Photoelectric Effect

Inelastic process: $\gamma + \text{atom} \rightarrow (\text{atom})^+ + e^-$

$$\Gamma = \frac{2\pi}{\hbar}\int |M_{fi}|^2 \cdot \frac{V\,d^3p}{(2\pi\hbar)^3}\,\delta(\underbrace{\hbar\omega}_{\text{photon energy}} - \underbrace{E_B}_{\text{binding energy}} - \underbrace{\frac{p_e^2}{2m}}_{\text{energy electron}}) = [\text{solve } \delta]$$

we are assuming that the atom does not go into an excited state.

$$= \frac{2\pi V}{\hbar}\int d\Omega\,\frac{m p_e}{(2\pi\hbar)^3}|M_{fi}|^2\Big|_{p_e = \tilde{p}_e} \qquad \text{where } \hbar\omega - E_B - \frac{\tilde{p}_e^2}{2m} = 0.$$

$$M_{fi} = \frac{e}{m}\sqrt{\frac{\hbar}{2\varepsilon_0\omega V}}\int d^3\vec{r}\,\psi_f^*(\vec{x})\,\vec{\varepsilon}\cdot\vec{p}\,e^{i\vec{k}\cdot\vec{r}}\,\psi_i(\vec{r})$$

The initial state w.f. is $\psi_i(\vec{r}) = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}$

We assume the energy of the final state electron is much larger than $-13{,}6\,Z^2\,eV \Rightarrow$ free solution ignoring Coulomb.

$$\psi_f(\vec{x}) = \frac{1}{\sqrt{V}}\,e^{i\vec{p}_e\cdot\vec{r}/\hbar}$$

The final state is an eigenstate of momentum $\rightarrow$

$$\rightarrow \langle f|\vec{\varepsilon}\cdot\vec{p}\,e^{i\vec{k}\cdot\vec{r}}|i\rangle = \vec{\varepsilon}\cdot\vec{p}_e\langle f|e^{i\vec{k}\cdot\vec{r}}|i\rangle \Rightarrow$$

$$|M_{fi}|^2 = \left(\frac{e}{m}\right)^2 \frac{\hbar}{2\varepsilon_0 \omega V} \frac{1}{V} \frac{1}{\pi}\left(\frac{Z}{a_0}\right)^3 (\bar{\varepsilon}\cdot\bar{p}_e)^2 \cdot \left| \int d^3r \, e^{i(\bar{k}-\bar{p}_e/\hbar)\cdot\bar{r}} \cdot e^{-Zr/a_0} \right|^2$$

We are interested in the cross section.

$\Rightarrow$ We need to multiply by $\frac{1}{J}$ J flux $1/J = \frac{V}{c}$

$$\frac{d\sigma}{d\Omega} = \frac{1}{j} M = \frac{\cancel{V}}{c} \frac{2\pi}{\hbar} \frac{\cancel{V} m p_e}{(2\pi\hbar)^3} \left(\frac{e}{m}\right)^2 \frac{\hbar}{2\varepsilon_0 \omega \cancel{V}} \frac{1}{\cancel{V}} \frac{1}{\pi}\left(\frac{Z}{a_0}\right)^3 (\bar{\varepsilon}\cdot\bar{p}_e)^2 \cdot$$

$$\cdot \underbrace{\left| \int d^3r \, e^{i(\vec{k}-\bar{p}_e)\cdot\bar{r}} e^{-Zr/a_0} \right|^2}_{=\ast}$$

$\vec{k}$ PHOTON MOMENTUM$/\hbar$ $\quad$ $\vec{p}_e$ EMITTED ELECTRON MOMENTUM

ω PHOTON ENERGY$/\hbar$ $\quad$ (remember, it is evaluated in $\tilde{p}_e$)

$\omega_{\vec{k}} = |\vec{k}|c$

– Fourier transform of Hydrogen-like w.f. (see other exercise)

$$\ast = \frac{8\pi (Z/a_0)}{\left[(\vec{k}-\vec{p}_e/\hbar)^2 + \left(\frac{Z}{a_0}\right)^2\right]^2}$$

Putting everything together:

$$\frac{d\sigma_\lambda}{d\Omega} = 32 Z^5 a_0^2 \left(\frac{p_e c}{\hbar\omega}\right)\left(\frac{\vec{\varepsilon}_\lambda\cdot\bar{p}_e}{mc}\right)^2 \frac{1}{\left(Z^2 + a_0^2\left(\frac{\bar{p}_\gamma-\bar{p}_e}{\hbar}\right)^2\right)^4} \qquad \vec{p}_\gamma = \hbar\vec{k}$$

