INTEGRATION BY PARTS ids

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RECAP (on 20.01.2022)

- $\int d^D\Omega = \Omega(D) = \frac{2\pi^{D/2}}{\Gamma(\frac{D}{2})}$

- GAMMA FUNCT. $\Gamma(x) = \int_0^\infty dt\, e^{-t} t^{x-1}$

$\Gamma(n) = (n-1)!$ if $n \in \mathbb{N}$

$\Gamma(1+x) = x\,\Gamma(x)$ Defining Property

- BETA FUNCTION $B(x,y) = \int_0^1 du\, u^{x-1}(1-u)^{y-1}$

$= \int_0^\infty dt\, \frac{t^{x-1}}{(1+t)^{x+y}} \overset{!}{\equiv} \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

1 LOOP TADPOLE

$$\bigcirc = (m^2)^{\frac{D-2}{2}}\,\Gamma\left(1-\frac{D}{2}\right) = (m^2)^{\frac{D-2}{2}}\,\frac{\Gamma\left(3-\frac{D}{2}\right)}{\left(\frac{2-D}{2}\right)\left(\frac{4-D}{2}\right)}$$

$$= 4\,(m^2)^{\frac{D-2}{2}}\,\Gamma\left(\frac{6-D}{2}\right)\,\frac{1}{(D-2)(D-4)}$$

important: not all divergences are the same in dim reg !

$$\int \frac{d^3k}{\pi^{3/2}}\,\frac{1}{k^2+m^2} = \frac{2\pi^{3/2}}{\Gamma(\frac{3}{2})\pi^{3/2}}\int_0^{\Lambda} dk\,\frac{k^2}{k^2+m^2} \sim \Lambda$$

Linear Divergence.

$\bigcirc^{(D=3)}$ computed in dim reg is NOT divergent!

$$\bigcirc = (m^2)^{\frac{3}{2}} \Gamma\left(1-\frac{3}{2}\right) = (m^2)^{\frac{1}{2}} \Gamma\left(-\frac{1}{2}\right)$$

$$= -2\sqrt{\pi}\, m$$ FINITE & NEGATIVE

Similar to $\sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s)$ Riemann Zeta

$$\zeta(-1) = \sum_{n=1}^{\infty} n = -\frac{1}{12} \quad !$$

⇒ In Dimensional Regularization the INTEGRAL REPRESENTATION represents the final result ONLY if it converges. Otherwise it has NOTHING to do with it!

What if integral representation never converges?

$\Rightarrow$ this can happen if there are also so called

IR divergences [SOFT $k \to 0$; COLLINEAR $k \parallel p$ with $p^2 = 0$]

typical example, massless TADPOLE

$$\int \frac{d^D k}{\pi^{D/2}} \frac{1}{k^2} = \frac{\Omega(D)}{\pi^{D/2}} \int_0^\infty dk \; k^{D-3}$$

($k = \infty$) UV div $D > 2$

IR div $D \leq 2$ ($k = 0$)

$\nexists$ D such that integral converges.

some people write $\int_0^\infty \frac{d^D k}{\pi^{D/2}} \frac{1}{k^2} = C \left[\frac{1}{\varepsilon_{UV}} - \frac{1}{\varepsilon_{IR}} \right]$

To solve the riddle, we need to supply more information, in particular we must specify what we expect from this "operation" of integrating in D dimensions —

* $\int d^D k\, f(\lambda k) = \lambda^{-D} \int d^D \tilde{k}\, f(\tilde{k}) \qquad \tilde{k} = \lambda k$

* $\int d^D k \left[f(k) + g(k) \right] = \int d^D k\, f(k) + \int d^D k\, g(k)$

usual properties that I am allowed to use when standard $\mathbb{R}^n$ integrals are CONVERGENT

they imply something interesting:

$$\int d^D k\, (k^2)^\alpha = 0 \qquad \forall\, \alpha \,!$$

and in general $\int d^D k\, f(k) = 0$ for $f(k)$ HOMOGENEOUS.

$f(\lambda k) = \lambda^2 f(k)$ homogeneous function

it means it does not depend on anything else $\Rightarrow$ integral said to be scaleless !

in fact

$$\int d^D k \; f(\lambda k) = \lambda^2 \int d^D k \; f(k)$$

but also $= \lambda^{-D} \int d^D k \; f(k)$

$\Rightarrow \lambda^2 = \lambda^{-D}$ or $\int d^D k \; f(k) = 0$!

not true in general

in particular $\int \frac{d^D k}{k^2} = 0$ massless TADPOLE

INTEGRATION BY PARTS

Important consequence of the fact that all integrals converge, one set of identities between integrals called integration by parts identities

IBP from standard analysis in $\mathbb{R}$

$$\int_a^b f(x)\,g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx$$

better rewritten as

$$\int_a^b \frac{d}{dx}\Big(\,h(x)\,\Big)\,dx = h(x)\Big|_a^b \qquad h(x) = f(x)\,g(x)$$

nothing but fundamental theorem of calculus.

