# Introduction & Notation

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SS 2022
TUM

Lectures on thursdays 08:15 – 09:45
or 08:30 – 10:00 ] PH 3344

Exercises Fridays 14:15 – 15:45 C.3202

Tomorrow Lecture to make up for
last week.

- Tutors Max Delto
  Niklas Syrrakos

- No BONUS BUT Exercises are important
  for final exam, questions can cover
  also that material.

BOOKS: First part mainly from // then review PAPERS!
J. Henn, J. C. Plefka, Scattering Amplitudes
in gauge theories

## WHAT WILL WE DO :

- One loop Amplitudes — { integrand reduction; Unitarity }

- Multiloop ampl. — { methods for tensor decomposition and reduction to master integrals }

- Integrals $\Longrightarrow$ Analytic structure; Differential Equations; Special Functions (Polylogs)

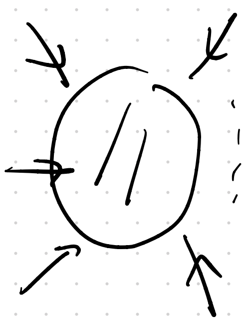
- Bonus / Extra
  - Riemann surfaces and Integrals; Elliptic functions, modular forms
  - Intersection theory (generalization of 1 loop Unitarity!)

# INTRODUCTION & NOTATION

In this course we'll deal with one- and multi-loop scattering amplitudes.

In the first part, we will be mainly concerned with ONE-LOOP Amplitudes and we will derive general results and methods that make it possible to simplify their calculation

## NOTATION



All incoming (or outgoing) kinematics

$p_i^\mu$ momenta external particles

$$\sum_{i=1}^{N} p_i = 0$$ momentum conservation

Minkowski metric $\eta_{\mu\nu} = g_{\mu\nu} = \mathrm{diag}(+ - - -)$

We use $\ell^\mu$ for loop momentum (sometimes $k^\mu$)

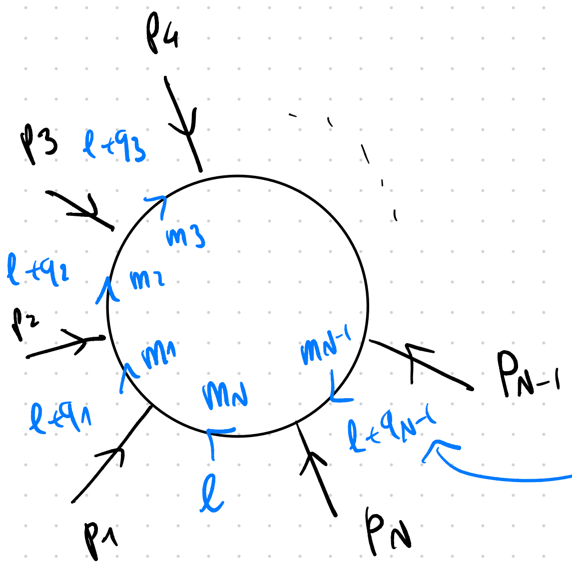
We use dimensional regularisation $\Rightarrow$ $\dfrac{d^D\ell}{i\pi^{D/2}}$ ints.

or $\dfrac{d^D\ell}{(2\pi)^D}$ ampl.

IMPORTANT

$$\int \frac{d^D\ell}{i\pi^{D/2}} \frac{1}{\ell^2 - m^2 + i\varepsilon} = -\int \frac{d^D\ell_E}{\pi^{D/2}} \frac{1}{\ell_E^2 + m^2}$$

Wick rotation to switch to euclidean variables.




there are $N$ different masses and $N-1$ different REGION MOMENTA

$$q_i^\mu = \sum_{k=1}^{i} p_k^\mu$$

Notice that

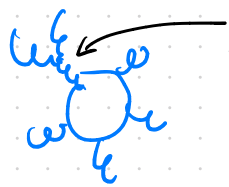
$$p_i^\mu = q_i^\mu - q_{i-1}^\mu \qquad \text{where} \qquad q_0^\mu = 0$$

Also, momentum conservation $\sum_{i=1}^{N} p_i^\mu = 0$ becomes

$\Longrightarrow$ $q_N^\mu = 0$ last region momentum $= q_0^\mu$

we allow for now external momenta to be off-shell

$p_i^2 = M_i^2 \neq 0$ in general.