Take $\hbar\omega \gg E_B$ $\rightsquigarrow$ $\tilde{p}_e \gg E_B$ $\Rightarrow$ E_B negligible

$$\hbar\omega - E_B - \frac{\vec{p}_e^{\,2}}{2m} \approx \hbar\omega - \frac{\vec{p}_e^{\,2}}{2m} = 0 \Rightarrow \hbar\omega \approx \frac{p_e^2}{2m}$$

$$\frac{1}{\hbar^2}(\vec{p}_\gamma - \vec{p}_e)^2 = \frac{1}{\hbar^2}\left[\left(\frac{\hbar\omega}{c}\right)^2 - 2\frac{\hbar\omega}{c} p_e \cos\theta + p_e^2\right]$$

$$\approx \frac{1}{\hbar^2}\left[\left(\frac{p_e^2}{2mc}\right)^2 - \cancel{2}\frac{p_e^3}{\cancel{2}mc}\cos\theta + p_e^2\right] \approx$$

non relativistic electron $\quad p_e \ll mc \Rightarrow$ neglect first term

$$\approx \frac{1}{\hbar^2}\left[p_e^2 - \frac{p_e^3}{mc}\cos\theta\right] \qquad \theta \text{ angle between } \vec{p}_e \text{ and } \vec{k}.$$

$$= \frac{p_e^2}{\hbar^2}\left[1 - \frac{v_e}{c}\cos\theta\right]$$

Take photon momentum along $\hat{z}$ axis:

$\Rightarrow \vec{\varepsilon}_{(1)} \parallel \hat{x} \quad \vec{\varepsilon}_{(2)} \parallel \hat{y} \qquad \vec{p}_e = p_e(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

$(\vec{p}_e \cdot \vec{\varepsilon}_{(1)})^2 = p_e^2 \operatorname{sen}^2\theta \cos^2\theta \qquad (\vec{p}_e \cdot \vec{\varepsilon}_{(2)})^2 = p_e^2 \sin^2\theta \sin^2\varphi$

So that we can average $\frac{d\sigma_\lambda}{d\Omega}$, $\left(\frac{d\sigma_1}{d\Omega} + \frac{d\sigma_2}{d\Omega}\right)\frac{1}{2}$

$$\sum_\lambda (\varepsilon_\lambda \cdot \vec{p}_e)^2 \frac{1}{2} = p_e^2 \operatorname{sen}^2\theta \frac{1}{2}$$

Everything together

$$\frac{d\sigma}{d\Omega} = 2\sqrt{2}\, Z^4 \alpha^8 \left(\frac{a_0}{Z}\right)^2 \left(\frac{E_e}{m_e c^2}\right)^{-7/2} \frac{\sin^2\theta}{\left(1 - \frac{v_e}{c}\cos\theta\right)^4}$$

where remember that $e^2 = 4\pi\varepsilon_0 \hbar c \alpha$

and $a_0 = \dfrac{\hbar}{m_e c \alpha}$

Riemann - Lebesgue Lemma

We want to prove that $\hat{f}(k) = \int_{\mathbb{R}} dx\, f(x)\, e^{-ikx}$ vanishes for $k \to \pm\infty$

$x \to x + \frac{\pi}{k} \implies \hat{f}(k) = \int_{\mathbb{R}} dx\, f(x)\, e^{-ikx} = \int_{\mathbb{R}} dx\, f(x + \frac{\pi}{k})\, e^{-ikx}\, e^{-i\pi}$

$= -\int_{\mathbb{R}} dx\, f(x + \frac{\pi}{k})\, e^{-ikx}$ → this minus sign is the crucial point

$$\hat{f}(k) = \frac{1}{2} \int_{\mathbb{R}} dx \left(f(x) - f(x + \frac{\pi}{k}) \right) e^{-ikx}$$

$$|\hat{f}(k)| \leq \frac{1}{2} \int_{\mathbb{R}} dx\, \underbrace{\left| f(x) - f(x + \frac{\pi}{k}) \right|}$$

↳ positive integrable functions $\xrightarrow[k \to \infty]{} 0 \quad \forall x$

$\implies$ I can find a integrable function that bounds it.
(at least from some k on)

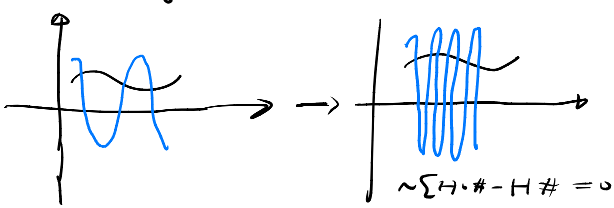
$\implies$ dominated convergence theorem $\longrightarrow \hat{f}(k) \xrightarrow[k \to \infty]{} \int 0 = 0$.