Now if $a = -\infty$, $b = +\infty$ and assuming that

$$\int_{-\infty}^{+\infty} h(x)\,dx < \infty \quad \text{convergent, well defined}$$

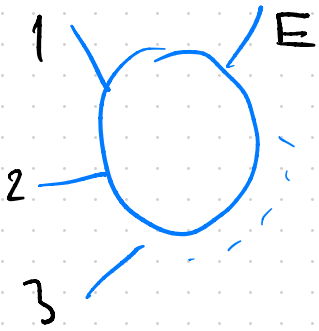
then IBP identity becomes

$$\int_{-\infty}^{+\infty} \frac{d}{dx}\big(h(x)\big)\,dx = 0 = h(+\infty) - h(-\infty)$$

necessary condition for $\int_{-\infty}^{+\infty} h(x)\,dx < 0$!

Now, in dimensional regularisation ALL loop ints converge [if they don't, they are zero ...] and are all integrated on whole $\mathbb{R}^D$ (Euclidean After Wick Rot)

Using this, we can write integration by parts identities for Feynman Integrals. At one-loop



$$= \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{D_1^{n_1} \cdots D_E^{n_E}}$$

$$\int \frac{d^D k}{i\pi^{D/2}} \left[\frac{\partial}{\partial k_\mu} v_\mu \frac{1}{D_1^{n_1} \cdots D_E^{n_E}} \right] = 0$$

HOW ARE THEY HELP FULL ?

Provide relations among Ints!

- Explicitly for 1 loop Tadpole



: more general Int of this type

$$I(n) = \int \frac{d^D k}{\pi^{D/2}} \frac{1}{(k^2+m^2)^n}$$

the only IBP I can write is

$$\int \frac{d^D k}{\pi^{D/2}} \left[\frac{\partial}{\partial k^\mu} k^\mu \frac{1}{(k^2+m^2)^n} \right] = 0$$

$$\Downarrow$$

$$\frac{\partial}{\partial k^\mu} k^\mu = D \quad ; \quad \frac{\partial}{\partial k^\mu} \frac{1}{(k^2+m^2)^n} = -\frac{n\, 2 k_\mu}{(k^2+m^2)^{n+1}}$$

so

$$= \frac{D}{(k^2+m^2)^n} - 2n \frac{k^\mu k_\mu}{(k^2+m^2)^{n+1}}$$

$$= \frac{D}{(k^2+m^2)^n} - 2n \frac{k^2}{(k^2+m^2)^{n+1}} = \frac{D}{(\;)^n} - \frac{2n}{(\;)^n} + \frac{2n m^2}{(\;)^{n+1}}$$

$$= \int \frac{D-2n}{(k^2+m^2)^n} + \frac{2n\,m^2}{(k^2+m^2)^{n+1}} = 0$$

$$\Rightarrow 2n\,m^2 \int \frac{d^D k}{\pi^{D/2}} \frac{1}{(k^2+m^2)^{n+1}} = (2n-D) \int \frac{d^D k}{\pi^{D/2}} \frac{1}{(k^2+m^2)^n}$$

$$I(n+1) = \frac{2n-D}{2n\,m^2} I(n)$$

which allows us to compute for free any Tadpole-like integral with $n > 1$, knowing only $I(1)$!

$$I(2) = \frac{2-D}{2m^2} I(1)$$

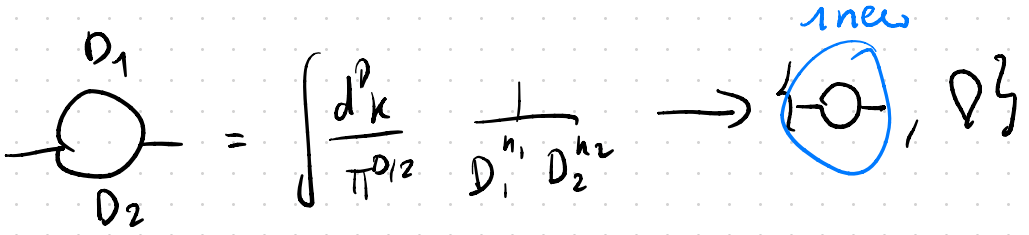
$$I(3) = \frac{(4-D)(2-D)}{8m^4} I(1) \quad \text{etc}$$

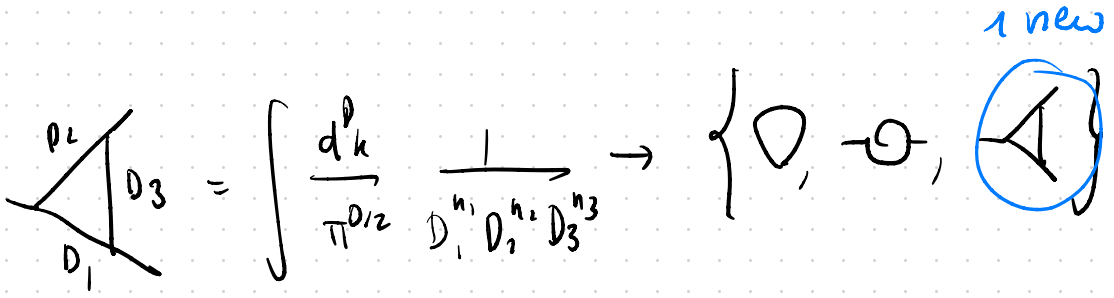
For this reason we say that the Tadpole "Family" of integrals has

