The Dirac delta function

$$\delta(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \frac{\varepsilon}{\varepsilon^2 + x^2}$$

We prove that

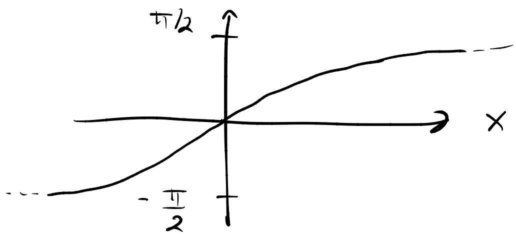
$$\lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{-a}^{a} dx \, \frac{\varepsilon}{x^2 + \varepsilon^2} f(x) = f(0)$$

$$\rightarrow \underbrace{\frac{\varepsilon}{\pi} \int_{-a}^{a} dx \, \frac{f(x) - f(0)}{x^2 + \varepsilon^2}}_{(\mathrm{II})} + \underbrace{\frac{\varepsilon f(0)}{\pi} \int_{-a}^{a} dx \, \frac{1}{x^2 + \varepsilon^2}}_{(\mathrm{I})}$$

(I) $\displaystyle \int dx \, \frac{1}{x^2 + 1} = \mathrm{Arctan}(x) \Rightarrow \mathrm{Arctan}'\left(\frac{x}{\varepsilon}\right) = \frac{1}{\left(\frac{x}{\varepsilon}\right)^2 + 1} \frac{1}{\varepsilon} =$

$$= \frac{\varepsilon}{x^2 + \varepsilon^2}$$

$$\Rightarrow (\mathrm{I}) = \frac{f(0)}{\pi} \mathrm{Arctan}\left(\frac{x}{\varepsilon}\right) \Big|_{-a}^{a} \overset{\text{odd}}{=} \frac{2}{\pi} f(0) \, \mathrm{Arctan}\left(\frac{a}{\varepsilon}\right)$$



$$\Rightarrow \lim_{\varepsilon \to 0^+} \frac{2}{\pi} f(0) \, \mathrm{Arctan}\left(\frac{a}{\varepsilon}\right) = f(0)$$

(II) $\displaystyle \varepsilon \int_{-a}^{a} dx \, \frac{\sum_{n \geq 1} a_n x^n}{x^2 + \varepsilon^2}$. We prove that $\displaystyle \int_{-a}^{a} dx \, \frac{x^n}{\varepsilon^2 + x^2}$ is finite

for $n \geq 1$ for $\varepsilon \to 0$, than the ε in front makes it vanish. $\forall n$.

$n=1$ $\quad \int_{-a}^{a} dx \, \frac{x}{\varepsilon^2+x^2} = 0$ for symmetry

$n=2$

$$\int_{-a}^{a} dx \, \frac{x^2}{\varepsilon^2+x^2} = \underbrace{\int_{-a}^{a} dx \, \frac{x^2+\varepsilon^2}{x^2+\varepsilon^2}}_{2a} - \varepsilon \int_{-a}^{a} dx \, \frac{\varepsilon}{x^2+\varepsilon^2} = 2a - 2\varepsilon \operatorname{Arctan}\left(\frac{a}{\varepsilon}\right) \text{ finite for } \varepsilon \to 0$$

$n > 2$?

For n odd is zero for symm.

For n even

$$\int_{-a}^{a} dx \, \frac{x^n}{\varepsilon^2+x^2} = \int_{-a}^{a} dx \, \frac{(x^2+\varepsilon^2-\varepsilon^2)\, x^{n-2}}{\varepsilon^2+x^2} = \overset{\text{finite}}{\overset{\|}{\int_{-a}^{a} dx \, x^{n-2}}} - \varepsilon^2 \int_{-a}^{a} dx \, \frac{x^{n-2}}{x^2+\varepsilon^2} =$$

$$= \text{iterate} = \text{finite integrals} + \varepsilon^{n-2} \int_{-a}^{a} dx \, \frac{x^2}{x^2+\varepsilon^2}$$

$\hookrightarrow$ which is finite.