(for $k \to -\infty$ nothing changes).

- For $\int_a^b dx\, f(x)\, e^{ikx} \implies f(x) \to f(x)\, \Theta(x-a)\, \Theta(x+b)$ and integrate from $-\infty, +\infty$.

- The point is that we are integrating with sin a cos

$$\int dx\, f(x) \sin(kx)$$



– That is why the shift worked. We went picking up the contribution from the region of the sin/cos with the opposite sign.

examples:

1) $$\int_a^b dx\, V_0 e^{ikx} = \frac{V_0}{ik}\left(e^{ikb} - e^{-ika}\right) \sim \frac{1}{k}$$

2) $$\int_0^b dx\, e^{-x} e^{ikx} = \frac{1-e^{ib(i+k)}}{1-ik} = \frac{1-e^{ik} e^{-b}}{1-ik} \sim \frac{1}{k}$$

For $b \to \infty \longrightarrow \dfrac{1}{1-ik} \sim \dfrac{1}{k}$

3) $$\int_{-\infty}^{+\infty} dx\, e^{-|x|} e^{ikx} = \frac{1}{1-ik} + \frac{1}{1+ik} = \frac{2}{1+k^2} \sim \frac{1}{k^2}$$

4) $$\int_{-\infty}^{+\infty} dx\, e^{-x^2+ikx} = \int_{-\infty}^{+\infty} dx\, e^{-(x-\frac{ik}{2})^2} e^{-k^2/4} = e^{-k^2/4}\sqrt{\pi} \sim e^{-k^2/4}$$

5) $$\int_{-\infty}^{+\infty} dx\, \delta(x-a)\, e^{+ikx} = e^{ika} \not\to 0$$

why?

6) Fourier Transform of hydrogen hatom wave function.

$$\psi_{100}(r,\theta,\varphi) = 2\left(\frac{Z}{a_0}\right)^{3/2} e^{-Z\cdot r/a_0} \frac{1}{4\pi}$$

$$\phi(\vec{p}) = \frac{1}{(2\pi\hbar)^{3/2}} \int d^3x\, e^{-i\vec{p}\cdot\vec{x}/\hbar}\, \psi_{100}(\vec{x}) =$$

$$= \frac{1}{2\pi}\left(\frac{Z}{a_0}\right)^{3/2} \frac{1}{(2\pi\hbar)^{3/2}} \int_0^{+\infty} dr\, r^2 \int_{-1}^{+1} d\cos\theta \int_0^{2\pi} d\varphi\, e^{-Zr/a_0 - i\vec{p}\cdot\vec{x}/\hbar}$$

$$= \frac{1}{2\pi}\left(\frac{Z}{2\pi a_0 \hbar}\right)^{3/2} \int_0^{+\infty} dr\, r^2 \int_{-1}^{+1} d\cos\theta \int_0^{2\pi} d\varphi\; e^{-rZ/a_0 - i\vec{p}\cdot\vec{x}/\hbar} =$$

choose $\vec{p} \parallel \hat{z}$

$$= \left(\frac{Z}{(2\pi a_0 \hbar)}\right)^{3/2} \cancel{\frac{1}{2\pi}}\, \cancel{2\pi} \int_0^{+\infty} dr\, r^2 \int_{-1}^{+1} d\cos\theta\; e^{-rZ/a_0 - ipr\cos\theta/\hbar} =$$

$$= \left(\frac{Z}{2\pi\hbar a_0}\right)^{3/2} \frac{\hbar}{p} \int_0^{+\infty} dr\, \frac{r^{\cancel{2}}}{\cancel{r}}\, e^{-rZ/a_0}\, \frac{-e^{-ipr/\hbar} + e^{ipr/\hbar}}{i}$$

$$= \left(\frac{Z}{2\pi\hbar a_0}\right)^{3/2} \frac{\hbar}{iP} \int_0^{+\infty} dr \left[e^{+\left(\frac{ip}{\hbar} - \frac{Z}{a_0}\right) r} + e^{\left(-\frac{ip}{\hbar} - \frac{Z}{a_0}\right) r} \right] r$$