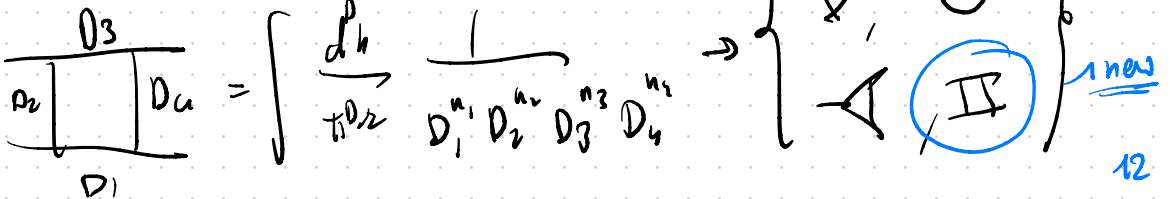
ONE MASTER INTEGRAL which we

can choose to be $I(1)$

SIMILARLY at 1 loop one finds

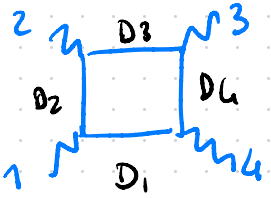






at 1 loop, every new graph with 1 more external leg, introduces always AT MOST one new master integral (not true at HIGHER LOOPS!)

Example at 1 loop: how to think about it?



4-photon (light-by-light) scattering in massive QED

get an "intuitive" picture of what integrals can appear in this calculation —

TOP TOPOLOGY, TOP GRAPH

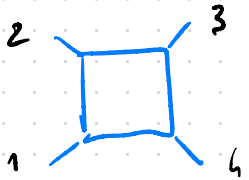
$$= \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{(k^2-m^2)\,((k-p_1)^2-m^2)\,((k-p_1-p_2)^2-m^2)\,((k-p_1-p_2-p_3)^2-m^2)}$$

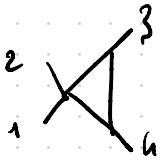
D_1 D_2 D_3 D_4

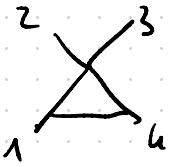
Real amplitude will contain complex numerator that originates from traces of fermion loop

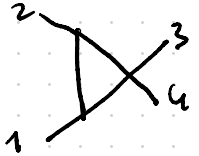
$$\int \frac{d^D k}{i\pi^{D/2}} \frac{N(k^2, l\cdot p_i)}{D_1 D_2 D_3 D_4} \qquad i = 1,2,3$$

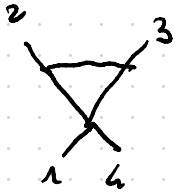
- Each scalar product can be written as linear comb of D_1, D_2, D_3, D_4 — I need to consider:



















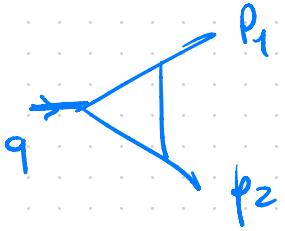
"FAMILY TREE"

"SUBTOPOLOGY TREE" of BOX

Again, in general, each of those graphs will require AT MOST 1 new master integral

(*) Note also that for special values of internal masses or external kinematics some graphs might become completely reducible !

YOU WILL PROVE IN AN EXERCISE THAT :



$p_1^2 = p_2^2 = 0 \qquad q^2 \neq 0$

massless propagators

is REDUCIBLE



COMPUTATION OF INTEGRALS :

We have seen only one example of Feynman integrals calculation

$$\Rightarrow \boxed{\text{○} = (m^2)^{\frac{D-2}{2}}\,\Gamma\left(1-\frac{D}{2}\right)}$$

this is a special example, it does not really depend on anything $\Rightarrow$ it's a number, up to overall dimensional scaling -

to have something more interesting, we need an object that depends at least on 2 dimensionful parameters, for example



$$= B(m^2, p^2) = \int \frac{d^D k}{\pi^{D/2}} \frac{1}{(k^2+m^2)\left((k-p)^2+m^2\right)}$$

From previous discussion

$$\int \frac{d^D k}{(i\pi^{D/2})} \frac{1}{(k^2+m^2)^\alpha ((k-p)^2+m^2)^\beta} \longrightarrow$$

$\alpha, \beta \in \mathbb{Z}$ all reduced to two master integral



Let's compute the Bubble. There are various methods, of increasing sofistication.