So, for every n:

$$\varepsilon \int_{-a}^{a} dx \, \frac{x^n}{x^2+\varepsilon^2} \xrightarrow[\varepsilon \to 0]{} 0 \implies (\#) \to 0 \quad \text{as } \varepsilon \to 0.$$

$$\delta(x) = \frac{1}{2\pi} \lim_{\eta \to \infty} \frac{4}{\eta x^2} \sin^2\left(\frac{x\eta}{2}\right)$$

$$\underbrace{\frac{1}{2\pi} \int_{-a}^{a} dx \, \frac{4}{\eta x^2} \sin^2\left(\frac{x\eta}{2}\right) f(0)}_{I} + \frac{1}{2\pi} \int_{-a}^{a} dx \, \frac{4}{\eta x^2} \sin^2\left(\frac{\eta x}{2}\right) [f(x) - f(0)]$$

$$I = \left[\frac{\eta x}{2} = t\right] = \frac{1}{\pi} \int_{-a\eta/2}^{a\eta/2} dt \, \frac{1}{t^2} \sin^2(t) f(0) \xrightarrow[\eta \to \infty]{} \underbrace{\frac{1}{\pi} \int_{-\infty}^{+\infty} dt \, \frac{1}{t^2} \sin^2(t)}_{1} f(0)$$

OPTION 1

$$\int_{-\infty}^{+\infty} dt \, \frac{\sin^2(t)}{t^2} \longrightarrow I(\alpha) \equiv \int_{-\infty}^{+\infty} dt \, \frac{\sin^2(\alpha t)}{t^2}$$

$I(1)$ will be the result

$$\frac{dI}{d\alpha} = \int_{-\infty}^{+\infty} dt \, \frac{\sin(2\alpha t)}{t} = [2\alpha t \to t]$$

Note that $I(0)=0$.

$$= \int_{-\infty}^{+\infty} dt \, \frac{\sin(t)}{t} = 2 \int_{0}^{\infty} dt \, \frac{\sin(t)}{t}$$

Consider $\tilde{I}(s) = \int_0^{+\infty} e^{-st} \frac{\sin(t)}{t}$

$\tilde{I}(\infty) = 0$

$\tilde{I}(0) = \frac{dI}{d\alpha} \frac{1}{2}$

$$\frac{d\tilde{I}}{ds} = - \int_0^{+\infty} dt \, e^{-st} \sin(t) = - \frac{1}{s^2+1}$$

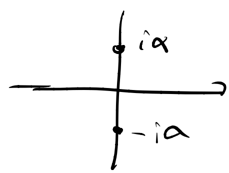
$$\tilde{I}(s) = -\arctan(s) + c \qquad \tilde{I}(\infty) = 0 = -\frac{\pi}{2} + c \Rightarrow c = \frac{\pi}{2}$$

$$\tilde{I}(s) = \frac{\pi}{2} - \arctan(s) \to \tilde{I}(0) = \frac{\pi}{2} = \frac{dI}{d\alpha} \frac{1}{2}$$

$$\Rightarrow I(\alpha) = \pi\alpha + c, \quad I(0) = 0 \Rightarrow c = 0.$$

$$I(\alpha) = \pi\alpha \Rightarrow I(1) = \pi$$

OPTION 2



$$\lim_{\alpha^2 \to 0} \int_{-\infty}^{+\infty} dt \, \frac{\sin^2(t)}{t^2+\alpha^2} =$$

$$= \lim_{\alpha \to 0} \int_{-\infty}^{+\infty} dt \, \frac{e^{2it} + e^{-2it} - 2}{(-4)(t+i\alpha)(t-i\alpha)} = -\frac{1}{4} \lim_{\alpha \to 0} \Bigg[-\frac{2}{\alpha} \arctan\left(\frac{t}{\alpha}\right)\Bigg|_{-\infty}^{\infty}$$

$$+ \frac{\cancel{2}\pi \cancel{i}}{\cancel{2i}\alpha} e^{-2\alpha} + \frac{\cancel{2}\pi \cancel{i}}{\cancel{2i}\alpha} e^{-2\alpha} \Bigg] = -\frac{1}{4} \lim_{\alpha \to 0} \left(-2\frac{\pi}{\alpha} + \frac{2\pi}{\alpha}(1 - 2\alpha + o(\alpha^2) \right)$$

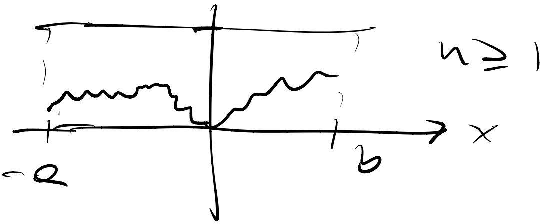
$$= \pi$$

$$\int_{-a}^{a} dx \ \frac{1}{\eta x^2} \ x^n \sin^2\left(\frac{\eta x}{2}\right) \qquad n \geq 1$$