Integrals of the form $\int_0^{+\infty} dx\; e^{-ax}\, x \qquad Re(a) > 0$

$$\Rightarrow -\frac{d}{da}\int_0^{+\infty} dx\; e^{-ax} = \int_0^{+\infty} dx\; e^{-ax}\, x \quad \Rightarrow$$

$$\Rightarrow \quad = \left(\frac{Z}{2\pi\hbar a_0}\right)^{3/2} \frac{\hbar}{p}\, 2\, \frac{1}{2i} \left[\frac{1}{\left(\frac{1}{a_0} - \frac{ip}{\hbar}\right)} - \frac{1}{\left(\frac{1}{a_0} + \frac{ip}{\hbar}\right)} \right] =$$

$$= \frac{2\hbar}{p}\left(\ldots\right)^{3/2} \frac{2p}{a_0 \hbar \left(\frac{1}{a_0^2} + \frac{p^2}{\hbar^2}\right)^2} = \frac{4}{a_0}\left(\frac{Z}{2\pi\hbar a_0}\right)^{3/2} \frac{1}{\left(\frac{1}{a_0^2} + \frac{p^2}{\hbar^2}\right)^2}$$

$$\sim \frac{1}{p^4}$$

9) Fourier transform of the Coulomb potential.

The Coulomb potential $V_e(\vec{x}) = \frac{1}{|x|}$ is a special case of $V_y^{\mu}(\vec{x}) = \frac{e^{-\mu|\vec{x}|}}{|\vec{x}|} V_0$ called Yukawa potential

$$V_e = \lim_{\mu \to 0} V_y^{\mu}$$

$$\frac{1}{V_0}\int d^3\vec{x}\, e^{i\vec{k}\cdot\vec{x}}\, V_Y^\mu(\vec{x}) = \int d^3\bar{x}\, e^{i\bar{k}\cdot\bar{x}}\, \frac{e^{-\mu x}}{x} =$$

$$= \int_0^{+\infty} dx\, x^2 \int_0^{2\pi} d\varphi \int_{-1}^{+1} dz\, e^{ikzx}\, \frac{e^{-\mu x}}{x} = 2\pi \int_0^{+\infty} dx\, x\, e^{-\mu x}\, \frac{e^{ikx} - e^{-ikx}}{ikx} =$$

$$= \frac{2\pi}{ik} \int_0^{+\infty} dx \left(e^{(ik-\mu)x} - e^{-(ik+\mu)x} \right) = \frac{2\pi}{ik}\left[\frac{-1}{ik-\mu} - \frac{-1}{-(ik+\mu)} \right] =$$

$$= \frac{2\pi}{ik}\left(-\frac{1}{ik+\mu} + \frac{1}{-ik+\mu} \right) = \frac{2\pi}{ik}\, \frac{2ik}{k^2+\mu^2} = \frac{4\pi}{k^2+\mu^2}$$

$$\hat{V}_Y^\mu(k) = V_0 \frac{4\pi}{k^2+\mu^2} \xrightarrow[\mu\to 0]{} V_0 \frac{4\pi}{k^2} \equiv \hat{V}_C(k) \underset{k\to\infty}{\sim} \frac{1}{k^2}$$

10) Consider $I(k\cdot r) = \int_0^{\pi} d\theta\, \sin\theta\, g(\cos\theta)\, e^{-ikr\cos\theta} = \int_{-1}^{+1} du\, g(u) e^{-ikru}$

where $g(\cos\theta)$ is strongly localised around $\theta = \theta_0$

(example $g = e^{-\alpha^2(\cos\theta - \cos\theta_0)^2}$ and $\alpha \gg 1$.)

Assume $g(u)$ and all its derivatives vanish at $u = \pm 1$. In this case, show that $I(k\cdot r)$ vanishes faster than any power of kr as $kr \to \infty$.

$kr = t$

$$\int_{-1}^{1} du\, g(u) e^{itu} = \int_{-1}^{+1} du\, g(u) \frac{1}{it}\left(\frac{d}{du} e^{itu}\right) \overset{\text{by parts}}{=}$$

$$= \frac{1}{it}\Big[\underbrace{g(u) e^{itu}\Big|_{-1}^{+1}}_{0} - \int_{-1}^{+1} du\, g'(u)\, e^{itu} \Big] = -\frac{1}{it}\int_{-1}^{+1} du\, g'(u)\, e^{itu}$$

Iterate ... $\frac{1}{t^n}\int_{-1}^{+1} du\, g^{(n)}(u)\, e^{itu}$ ...

