to 0$ as $z \to \infty$

(circle at infinity)

then :

$$\oint_{C} \frac{f(z)}{z-y}\, dz = \int_{\Gamma^-} \frac{f(z)}{z-y}\, dz + \int_{\Gamma^+} \frac{f(z)}{z-y}\, dz$$

$$= 2\pi i\, f(y)$$

or, turning this around

$$f(y) = \frac{1}{2\pi i}\left(\int_{\Gamma^-} + \int_{\Gamma^+}\right) \frac{f(z)}{z-y}\, dz$$

$$\to \frac{1}{2\pi i} \int_{z_0}^{\infty} dz\, \frac{f^+(z) - f^-(z)}{z-y} = \frac{1}{2\pi i}\int_{z_0}^{\infty} dz\, \frac{\mathrm{Disc}\, f(z)}{z-y}$$

finally notice that $\underline{\text{Disc}}\, f(z) = \underline{2i\,\mathrm{Im}(f(z))}$

$$\boxed{f(y) = \frac{1}{\pi}\int_{z_0}^{\infty} \frac{dz}{z-y}\,\mathrm{Im}(f(z))}$$

Dispersion Relation for $f(y)$!

note that I assumed y far from branch cut, so it is $\underline{\text{NOT}}$ on integration contour.

We also assumed that $\frac{f(z)}{z-y} \to 0$ at infinity.

If it doesn't, one needs to build so-called $\underline{\text{subtracted}}$ $\underline{\text{dispersion relations}}$;

Suppose $f(z)$ goes too fast at infinity; then consider

$$f(y) - f(y_0) = \frac{1}{\pi}\int_{z_0}^{\infty} dz \left[\frac{\mathrm{Im}(f(z))}{z-y} - \frac{\mathrm{Im}(f(z))}{z-y_0}\right]$$

$$= \frac{1}{\pi}\int_{z_0}^{\infty} dz\, \mathrm{Im}(f(z)) \frac{[y-y_0]}{\underbrace{(z-y)(z-y_0)}}$$

this goes now as $\frac{1}{z^2}$

so it kills $f(z)$ more!

$$\Rightarrow f(y) = f(y_0) + \frac{(y-y_0)}{\pi}\int_{z_0}^{\infty} \frac{dz}{(z-y)} \frac{\mathrm{Im}(f(z))}{(z-y_0)}$$

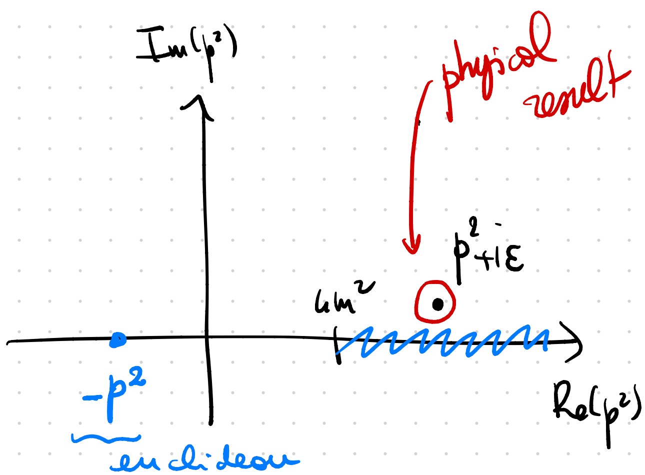
Let's apply this immediately to our 1-loop bubble in $D \sim 2$ (CONVERGENT!)

 $\propto \mathrm{Im}\Big($  $\Big)$

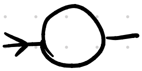
$$= -\frac{2\pi}{m^2}\,(m^2)^{-\varepsilon}\,\Gamma(1+\varepsilon)\,\frac{x}{1-x^2}\,\underbrace{\theta(x)\,\theta(1-x)}$$

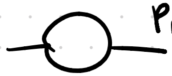
$\neq 0$ for $0 < x < 1$

or $p_M^2 > 4m^2$!

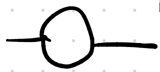


so we should have

 $-p^2$

$= \frac{1}{\pi}\int_{4m^2}^{\infty}\frac{d p_M^2}{p_M^2-(-p^2)}\,\mathrm{Im}\Big($  $p_M^2\Big)$

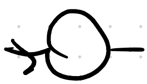
↑ euclidean momentum is negative!

$= \frac{1}{\pi}\int_{4m^2}^{\infty}\frac{d p_M^2}{p_M^2+p^2}\,\mathrm{Im}\Big($  $p_M^2\Big)$

⇑ expressed conveniently with $p_M^2 = m^2\frac{(1+x)^2}{x}$

$$d p_M^2 = m^2\left[\frac{2(1+x)}{x}-\frac{(1+x)^2}{x^2}\right]dx = m^2\left[\frac{2x+2x^2-1-x^2-2x}{x^2}\right]$$

$$= m^2\left[\frac{x^2-1}{x^2}\right]dx = -m^2\,\frac{1-x^2}{x^2}\,dx \qquad (0<x<1)$$



$$= \frac{1}{\cancel{\pi}} \left(+ \frac{2\cancel{\pi}}{m^2} (m^2)^{-\varepsilon} \Gamma(1+\varepsilon) \right)$$

$$\times \int_0^1 dx \, \frac{1-x^2}{x^2} m^2 \quad \frac{1}{m^2 \frac{(1+x)^2}{x} + p^2} \quad \frac{x}{1-x^2}$$

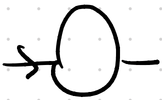
$$= + \frac{2}{m^2} (m^2)^{-\varepsilon} \Gamma(1+\varepsilon) \int_0^1 dx \, \frac{1}{(1+x)^2 + \frac{p^2}{m^2} x}$$

Use also parametrization for $\frac{p^2}{m^2} = \frac{(1-\xi)^2}{\xi}$

then rational function becomes

$$\frac{\xi}{(1+x)^2 \xi + (1-\xi)^2 x} = \frac{\xi}{(1+x^2+\cancel{2x})\xi + (1+\xi^2-\cancel{2\xi})x}$$

$$= \frac{\xi}{\xi + x^2\xi + x + \xi^2 x} = \frac{\xi}{(x+\xi)(1+\xi x)} \quad !$$



$$= + \frac{2}{m^2}(m^2)^{-\varepsilon}\,\Gamma(1+\varepsilon)\int_0^1 dx\,\frac{\xi}{(x+\xi)(1+x\xi)}$$

$$\frac{1}{(x+\xi)(1+x\xi)} = \left(\frac{1}{x+\xi} - \frac{\xi}{1+x\xi}\right)\frac{1}{(1-\xi^2)}$$

$$= + \frac{2}{m^2}(m^2)^{-\varepsilon}\,\Gamma(1+\varepsilon)\left[\frac{\xi}{1-\xi^2}\right]\left(\int_0^1 \frac{dx}{x+\xi} - \int_0^1 dx\,\frac{\xi}{1+x\xi}\right)$$

$$= + \frac{2}{m^2}(m^2)^{-\varepsilon}\,\Gamma(1+\varepsilon)\,\frac{\xi}{1-\xi^2}\left[\cancel{\ln(1+\xi)} - \ln(\xi) - \cancel{\ln(1+\xi)} + \ln(1)\right]$$

$$= - \frac{2}{m^2}(m^2)^{-\varepsilon}\,\Gamma(1+\varepsilon)\,\frac{\xi}{1-\xi^2}\ln(\xi)$$

Exactly result we started from!