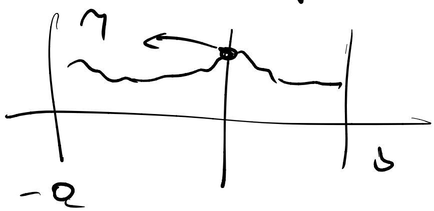
for $n \geq 1$ this trivially vanishes. as $\frac{1}{\eta} \to \infty$

In fact, the integrand, for $n \geq 1$ is bounded by some ~~constante~~ for $x \sim 0$. $\forall \eta$

$$\frac{x^n}{\eta x^2} \sin^2\left(\frac{\eta x}{2}\right) \longrightarrow$$



for $n = 1 \Rightarrow$

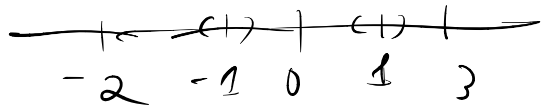


and for $\eta \to \infty$ it explode in $0 \to$ contribution of δ!

Cauchy Principal Value

1) $PV\int_{-2}^{3}\frac{2x}{x^2-1}\,dx = ?$

x^2-1 vanishes for $x=\pm 1$



$$= \int_{-2}^{-1-\varepsilon} dx\,\frac{2x}{(x-1)(x+1)} + \quad 1$$

$$+ \int_{-1+\varepsilon}^{1-\varepsilon} dx\,\frac{2x}{(x-1)(x+1)} \quad 2 \quad + \quad 3 \quad \int_{1+\varepsilon}^{3} dx\,\frac{2x}{(x-1)(x+1)}$$

$$3 \quad \int_{1+\varepsilon}^{3} dx\,\frac{2x}{(x+1)(x-1)} = \int_{1+\varepsilon}^{3} dx\left(\frac{1}{(x+1)} + \frac{1}{(x-1)}\right) =$$

$$= Log(x+1)\Big|_{1}^{3} + Log(x-1)\Big|_{1+\varepsilon}^{3} = 2Log(2) - Log(2) + Log(2) - Log(\varepsilon)$$

$$= 2Log(2) - Log(\varepsilon)$$

$$2 \quad Log(1+x)\Big|_{-1+\varepsilon}^{1-\varepsilon} + Log(1-x)\Big|_{-1+\varepsilon}^{1-\varepsilon} = Log(2) - Log(\varepsilon) + Log(\varepsilon) - Log(2) = 0$$

$$1 \quad Log(-1-x)\Big|_{-2}^{-1-\varepsilon} + Log(1-x)\Big|_{-2}^{-1-\varepsilon} = Log(\varepsilon) + Log(2) - Log(3)$$

$$\boxed{1+2+3 = 2\,\mathrm{Log}(2) - \mathrm{Log}(3)}$$

2) $$\frac{1}{x \pm i\varepsilon} = PV\left(\frac{1}{x}\right) \mp i\pi\,\delta(x)$$

$$-\frac{1}{x+i\varepsilon} + \frac{1}{x-i\varepsilon} = +\frac{2i\varepsilon}{x^2+\varepsilon^2} \xrightarrow[\varepsilon\to 0]{} 2i\pi\,\delta(x)$$

previous exercise

$$A_\varepsilon = \frac{1}{x-i\varepsilon} \qquad B_\varepsilon = \frac{1}{x+i\varepsilon}, \qquad A_\varepsilon - B_\varepsilon \to 2i\pi\,\delta(x)$$

$A_\varepsilon + B_\varepsilon$? $\quad A_\varepsilon + B_\varepsilon = \dfrac{2x}{x^2+\varepsilon^2}$

f regular in $x=0$

$$\int_{-a}^{b} dx\, \frac{2x}{x^2+\varepsilon^2} f(x) =$$

$$= \int_{-a}^{-\eta} dx\, \frac{2x}{x^2+\varepsilon^2} f(x) + \int_{-\eta}^{\eta} dx\, \frac{2x}{x^2+\varepsilon^2} f(x) + \int_{\eta}^{b} dx\, \frac{f(x)\,2x}{x^2+\varepsilon^2}$$

$$\xrightarrow[\varepsilon\to 0]{} 2\left[\int_{-a}^{-\eta} dx\, \frac{1}{x} f(x) + \int_{\eta}^{b} dx\, \frac{1}{x} f(x)\right] + \int_{-\eta}^{\eta} dx\, \frac{2x f(x)}{x^2+\varepsilon^2}$$