7) Potential Well

$$V(\vec{x}) = \begin{cases} -V_0 & \text{for } |\vec{x}| < R \\ 0 & \text{otherwise} \end{cases}$$

$$V(\vec{K}) = \int_0^{+\infty} dr\, r^2 \int_{-1}^{+1} dz \int_0^{2\pi} d\phi\; -V_0\, \theta(R-r)\, e^{iKrz} =$$

$$= -V_0 \int_0^R dr\, r^2 \cdot 2\pi \frac{e^{iKr} - e^{-iKr}}{iKr} = \frac{-2\pi V_0}{iK} \int_0^R dr\, r\,(e^{iKr} - e^{-iKr})$$

$$= \frac{4\pi V_0}{K^3}\left(KR\cos(KR) - \sin(KR)\right) \underset{K\to\infty}{\sim} \frac{1}{K^3}$$

8)

$$V(\vec{x}) = V_0\, e^{-r^2/a^2} \qquad r = |\vec{x}|$$

$$V(K) = V_0\, 2\pi \int_0^{+\infty} dr\, r^2\, e^{-r^2/a^2} \frac{e^{iKr} - e^{-iKr}}{iKr} = \frac{2\pi V_0}{iK} \int_0^{+\infty} dr\, r\, e^{-r^2/a^2} (e^{(\dots)} - e^{(\dots)})$$

$$= \frac{\pi V_0}{iK} \int_{-\infty}^{+\infty} dr\, r e^{-r^2/a^2} (e^{iKr} - e^{-iKr}) = \pi V_0\, a^3 e^{-\frac{1}{4}a^2K^2} \sqrt{\pi}$$

Amplitude for potential Well

$$f(\vec{K},\vec{K}') \propto \frac{4\pi V_0}{\Delta K^3}\left(\Delta K R\cos(\Delta K R) - \sin(\Delta K R)\right)$$

$$\Delta K = |\vec{K}-\vec{K}'| = \sqrt{K^2(2-2\cos\theta)} = K\sqrt{2-2\cos\theta} =$$

$$= \left[\cos\theta = 1-2\sin^2\frac{\theta}{2}\right] = 2K\sin\left(\frac{\theta}{2}\right)$$

For $\Delta K R \ll 1$ (only spatial size is resolved)

$$f \propto R^3 \frac{1}{(\Delta K R)^3}\left(\Delta K R\cos(\Delta K R) - \sin(\Delta K)\right) \sim$$

$$\sim R^3 \frac{1}{(\Delta K R)^3}\left(-\frac{(\Delta K R)^3}{3} + \frac{(\Delta K R)^5}{30} + \cdots\right) \sim$$

$$\sim V_0 R^3\left(1 - \frac{1}{5}\underbrace{(KR)^2}_{\text{suppressed}}(1-\cos\theta) + \cdots\right)$$ does not change sign.

For $\Delta K R \gg 1$ many oscillation and change of sign!

Cross section will have many zeroes.

Scattering theory, Born approximation

In class: $f_k(\theta,\varphi) = -\frac{\mu}{2\pi\hbar^2}\int d^3y\, e^{i\overbrace{(\vec{k}_i-\vec{k}_f)}\cdot\vec{y}}\, V(\vec{y})$

BORN APPROXIMATION

momentum difference

interaction potential

For $V(\bar{y})$ spherically symmetric, $V(\vec{y}) = V(|\vec{y}|)$

$$\int d^3\bar{y}\, e^{i\Delta\vec{K}\,\bar{y}}\, V(\bar{y}) = \int d^3\bar{y}\, e^{i\Delta\bar{K}\cdot\bar{y}}\, V(\underbrace{|\bar{y}|}_{r}) = [\text{choose axis s.t.} \ldots]$$

$$= \int_0^{+\infty} dr\, r^2 \int_{-1}^{+1} dz \int_0^{2\pi} d\varphi\, e^{i|\Delta K| r z}\, V(r) = 4\pi \frac{1}{\Delta K}\int_0^{+\infty} dr\, r\, V(r)\sin(\Delta K\, r)$$

$$\Rightarrow f_k(\theta,\varphi) = -\frac{\mu}{\hbar^2}\frac{2}{\Delta K}\int_0^{+\infty} dr\, r\, V(r)\sin(\Delta K\cdot r)$$

Born cross section for Yukawa an e^{-r^2}

Cross section (m, Z_1) off Coulomb potential, $Z_2, M \to \infty$.