Standard one is FEYNMAN PARAMETERS

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[Ax + B(1-x)]^2} \Leftrightarrow \int_0^\infty dx \int_0^\infty dy \frac{\delta(1-x-y)}{(Ax+By)^2}$$

Textbooks

this is a so-called projective integral

PROJECTIVE INT : Int of some function homogeneous of degree zero including integration measure

$$\Delta = \{(x,y) \in \mathbb{R};\ x+y=1\}$$

$$\int_\Delta dx\,dy\ \frac{1}{(Ax+By)^2}$$

$$x \to \lambda \tilde{x}$$
$$y \to \lambda \tilde{y}$$

$$\int_\Delta \lambda^2\, d\tilde{x}\,d\tilde{y}\ \frac{1}{\lambda^2 (A\tilde{x}+B\tilde{y})^2} = \text{hom degree zero.}$$

Indeed this is integral over projective line

One can prove that any parametrization of the projective space is equivalent to solve this integral

CHENG-WU theorem (in particle physics)

Various parametrisations of projective line (x, y)

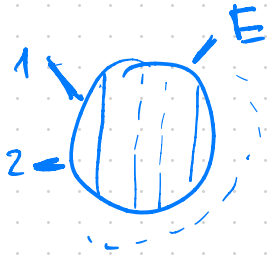
$\{ x = 1,\ y = \text{free} \}$ $\{ x = \text{free},\ y = 1 \}$ $\{ x + y = 1 \}$

etc ... all these parametrisations are equivalent, in fact

$$\int_0^1 dx \frac{1}{(Ax + B(1-x))^2} = \int_0^\infty dx \frac{1}{(Ax+B)^2} = \int_0^\infty dy \frac{1}{(A+By)^2}$$

which choice we make, might simplify substantially the integral, in particular when dealing with more propagators

GENERALISATION OF FEYN. PARAMETRISATION
(valid @ L-loops!)



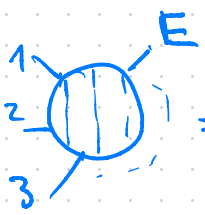
$$= \int \prod_{\ell=1}^{L} \frac{d^D k_\ell}{\pi^{D/2}} \frac{1}{D_1 D_2 \dots D_E}$$

$$\frac{1}{D_1^{a_1} \dots D_E^{a_E}} = \frac{\Gamma(a_1 + \dots + a_E)}{\Gamma(a_1) \dots \Gamma(a_E)}$$

$$\times \int_{\Delta} \prod_{j=1} dx_j \frac{x_1^{a_1-1} \dots x_E^{a_E-1}}{\left(\sum_k x_k D_k\right)^{\sum_{\ell=1}^{E} a_\ell}}$$

integral can be done putting any subset Σ of Feynman Parameters to 1 $\sum_{j \in \Sigma} a_j = 1$ and integrating the rest from zero to infinity

Substituting this back in Feynman Integral



$$= \frac{\Gamma(d_1 + \dots + d_E)}{\Gamma(d_1) \dots \Gamma(d_E)} \int_\Delta \prod_{j=1} dx_j \; x_1^{d_1 - 1} \dots x^{d_E - 1}$$

$$\times \int \prod_{e=1}^{L} \frac{d^D k_e}{\pi^{D/2}} \; \frac{1}{\left[x_1 D_1 + x_2 D_2 + \dots + x_E D_E \right]^{\sum_{n=1}^{E} d_k}}$$

this is a quadratic form in the loop momenta k_e

$$x = \{x_k\} \qquad p = \{p_k\} \qquad m = \{m_k\}$$

$$Q = k_i A_{ij}(x) k_j + 2 B_i(x, p) k_i + C(x, p, m)$$

i counts the loop, not the components!

put also $N = \sum_{k=1}^{E} d_k$

the integral can be performed by diagonalising this quadratic form — exercise —

$$= \Gamma\left(N - \frac{LD}{2}\right) \int_{\Delta} \prod_{j=1}^{N} \left[dx_j \frac{x_j^{d_j - 1}}{\Gamma(d_j)} \right] \frac{(\mathcal{U}_G)^{N-(L+1)\frac{D}{2}}}{(\mathcal{F}_G)^{N-L\frac{D}{2}}}$$

$\mathcal{U}_G, \mathcal{F}_G$ are called Symanzik Polynomial

or ~ GRAPH POLYNOMIALS

[Can be defined graph-theoretically, without having to go through explicit representation as loop integral!]

$$\mathcal{U}_G = \det A_{ij}$$

$$\mathcal{F}_G = \mathcal{U}_G \, C(x,p,m) - B_i(x,p) \, a_{ij} \, B_j(x,p)$$

with a_{ij} = minors A_{ij} $\Rightarrow$ $A^{-1}_{ij} = \frac{a_{ij}}{\det A_{ij}} = \frac{a_{ij}}{\mathcal{U}_G}$

N.B. Since $\mathcal{F}_G$, $\mathcal{U}_G$ can be uniquely defined from the "graph" associated to the integral, we can take the mathematician attitude and "DEFINE" a Feynman integral through it's Feyn. Parameter representation



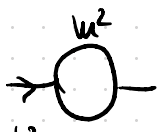
$$= \Gamma\left(N - \frac{LD}{2}\right) \int_{\Delta} \prod_{j=1}^{N} \left[dx_j \frac{x_j^{d_j - 1}}{\Gamma(d_j)} \right] \frac{(\mathcal{U}_G)^{N-(L+1)\frac{D}{2}}}{(\pm \mathcal{F}_G)^{N - L\frac{D}{2}}}$$

D is here just an exponent, it can be DEC

$\pm$ sign depends if EUCLIDEAN (+) or MINKOWSKI (−)
Vectors Vectors

⇒ pay attention to this !!!