$$= 2\,P\left[\frac{1}{x} f(x)\right] + \lim_{\varepsilon\to 0} \int_{-\eta}^{\eta} dx\, \frac{2x\, f(x)}{x^2+\varepsilon^2}$$

$$\int_{-\eta}^{\eta} dx \, \frac{x}{x^2+\varepsilon^2} f(x)$$

$$f(x) = \sum_{n\geq 0} x^n c_n$$

For $n \geq 2$

$$\int_{-\eta}^{\eta} dx \, \frac{x^{n+1}}{x^2+\varepsilon^2} \xrightarrow[\varepsilon\to 0]{} \int_{-\eta}^{\eta} dx \; x^{n-1} = \frac{x^n}{n}\Big|_{-\eta}^{\eta}.$$

$$= \frac{\eta^n - (-\eta)^n}{n}.$$

For $n = 1$ $\displaystyle\int_{-\eta}^{\eta} dx \, \frac{x^2}{x^2+\varepsilon^2} = \left(x - \varepsilon \operatorname{ArcTan}\left(\frac{x}{\varepsilon}\right)\right)\Big|_{-\eta}^{\eta} =$

$$= 2\eta - 2\varepsilon \operatorname{ArcTan}\left(\frac{\eta}{\varepsilon}\right) \xrightarrow[\varepsilon\to 0]{} 2\eta$$

For $n = 0$

$$\int_{-\eta}^{\eta} dx \, \frac{x}{x^2+\varepsilon^2} = 0 \text{ by symm.}$$

Selection Rules

Interaction with e.m. field $\langle f | e^{-i\vec{k}\cdot\bar{r}} \bar{\varepsilon}\cdot\vec{p} | i \rangle$

In class we have seen that $\langle f | \bar{\varepsilon}\cdot\bar{p} | i \rangle$ (dipole) $\Longrightarrow$

$\Longrightarrow \Delta \ell = \pm 1$.

Let's expand more: $\langle f | \bar{k}\cdot\bar{r} \ \bar{\varepsilon}\cdot\bar{p} | i \rangle$

$$\bar{k}\cdot\bar{r} \ \bar{\varepsilon}\cdot\bar{p} = \frac{1}{2}\left(\bar{\varepsilon}\cdot\bar{p} \ \bar{k}\cdot\bar{r} + \bar{\varepsilon}\cdot\bar{r} \ \bar{k}\cdot\bar{p}\right) + \frac{1}{2}\left(\bar{\varepsilon}\cdot\bar{p} \ \bar{k}\cdot\bar{r} - \bar{\varepsilon}\cdot\bar{r} \ \bar{k}\cdot\bar{p}\right) =$$

$$= \underbrace{\frac{1}{2}\left(\bar{\varepsilon}\cdot\bar{p} \ \bar{k}\cdot\bar{r} + \bar{\varepsilon}\cdot\bar{r} \ \bar{k}\cdot\bar{p}\right)}_{\text{electric quadrupole}} + \frac{1}{2}\underbrace{(\vec{k}\times\bar{\varepsilon})\cdot(\bar{r}\times\bar{p})}_{\text{magnetic dipole}}$$

These operators are parity even $\Longrightarrow$ cannot have change of parity in the angular part.

$$\langle n,\ell,m | \left(\bar{\varepsilon}\cdot\bar{p} \ \bar{k}\cdot\bar{r} + \bar{\varepsilon}\cdot\bar{r} \ \bar{k}\cdot\bar{p}\right) | 0,0,0 \rangle = ?$$

$$= -i m \omega_{fi} \langle n,\ell,m | \left(\bar{\varepsilon}\cdot\bar{r} \ \bar{k}\cdot\bar{r} + \bar{\varepsilon}\cdot\bar{r} \ \bar{k}\cdot\bar{r}\right) | 000 \rangle =$$

$$\bar{\varepsilon} H_0 \vec{r} \ \bar{k}\cdot\bar{r} - \cancel{\bar{\varepsilon}\cdot\bar{r} H_0 \ \bar{k}\cdot\bar{r}} + \cancel{\bar{\varepsilon}\cdot\bar{r} H_0 \ \bar{k}\cdot\bar{r}} - \bar{\varepsilon}\cdot\bar{r} \ \bar{k}\cdot\bar{r} H_0$$

$$-2 i m \omega_{fi} \langle n,\ell,m | \vec{\varepsilon}\cdot\bar{r} \ \vec{k}\cdot\bar{r} | 0,0,0 \rangle$$

$$= \int_0^{+\infty} dr \, r^2 \int d\Omega \, R^*_{n\ell} \, Y^*_{\ell m} \, \bar{\varepsilon}\cdot\bar{r} \ \bar{k}\cdot\bar{r} \, R_{0,0} \, \underset{1}{\overset{}{Y_{00}}}$$