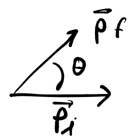
$$V(\vec{r}) = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\,\frac{e^{-r/a}}{r} \qquad a \text{ Yukawa regulator}$$

Already computed the Fourier transf.

$$\hat{V}(\Delta K) = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\,\frac{4\pi}{\Delta K^2 + 1/a^2}$$

$$\Delta K^2 = |\vec{K}_f - \vec{K}_i|^2 = \frac{1}{\hbar^2}\big(\underbrace{\vec{p}_f^{\,2} + \vec{p}_i^{\,2}}_{}- 2\vec{p}_i\cdot\vec{p}_f\big) = \frac{2p^2}{\hbar^2}(1-\cos\theta)$$

= elastic scattering



$$\frac{d\sigma}{d\Omega} = \frac{m}{4\pi^2\hbar^4}\left(\frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\right)^2 \frac{16\pi^2}{\left[(2p^2/\hbar^2)(1-\cos\theta) + (1/a^2)^2\right]^2} =$$

use $1-\cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)$ and $p^2 = 2mE$

$$= \left(\frac{\frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}}{4E\sin^2(\theta/2) + (\hbar^2/2ma^2)}\right)^2 \xrightarrow[\text{coulomb}]{} \left(\frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\right)^2\left(\frac{1}{4E\sin^2(\theta/2)}\right)^2$$

Let $\left(\frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0}\right) \longrightarrow V_0 b \Rightarrow \frac{d\sigma}{d\Omega} = V_0^2\left(\frac{b}{4E^2\sin^2\left(\frac{\theta}{2}\right) + \left(\hbar^2/2mb^2\right)}\right)^2$

$a \longrightarrow b$

namely, the potential is now $V_0 b\, e^{-r/b}/r$

or, in the form above:

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}V_0^2 b^2\,\frac{16\pi^2}{(\Delta K^2 + 1/b^2)^2} = \frac{4m^2V_0^2b^2}{\hbar^4\left(\Delta K^2 + \frac{1}{b^2}\right)^2}$$

We compute also the cross section for $V_0 e^{-r^2/a^2}$

$$V(\Delta k) = \pi V_0 a^3 e^{-\frac{1}{4}a^2 \Delta k^2} \sqrt{\pi}$$

$$\frac{d\tilde{\sigma}}{d\Omega} = \frac{m^2}{4 \cancel{\pi^2} \hbar^4} \cancel{\pi^2} \pi V_0^2 a^6 e^{-\frac{a^2}{2}\Delta k^2} = \frac{\pi m^2 V_0^2 a^6}{4 \hbar^4} e^{-\frac{a^2}{2}\Delta k^2}$$

In FORWARD limit $\Delta K = 0$ $\quad \frac{d\sigma}{d\Omega} = \frac{4 m^2 V_0^2 b^6}{\hbar^4}$

$$\frac{d\tilde{\sigma}}{d\Omega} = \frac{\pi m^2 \tilde{V}_0^2 a^6}{4 \hbar}$$

$\Rightarrow$ $a = b$ To have same cross section for $\Delta K = 0$ -

$$16 V_0^2 b^6 = \tilde{V}_0^2 a^6 \pi \qquad (\mathrm{I})$$

$$\frac{\partial}{\partial \Delta k} \frac{d\sigma}{d\Omega}\bigg|_{\Delta k = 0} = -2 b^2 4 V_0^2 b^6$$

$$-8 V_0^2 b^8 = -\frac{\pi}{8} \tilde{V}_0^2 a^8 \qquad (\mathrm{II})$$

$$\frac{\partial}{\partial \Delta K} \frac{d\tilde{\sigma}}{d\Omega}\bigg|_{\Delta K = 0} = -\frac{a^2}{8} \pi a^6 \tilde{V}_0^2$$

$$(\mathrm{I}) \Rightarrow V_0^2 = \frac{\tilde{V}_0^2 a^6}{b^6} \frac{\pi}{16}$$

$$(\mathrm{II}) \rightsquigarrow -8 \frac{\cancel{\pi}}{4} a^6 b^2 \cancel{\tilde{V}_0^2} = -\frac{\cancel{\pi}}{2} \cancel{\tilde{V}_0^2} a^8$$