LET'S GO BACK TO ONE-LOOP BUBBLE



$$= \int \frac{d^D k}{\pi^{D/2}} \frac{1}{(k^2+m^2)\,((k-p)^2+m^2)}$$

$$= \int_\Delta dx\,dy \int \frac{d^D k}{\pi^{D/2}} \frac{1}{\left[(k^2+m^2)x + ((k-p)^2+m^2)y\right]^2}$$

only 1 loop mom

$N = 2$

$$Q = k^2(x+y) - 2k\cdot p\, y + \underbrace{m^2(x+y) + p^2 y}_{\subset (x,\, p,\, m^2)}$$

$A_{11} = (x+y)$ $\quad B_1 = (p\,y)$

to $U_G = \det A_{11} = x+y$ simple

$\Theta_{ij} = A_{ij}^{-1} \cdot \det A_{ij} = \frac{1}{x+y}(x+y) = 1$ out

$$F_G = (x+y)\left(m^2(x+y) + p^2 y\right) - B_1 \cdot B_1 \cdot 1$$

$$= (x+y)\left[m^2(x+y) + p^2 y\right] - p^2 y^2$$

$$F_a = m^2(x+y)^2 + p^2 xy$$

and so fully integral becomes :

$$\text{(loop)} = \Gamma\left(2-\frac{D}{2}\right)\int_\Delta dx\,dy\ \frac{(x+y)^{2-D}}{\left[m^2(x+y)^2 + p^2 xy\right]^{2-\frac{D}{2}}}$$

$$= \Gamma\left(\frac{4-D}{2}\right)(m^2)^{\frac{D-4}{2}}\int_\Delta dx\,dy\ \frac{(x+y)^{2-D}}{\left((x+y)^2 + \frac{p^2}{m^2}xy\right)^{\frac{4-D}{2}}}$$

SKIP THIS

$D = 4-2\varepsilon$ and choose $x+y=1$ parametrization of projective line

$$= \Gamma(\varepsilon)(m^2)^{-\varepsilon}\int_0^1 dx\ \left(1+\frac{p^2}{m^2}x(1-x)\right)^{-\varepsilon}$$

$$= \frac{\Gamma(1+\varepsilon)(m^2)^{-\varepsilon}}{\varepsilon}\int_0^1 dx\ \left(1+\frac{p^2}{m^2}x(1-x)\right)^{-\varepsilon}$$

$$= (m^2)^{-\varepsilon}\,\frac{\Gamma(1+\varepsilon)}{\varepsilon}\left\{1 - \varepsilon\int_0^1 \ln\left(1+\frac{p^2}{m^2}x(1-x)\right)dx + O(\varepsilon^2)\right\}$$

$$= \Gamma(1+\varepsilon)\,(m^2)^{-\varepsilon}\left\{\frac{1}{\varepsilon} - \int_0^1 \ln\left(1+\frac{p^2}{m^2}x(1-x)\right)dx + O(\varepsilon)\right\}$$

pole

Finite piece of bubble

to do the integral, use

$$\left[1+\frac{p^2}{m^2}x(1-x) = 0 \Rightarrow -\frac{p^2}{m^2}\left[x^2 - x - \frac{m^2}{p^2}\right] = 0\right.$$

$$\left. x_{1/2} = \frac{1 \pm \sqrt{1+\frac{4m^2}{p^2}}}{2} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+\frac{4m^2}{p^2}}\right]$$

define

$$p^2 = m^2\,\frac{(1-\xi)^2}{\xi}$$

$$0 < \xi < 1 \qquad p^2 > 0$$

$$\left\{\begin{array}{l} x_1 = \dfrac{1}{1-\xi} > 0\ ! \\ x_2 = -\dfrac{\xi}{1-\xi} < 0\ ! \end{array}\right.$$

So integral becomes

$$\int_0^1 dx \ln\left(1 + \frac{p^2}{m^2} x(1-x)\right) = -2 - \frac{1+\xi}{1-\xi} \ln(\xi)$$

so we finally get

$$\text{—O—} = (m^2)^{-\varepsilon} \Gamma(1+\varepsilon) \left\{ \frac{1}{\varepsilon} + 2 + \frac{1+\xi}{1-\xi} \ln(\xi) + O(\varepsilon) \right\}$$

DON'T EXPAND THIS ! (Seriously!)

the 1-loop bubble close to $D = 2$ dimensions $\equiv$ FINITE

$$= \Gamma\left(2 - \frac{D}{2}\right) \int_\Delta dx\,dy \; \frac{(x+y)^{2-D}}{\left[m^2(x+y)^2 + p^2 xy\right]^{2-\frac{D}{2}}}$$

$$= \Gamma\left(\frac{4-D}{2}\right)(m^2)^{\frac{D-4}{2}} \int_\Delta dx\,dy \; \frac{(x+y)^{2-D}}{\left((x+y)^2 + \frac{p^2}{m^2} xy\right)^{\frac{4-D}{2}}}$$