$Y_{00} = 1$

$$\bar{\varepsilon}\cdot\hat{r} = \# \left(\varepsilon_z Y_{10} + \frac{-\varepsilon_x + i\varepsilon_y}{\sqrt{2}} Y_{1,1} + \frac{\varepsilon_x + i\varepsilon_y}{\sqrt{2}} Y_{1,-1} \right)$$

$$\bar{k}\cdot\bar{r} = \# \left(\varepsilon \leftrightarrow k \right)$$

Angular part:

$$\int d\Omega\, Y^*_{\ell_f m_f}(\theta,\varphi)\, Y_{1m}\, Y_{1m'}\, \overset{\frac{1}{4}}{Y_{00}}$$

$$d\varphi: \quad \int_0^{2\pi} d\varphi\; e^{i\varphi(m+m'-m_f)} = 2\pi\, \delta_{m+m'-m_f}$$

$$m, m \in \{1, 0, -1\} \Rightarrow m_f \in \{0, \pm 1, \pm 2\} \qquad \Delta m = 0, \pm 1, \pm 2$$

$$\int d\cos\theta\; Y^*_{\ell_f m_f}(\theta,\varphi)\, \underbrace{Y_{1m}\, Y_{1m'}}_{\cos^2\theta} \Rightarrow$$

$$\Rightarrow \int d\cos\theta\, \cos^2(\theta)\, Y^*_{\ell f}$$

$\ell_f = 0$ allowed ✓

$\ell_f = 1 \quad \int_{-1}^{+1} dx\, x^3 = 0$

$\ell_f = 2 \quad \int_{-1}^{1} dx\, x^2 \sqrt{1-x^2} \neq 0$ allowed ✓

$$\Rightarrow \boxed{\Delta \ell = 0, 2}$$

The radial contribution will anyway suppress these transitions.

The magnetic dipole:

$$\vec{r} \times \vec{p} \propto \vec{L}$$

$$(\vec{k} \times \vec{\varepsilon}) \langle n_f, \ell_f, m_f | \vec{L} | 0, 0, 0 \rangle$$

$\vec{L}$ can be decomposed int $L_+, L_-, L_z \Rightarrow$ change m of $0, \pm 1$.
change ℓ of 0

$$\boxed{\begin{array}{l} \Delta \ell = 0 \\ \Delta m = \pm 1, 0 \end{array}}$$

Mössbauer Effect

How do we induce nuclear/atomic transition?

- using radiation from same sources! But DOPPLER broadening can cause trouble, in particular for gas (thermal motion)
  (Atoms needs to be cooled)

- Use solids

Consider emission from Iridium nucleus ($_{77}Ir^{191}$) of γ-rays.
$E_\gamma \simeq 100\,KeV$ ($\mathcal{O}(10^5$ energy of atomic transition!)
Lifetime $\tau \sim 10^{-10}\,s \implies \frac{\Delta\omega}{\omega} \sim \frac{\Delta E}{E} \sim \frac{\hbar/\tau}{E} \simeq \frac{10^{-34}/10^{-10}}{10^5 \cdot 1{,}6 \cdot 10^{-19}} \simeq 0{,}6 \cdot 10^{-10}$

Unfortunately, a new problem arises.

γ-rays carry away momentum $\frac{\hbar\omega}{c} \Rightarrow \Delta E_{recoil} = \frac{P_{recoil}^2}{2M} = \frac{1}{2M}\left(\frac{\hbar\omega}{c}\right)^2$

$\Rightarrow \frac{\Delta E}{\hbar\omega} \simeq \frac{\hbar\omega}{2Mc^2} = \frac{10^{-1}\,MeV}{2 \cdot 940 \cdot 191\,MeV} \simeq 3 \cdot 10^{-7}$

Absorber $\rightarrow$ a same recoil $\implies \Delta E/\hbar\omega \simeq 6 \cdot 10^{-7}$

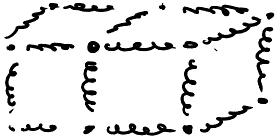
$\begin{cases} \text{width} \sim 10^{-10} \\ \text{shift} \sim 10^{-7} \end{cases}$ $\hat{\wedge}$

How do we overcome this? Recoilless motion

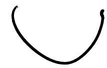
In a crystal $\rightarrow$ the crystal takes the recoil $\rightsquigarrow$ single nucleus does not.
(factor 10^{-22})

How can we make the whole crystal taking the recoil?