$$+4 b^2 = a^2 \Rightarrow b = \frac{a}{2}$$

$$\begin{cases} b = \frac{a}{2} \\ V_0^2 = 4 \pi \tilde{V}_0^2 \end{cases}$$

With this parameters:

$$f_1(\Delta K) = \frac{\pi m^2 \tilde{V}_0^2 a^6}{\hbar^4} \frac{1}{((\frac{a}{2})^2 \Delta K^2 + 1)^2} = C \frac{1}{\left(1 + (\frac{a}{2}\Delta K)^2\right)^2}$$

$$f_2(\Delta K) = \frac{\pi m^2 \tilde{V}_0^2 a^6}{\hbar^4} e^{-\frac{\Delta K^2 a^2}{2}} = C \exp\left(-\frac{1}{2}(\Delta K a)^2\right)$$

For $\Delta K \to 0$ $\left(\Delta K \sim \sin^2(\frac{\theta}{2})\right)$

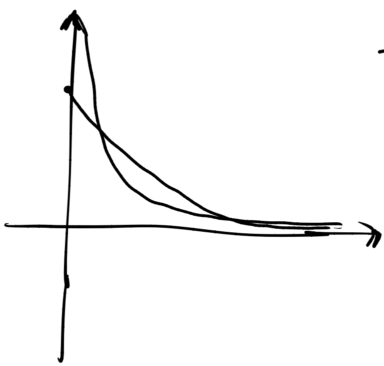
$\sim K$

$f_1 \sim f_2$

But for $\Delta K \gg 1$ the behaviour is very different

When you don't resolve the detail (not high energy enough) "any model" is fine.

For large momentum transfer, $f_2 \to 0$ faster



→ For large momentum transfer only region close to $2 \to 0$ matter and there the potentials are qualitatively differents

Total cross section an optical theorem

The class $d\sigma(\theta,\varphi) = |f_k(\theta,\varphi)|^2 d\Omega$ elastic scattering

where

$$f_k(\theta,\varphi) = -\frac{mL^3}{2\pi\hbar^2}\langle \vec{k}'|V|\psi^+\rangle = -\frac{mL^3}{2\pi\hbar^2}\langle \vec{k}'|T|\vec{k}\rangle$$

$\vec{k}$ incoming $\vec{k}'$ outgoing state, $|\vec{k}| = |\vec{k}'| = k$

T defined by $T|\vec{k}\rangle = V|\psi^+\rangle$

$$T = V + V\frac{1}{E-H_0+i\varepsilon}V + \dots = V\sum_{n\geq 0}\left(\frac{1}{E-H_0+i\varepsilon}V\right)^n$$

<u>BORN</u> <u>APPROXIMATION</u> $n=0$

$$f^{(1)}(\vec{k}',\vec{k}) = -\frac{mL^3}{2\pi\hbar^2}\int d^3x\, d^3x' \underbrace{\langle\vec{k}'|\vec{x}'\rangle}_{\frac{1}{V}e^{-i\vec{k}'\vec{x}'}}\underbrace{\langle\vec{x}'|V|\vec{x}\rangle}_{V(\vec{x})\delta^{(3)}(\vec{x}-\vec{x}')}\underbrace{\langle\vec{x}|\vec{k}\rangle}_{\frac{1}{V}e^{i\vec{k}\cdot\vec{x}}}$$

$$= -\frac{m}{2\pi\hbar^2}\int d^3x\, e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}\, V(\vec{x})$$

$$\frac{d\sigma}{d\Omega} = |f^{(1)}(\vec{k}',\vec{k})|^2 = \left(\frac{m}{2\pi\hbar^2}\right)^2\int d^3x\, d^3x'\, e^{i(\vec{k}-\vec{k}')(\vec{x}-\vec{x}')}\, V(\vec{x})V(\vec{x}')$$

Remember $|\vec{k}| = |\vec{k}'|$.