$D = 2 - 2\varepsilon$ and $x + y = 1$

$$= \Gamma(1+\varepsilon)(m^2)^{-1-\varepsilon} \int_0^1 dx \; \left[1 + \frac{p^2}{m^2} x(1-x)\right]^{-1-\varepsilon}$$

$$\cong (m^2)^{-\varepsilon} \Gamma(1+\varepsilon) \left[\frac{1}{m^2} \int_0^1 dx \; \frac{1}{\left[1 + \frac{p^2}{m^2} x(1-x)\right]} + O(\varepsilon)\right]$$ no poles!

$$= \frac{1}{m^2} \int_0^1 dx \; \frac{1}{\frac{p^2}{m^2}(x-x_2)(x_1-x)}$$

with $x_1 = \frac{1}{1-\xi}$, $x_2 = -\frac{\xi}{1-\xi}$

$$= \frac{1}{m^2} \frac{\xi}{(1-\xi)^2} \int_0^1 dx \left(\frac{1}{x-x_2} + \frac{1}{x_1-x} \right) \frac{1}{x_1-x_2} \qquad x_1 - x_2 = \frac{1+\xi}{1-\xi}$$

$$= \frac{1}{m^2} \frac{\xi}{(1-\xi)^2} \frac{1-\xi}{1+\xi} \int_0^1 dx \left(\frac{1}{x-x_2} - \frac{1}{x-x_1} \right)$$

$$= \frac{1}{m^2} \left(\frac{\xi}{1-\xi^2} \right) \left[\ln(x-x_2) \Big|_0^1 - \ln(x_1-x) \Big|_0^1 \right]$$

$$= \frac{1}{m^2} \frac{\xi}{1-\xi^2} \left[2\ln\left(\frac{1}{1-\xi}\right) - 2\ln\left(\frac{\xi}{1-\xi}\right) - \ln\left(\frac{\xi}{1-\xi}\right) + \ln\left(\frac{1}{1-\xi}\right) \right]$$

$$= \frac{2}{m^2}\,\frac{\xi}{1-\xi^2}\,\ln\left(\frac{1}{\xi}\right) = -\frac{2}{m^2}\,\frac{\xi}{1-\xi^2}\,\ln(\xi)$$



$$= (m^2)^{-\varepsilon}\,\Gamma(1+\varepsilon)\left[\underbrace{-\frac{2}{m^2}\,\frac{\xi}{1-\xi^2}\,\ln\xi}_{\text{same function as DN4}} + O(\varepsilon)\right]$$

manifestation of relation $D \to D \pm 2$ dimensions!

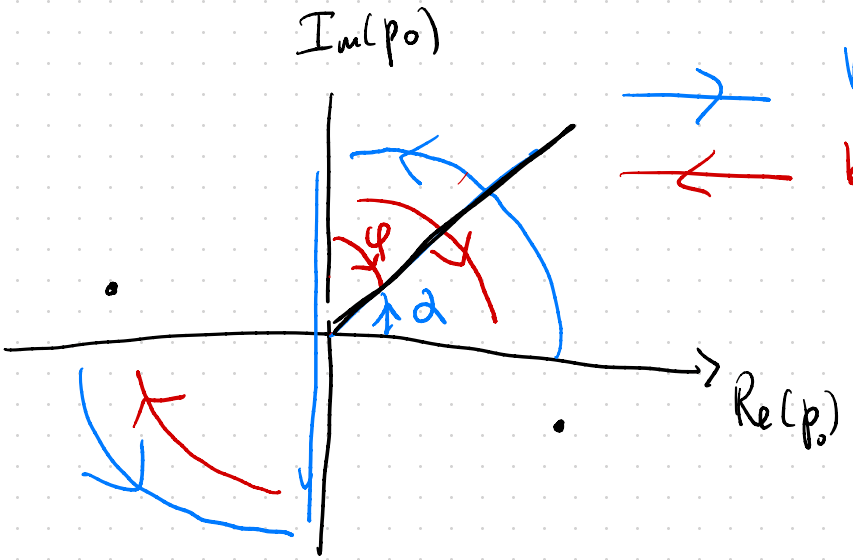
- note that integral is real $(0 < \xi < 1)$ as long as $p^2 > 0$ (EUCLIDEAN!)

In order to get physical result in Minkowski kinematics we need to Wick Rotate back!

$$p^2 \longrightarrow -(p_M^2 + i\varepsilon)$$

Minkowski momentum

comes from Feyn. Prescription, or, how we avoid the poles!



Wick Rotation

backward rotation

$p_4^2 = p_0^2 - \vec{p}^2$ $\quad p_0$ energy component

rotation is, literaly $p_0 \to e^{i\alpha} p_0 \quad | \; 0 < \alpha < \frac{\pi}{2}$

and $p_4 = e^{i\frac{\pi}{2}} p_0 = i p_0$

so that $p_0 = e^{-i\varphi} p_4 \qquad 0 < \varphi < \frac{\pi}{2}$

$= \cos\varphi \, p_4 - i \sin\varphi \, p_4$

imaginary part

20

$$p^2 = p_4^2 + \vec{p}^{\,2} = \cos^2\varphi\, p_4^2 - \sin^2\varphi\, p_4^2 - 2i\sin\varphi\cos\varphi\, p_4 + \vec{p}^{\,2}$$

if $\varphi = \frac{\pi}{2} - \varepsilon \Rightarrow$ $p^2 = -p_4^2 + \vec{p}^{\,2} - 2i\sin\left(\frac{\pi}{2} - \varepsilon\right) p_4$

$$= -\left(p_4^2 - \vec{p}^{\,2}\right) - i\tilde{\varepsilon}$$

$$= -\left(p_M^2 + i\tilde{\varepsilon}\right)$$

From this the following is (almost) always true:

[every invariant that changes sign going from EUCLIDE → MINKOWSKI takes a $+i\varepsilon$ to be properly continued !]