off-shell

Feynman Diagrams generate then expressions of this form

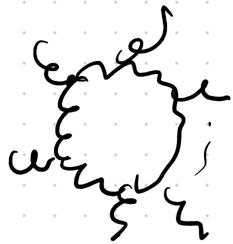
$$I = \int \frac{d^D l}{(2\pi)^D} \, \frac{N(l^\mu, p_j^\mu, \varepsilon_j^\mu)}{\underbrace{((l+q_1)^2 - m_1^2 + i\varepsilon)}_{D_1} \cdots \underbrace{((l+q_{N-1})^2 - m_{N-1}^2 + i\varepsilon)}_{D_{N-1}} \underbrace{(l^2 - m_N^2 + i\varepsilon)}_{D_N}}$$

"physical" integration measure $\neq \frac{1}{i\pi^{D/2}}$

alternative compact notation

$N(\ell^\mu)$ is a polynomial in loop momentum
external momenta
AND all remaining external data (polarisations!)

- For pure Yang Mills diagrams



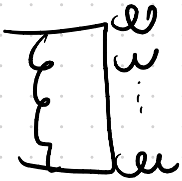
etc

- Fermion loops as well $\Rightarrow \dfrac{i}{\not\ell - m} \Rightarrow \dfrac{\not\ell + m}{\ell^2 - m}$ $\boxed{\text{traced}}$

- External Fermion lines give similar structures + $\gamma^\mu$ matrices


$$= \bar{u}(p_2)\left[\frac{\mathcal{N}(\ell^\mu, \varepsilon_j^\mu, p_j^\mu)}{D_1 D_2 D_3 \cdots D_N}\right] u(p_1)$$

contains also $\gamma^\mu$



$$= \bar{u}(p_2)\,\Gamma^{\mu_1\ldots\mu_R}\,u(p_1)\int\frac{d^D\ell}{(2\pi)^D}\,\frac{\mathcal{N}(\ell^\mu, p_j^\mu, \varepsilon_j^\mu)}{D_1\ldots D_N}$$

$\Gamma^{\mu_1\ldots\mu_R}$: all $\gamma^\mu$ matrices

**QUESTION** can we put a upper limit on powers of loop momentum at numerator?

$\Rightarrow$ Yes! in Renormalizable Field theories, as YM, it turns out that at **one loop** we can have **at most** $\mathcal{N}(\ell^\mu) \sim [\ell]^N$

$N$ = # external legs

numerator momenta come from
- 3-gluon vertex
- fermion props
- ghost vertices

$$I_N^{(r)} = \int \frac{d^D\ell}{(2\pi)^D} \frac{\prod_{j=1}^{r} (u_j \cdot \ell)}{\left((\ell+q_1)^2 - m_1^2\right) \dots \left(\ell^2 - m_N^2\right)}$$

N-point

we call this a TENSOR INTEGRAL of rank $r$

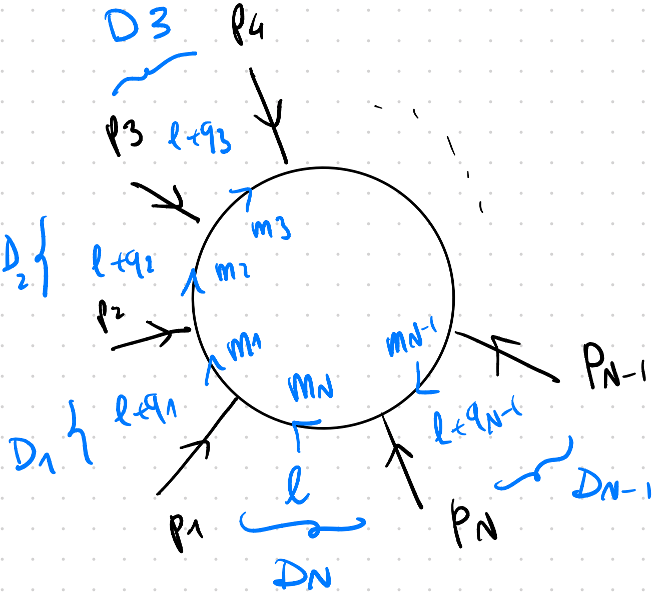
power counting shows that $\boxed{r \leq N}$

if $r = 0$ then we talk about SCALAR INTEGRAL

reason "tensor" → because $u_j$ = could be anything ($\gamma^\mu, \varepsilon^\mu, \dots$)

$$I_N^{(r)} = \prod_{j=1}^{r} u_j^{\mu_j} \underbrace{\int \frac{d^D\ell}{(2\pi)^D} \frac{\ell_{\mu_1} \dots \ell_{\mu_r}}{D_1 D_2 \dots D_N}}_{\text{this is a tensor of rank } r}$$