Define $\dfrac{\vec{k}\cdot(\vec{x}-\vec{x}')}{k|\vec{x}-\vec{x}'|} = \cos\theta$ $\quad - \dfrac{\vec{k}'\cdot(\vec{x}-\vec{x}')}{k|\vec{x}-\vec{x}'|} = \cos\theta'$

and compute

$$\langle\sigma_{TOT}\rangle = \frac{1}{4\pi}\int d\Omega\, d\Omega' \left(\frac{d\sigma}{d\Omega}\right) =$$

$$= \frac{m^2}{4\pi^2\hbar^4}\cancel{2\pi}\,\cancel{2\pi}\,\frac{1}{\cancel{4\pi}}\int d^3x\,d^3x'\,V(\bar{x})V(\bar{x}')\int_{-1}^{+1}d\cos\theta\; e^{ik|\bar{x}-\bar{x}'|\cos\theta}\cdot$$

$$\int_{-1}^{+1}d\cos\theta'\; e^{ik|\bar{x}-\bar{x}'|\cos\theta'}$$

$$= \frac{m^2}{4\pi\hbar^4}\int d^3x\,d^3x'\,V(\bar{x})V(\bar{x}')\;4\left(\frac{\sin(k|\bar{x}-\bar{x}'|)}{k|\bar{x}-\bar{x}'|}\right)^2$$

$$\langle\sigma_{TOT}\rangle = \underset{\substack{\text{isotropic}\\ \text{potential}}}{\sigma_{TOT}} = \frac{m}{\pi\hbar^4}\int d^3x\int d^3x'\,V(r)V(r')\left(\frac{\sin(k|\bar{x}-\bar{x}'|)}{k|\bar{x}-\bar{x}'|}\right)^2.$$

OPTICAL THEOREM

$$\operatorname{Im} f(\bar{k},\bar{k}) = \frac{k}{4\pi}\sigma_{TOT}$$

notice that $f^{(1)}(\bar{k},\bar{k})$ is always real for even potential. In particular then, for isotropic potentials.

$\Rightarrow$ 2° order

$$f^{(2)}(\bar{k},\bar{k}') = -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3\int d^3x\int d^3x''\langle\bar{k}'|\bar{x}'\rangle V(\bar{x}')\cdot$$

$$\cdot\langle x'|\frac{1}{E-H_0+i\varepsilon}|\bar{x}''\rangle V(\bar{x}'')\langle\bar{x}''|\bar{k}\rangle =$$

$$= -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d^3x'\int d^3x''\,e^{-i\bar{k}\cdot\bar{x}'}V(\bar{x}')\;\frac{2m}{\hbar^2}\,G_+(\bar{x}',\bar{x}'')V(\bar{x}'')\,e^{i\bar{k}\cdot\bar{x}''}$$

$$G_+(\bar{x}',\bar{x}) = -\frac{1}{4\pi}\frac{e^{i\bar{k}|\bar{x}-\bar{x}'|}}{|\bar{x}'-\bar{x}|} \qquad \cos\theta = \frac{\bar{k}\cdot(\bar{x}'-\bar{x})}{k|\bar{x}-\bar{x}'|}$$

$$f^{(2)}(\bar{k},\bar{k}) = \left(-\frac{1}{4\pi}\right)^2 \frac{4m^2}{\hbar^4} \int d^3x \int d^3x\, V(\bar{x}')V(\bar{x}) \frac{e^{i(1-\cos\theta)k|\bar{x}-\bar{x}'|}}{|\bar{x}'-\bar{x}|}$$

Compute again:

$$\langle f^{(2)}(\bar{k},\bar{k})\rangle = \int d\Omega\, f^{(2)}(\bar{k},\bar{k}) \frac{1}{4\pi}$$

$$\Rightarrow \frac{2\pi}{4\pi}\int_{-1}^{+1} d\cos\theta \frac{e^{i(1-\cos\theta)k|\bar{x}-\bar{x}'|}}{|\bar{x}-\bar{x}'|} = \frac{i}{2k|\bar{x}-\bar{x}'|^2}\left(1-e^{i2k|\bar{x}-\bar{x}'|}\right)$$

$$\mathrm{Im}(\dots) = \frac{\sin^2(k|\bar{x}-\bar{x}'|)}{k|\bar{x}-\bar{x}'|^2}$$

$$\mathrm{Im}(\langle f^{(2)}(k,k)\rangle) = \frac{k}{4\pi}\left(\frac{m^2}{\hbar\hbar^4}\right)\int d^3x' \int d^3x\, V(\bar{x})V(\bar{x}') \frac{\sin^2(k|\bar{x}-\bar{x}'|)}{(k|\bar{x}-\bar{x}'|)^2}$$

and using optical theorem and isotropic pot. $\Rightarrow$

$$\Rightarrow \mathrm{Im}(\langle f^{(2)}\rangle) = \mathrm{Im}(f^{(2)}) .$$

$\blacksquare$