What happens to our 1-loop Bubble ?

$$= (m^2)^{-\varepsilon}\,\Gamma(1+\varepsilon)\left\{\frac{1}{\varepsilon} + 2 + \frac{1+\xi}{1-\xi}\ln(\xi) + O(\varepsilon)\right\}$$

in $D = 4$



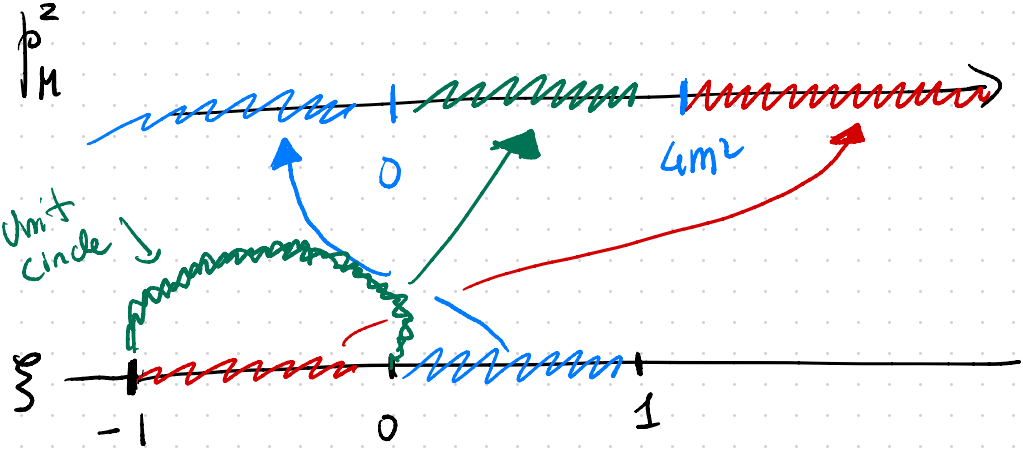
$$= -\frac{2}{m^2}\,\frac{\xi}{1-\xi^2}\ln(\xi) \quad \text{in} \quad D = 2$$

$$p_E^2 = m^2\,\frac{(1-\xi)^2}{\xi} \qquad \text{with} \quad 0 < \xi < 1 \qquad p^2 > 0$$

how we want to continue this for $p^2 < 0$

$\Rightarrow$ $p_M^2 > 0$ positive minkowski momentum

only "non trivial" continuation is for $\log(\xi)$!



~~~ = euclidean region $0 < \xi < 1$

~~~ = physical region $p^2 > 4m^2$ $-1 < \xi < 0$

~~~ = $0 < p^2 < 4m^2$ needs $\xi = e^{i\varphi}$

$$p^2 = m^2 \frac{(1-\xi)^2}{\xi} = m^2 \frac{(1-e^{i\varphi})^2}{e^{i\varphi}} = m^2 \left[e^{-\frac{i\varphi}{2}} - e^{\frac{i\varphi}{2}} \right]^2$$

$$= m^2 \left[-2i \sin\frac{\varphi}{2} \right]^2 = -4m^2 \sin^2\frac{\varphi}{2}$$
~~~

Let's see what happens to the bubble

- $0 < p_H^2 < 4m^2 \Rightarrow \xi = e^{i\varphi} \qquad 0 < \varphi < \pi$

$$= (m^2)^{-\varepsilon}\,\Gamma(1+\varepsilon)\left\{\frac{1}{\varepsilon} + 2 + \frac{1+\xi}{1-\xi}\ln(\xi) + O(\varepsilon)\right\}$$

$$\to (m^2)^{-\varepsilon}\Gamma(1+\varepsilon)\left\{\frac{1}{\varepsilon} + 2 + \frac{1+e^{i\varphi}}{1-e^{i\varphi}}\ln(e^{i\varphi}) + O(\varepsilon)\right\}$$

$$= (m^2)^{-\varepsilon}\Gamma(1+\varepsilon)\left\{\frac{1}{\varepsilon} + 2 - \varphi\tan\left(\frac{\varphi}{2}\right) + O(\varepsilon)\right\}$$

REAL



$$= (m^2)^{-\varepsilon}\Gamma(1+\varepsilon)\left[-\frac{2}{m^2}\,\frac{\xi}{1-\xi^2}\ln\xi + O(\varepsilon)\right]$$

$$= (m^2)^{-\varepsilon}\Gamma(1+\varepsilon)\left[-\frac{2}{m^2}\,\frac{e^{i\varphi}}{1-e^{2i\varphi}}\,i\varphi + O(\varepsilon)\right]$$

$$= (m^2)^{-\varepsilon}\Gamma(1+\varepsilon)\left[\frac{\varphi}{m^2}\,\frac{1}{\sin\varphi} + O(\varepsilon)\right] \quad \text{REAL}$$