Finally notice that given general graph




$$l\cdot q_i = \frac{1}{2}\left[\left((l+q_i)^2 - m_i^2\right) - \left(l^2 - m_N^2\right) - \left(q_i^2 - m_i^2\right) - m_N^2\right]$$

$$= \frac{1}{2}\left[D_i - D_N - q_i^2 + m_i^2 - m_N^2\right]$$

scalar prods between loop mom and region momenta are, of course, linear comb of PROPAGATORS

# COMMENTS ON DIVERGENCES

UV divergences come from regions $l^\mu \to \infty$

Power counting is straightforward way to check if an integral diverges or not —

$$I_N^{(r)} \sim \int \frac{d^D l}{(2\pi)^D} \frac{(l)^r}{[l^2]^N}$$ in UV limit

so we have total power from

$$\sim \int d\Omega_D \int_\Lambda^\infty dl \left\{ \frac{l^{D-1} \cdot l^r}{(l^2)^N} = l^{r+D-1-2N} \right\}$$

if $r + D - 1 - 2N \geq -1 \Rightarrow r \geq 2N - D$ UV divergence

In our case $\underline{\underline{D=4}}$ $\quad r \geq 2N-4$

since $r \leq N$ for a given number of "points"

$$\Rightarrow N \geq 2N - 4 \Rightarrow N \leq 4$$

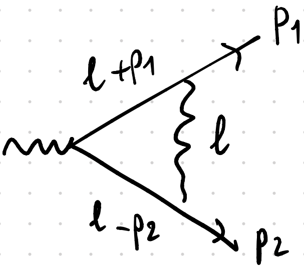
So only integrals up to 4 points (boxes) can generate UV divergences in a RENORMALISABLE QFT like QCD, $N=4$ SYM, SM etc...

IR divergences are trickier in general

- SOFT ones $\ell^\mu \to 0$
- COLLINEAR ones $\ell^\mu \parallel p^\mu \quad p^2 = 0$

already at one loop, they can overlap generating up to $\frac{1}{(D-4)^2} \sim \frac{1}{\varepsilon^2}$ poles

$D = 4 - 2\varepsilon \Leftarrow$ usual def in DIM REG




# QED form factor

$p_1^2 = m^2$ on-shell electrons!

$$\int \frac{d^D\ell}{(2\pi)^D} \frac{1}{\ell^2\left((\ell-p_2)^2-m^2\right)\left((\ell+p_1)^2-m^2\right)}$$

$$= \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{\ell^2\left(\ell^2-2\ell\cdot p_2\right)\left(\ell^2+2\ell\cdot p_1\right)}$$

- if $\ell^\mu \to 0$ then $\sim \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{\ell^2 \cdot \ell \cdot \ell} \sim \int_0^\Lambda \frac{d\ell}{\ell}$

log divergence

Soft singularity @ 1 loop gives $\frac{1}{\varepsilon}$ pole

IMAGINE $p_1^2 = 0$ (no mass, $m=0$ too)

- if $l \parallel p_1$ then $l \cdot p_1 \sim c\, p_1^2 \sim 0$ and we get

$l^\mu = c\, p_1^\mu$ $\qquad l^2 \sim c\, p_1^2 \sim 0$ as well

this means

$$\int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2 (l-p_2)^2 (l+p_1)^2} \sim \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2 (l^2 - 2p_2 \cdot l)(l^2 + 2p_1 \cdot l)}$$

$$\approx \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2 (l^2 - 2p_2 \cdot l)(l^2)}$$

bcause

$l^2 \sim p_1^2 \sim 0$

$l \cdot p_1 \sim p_1^2 \sim 0$

Collinear + soft if $l^\mu \to 0$ at the same time, gives a $\frac{1}{\varepsilon^2}$