. Analytic continuation above threshold

$p_M^2 > 4m^2$ $\qquad p^2 = m^2 \dfrac{(1-\xi)^2}{\xi}$ $\qquad \xi < 0$

↑ euclidean

to go there, we should send $\xi \to -x \pm i\delta$

such that $(p_M^2 + i\varepsilon)$: right choice is

$\xi \to -(x - i\delta)$ in fact

$$p^2 = m^2 \frac{(1-\xi)^2}{\xi} \to -m^2 \frac{(1+x-i\delta)^2}{x-i\delta} =$$

$$\simeq -\left[m^2 \frac{(1+x)^2}{x} + \underbrace{i\delta\, m^2 \frac{(1-x^2)}{x^2}}_{i\varepsilon \text{ a small number}} \right]$$

so if I send $\xi \to -x - i\delta$ it means

$$p^2 \to -(p_M^2 + i\varepsilon) = -\left[\underbrace{m^2 \frac{(1+x)^2}{x}}_{p_M^2} + i\varepsilon \right]$$

And the integral becomes :

$$\text{D=4}\quad \bigcirc = (m^2)^{-\varepsilon}\,\Gamma(1+\varepsilon)\left\{\frac{1}{\varepsilon} + 2 + \frac{1+\xi}{1-\xi}\ln(\xi) + O(\varepsilon)\right\}$$

$$= (m^2)^{-\varepsilon}\,\Gamma(1+\varepsilon)\left\{\frac{1}{\varepsilon} + 2 + \frac{1-X}{1+X}\ln(-X+i\varepsilon) + O(\varepsilon)\right\}$$

$$= (m^2)^{-\varepsilon}\,\Gamma(1+\varepsilon)\left\{\frac{1}{\varepsilon} + 2 + \frac{1-X}{1+X}\ln X + i\pi\left(\frac{1-X}{1+X}\right) + O(\varepsilon)\right\}$$

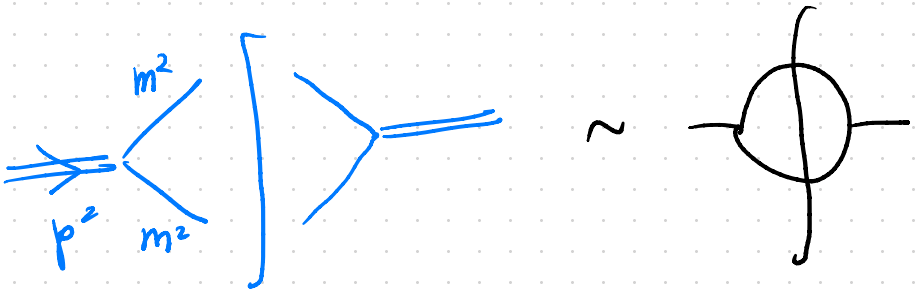
the Bubble has developed an imaginary part !

Similarly for D=2

$$\bigcirc^{\text{D=2}} = (m^2)^{-\varepsilon}\,\Gamma(1+\varepsilon)\left[-\frac{2}{m^2}\,\frac{\xi}{1-\xi^2}\ln\xi + O(\varepsilon)\right]$$

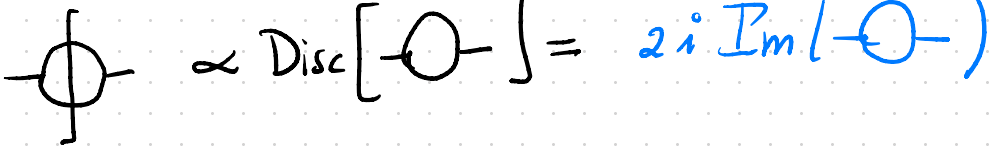
$$= (m^2)^{-\varepsilon}\,\Gamma(1+\varepsilon)\left[\frac{2}{m^2}\,\frac{X}{1-X^2}\ln(X) + \frac{2}{m^2}\left(\frac{X}{1-X^2}\right) i\pi + O(\varepsilon)\right]$$

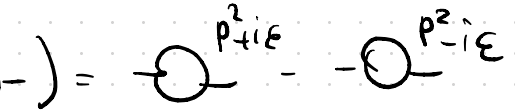
the existence of an imaginary part when $p_n^2 > 4m^2$ is a fundamental consequence of UNITARITY and the optical theorem !



the Bubble integral develops an imaginary part (a "discontinuity") in p^2 when $p^2 > 4m^2$, ie when there is enough energy to produce two particles not ONLY as VIRTUAL particles

in fact, we write



Since Disc  =

$$\mathrm{Im}\left(\text{-O-}\right)^{D\sim 4} = +\pi\,(m^2)^{-\varepsilon}\,\Gamma(1+\varepsilon)\left(\frac{1-x}{1+x}\right) + O(\varepsilon)$$

$$\boxed{\mathrm{Im}\left(\text{-O-}\right)^{D\sim 2} = +\frac{2}{m^2}\pi(m^2)^{-\varepsilon}\Gamma(1+\varepsilon)\,\frac{x}{1-x^2} + O(\varepsilon)}$$

important in what follows...