Suppose the nucleus bound by an harmonic potential.



This is a good approximation if the nucleus is not in a very excited state.



$V(x) \sim$ quadratic

$$\hat{H}_0 = \left(\frac{\hat{P}_{nucl}^2}{2M_{nucl}} + \frac{1}{2} M \omega_0^2 \hat{x}^2 \right)$$

$$E_n = \hbar\omega_0 \left(n_x + n_y + n_z + \frac{3}{2} \right)$$

ω_0 large $\Rightarrow$ stiff "SPRINGS"

If our nucleus has N nucleons $\Rightarrow \psi = \psi(\vec{r}_1, \ldots, \vec{r}_N)$

and the interaction is $-\frac{e}{M} \sum_{protons} \vec{P}_\kappa \cdot \vec{A}_\kappa(\vec{r}_\kappa, t)$

The transition has matrix element for firs order pert. theory

$$\langle f | (-\frac{e}{M} \Sigma \text{ dipoles} \quad | i \rangle =$$

$$= -\frac{e}{M} \int \cdots \int d^3\vec{r}_1 \cdots d^3\vec{r}_N \, \psi_f^*(\vec{r}_1, \ldots, \vec{r}_N) \sum_\kappa \vec{\varepsilon} \cdot \vec{P}_\kappa \, e^{-i\vec{k}\cdot\vec{r}_\kappa} \, \psi_i(\vec{r}_1, \ldots, \vec{r}_N)$$

Consider the center of mass $\vec{R} = \frac{1}{N} \sum_i \vec{r}_i$ and relative coordinates

$$\vec{\rho}_i = \vec{r}_i - \vec{R} \Rightarrow \psi(\vec{r}_1, \ldots, \vec{r}_N) = \underbrace{\psi_{n_x n_y n_z}(\vec{R})}_{\text{subject to harmonic potential}} \phi(\vec{\rho}_1, \vec{\rho}_2, \ldots, \vec{\rho}_{N-1})$$

$$\Rightarrow -\frac{e}{M} \left[\int d^3\vec{R} \, \psi_{n_f}^*(\vec{R}) \, e^{-i\vec{k}\cdot\vec{R}} \, \psi_{n_i}(\vec{R}) \right] \cdot \left[\int d^3\vec{\rho}_1 \cdots d^3\vec{\rho}_{N-1} \, \phi_f^*(\vec{\rho}_i) \sum_{protons} \vec{\varepsilon} \cdot \vec{P}_\kappa \, e^{-i\vec{k}\cdot\vec{\rho}_\kappa} \cdot \phi_i(\vec{\rho}_i) \right] =$$

$$= M_{internal} \cdot \int d^3\vec{R} \, \psi_{n_f}^*(\vec{R}) \, e^{-i\vec{k}\cdot\vec{R}} \, \psi_0(\vec{R})$$

Probability to remain in ground state is:

$$P_0(\vec{k}) = \frac{|M_{int}|^2 \left| \int d^3\vec{R}\, \varphi_0^*(R)\, e^{-i\vec{k}\cdot\vec{R}}\, \varphi_0(\vec{R}) \right|^2}{|M_{int}|^2 \underbrace{\sum_{nf} \left| \int d^3\bar{R}\, \varphi_{nf}^*(R)\, e^{-i\bar{k}\cdot\bar{R}}\, \varphi_0(\bar{R}) \right|^2}_{=1}} =$$

$$\hookrightarrow \sum_{nf} |\langle f| e^{i\vec{k}\cdot\vec{r}} |0\rangle|^2 = \sum_{nf} \langle 0| e^{-i\vec{k}\vec{R}} \underbrace{|f\rangle\langle f|}_{\mathbb{I}} e^{i\vec{k}\cdot\vec{R}} |0\rangle$$

$$= \langle 0|0\rangle = 1$$

$$= \left| \int d^3R\, \varphi_0^*(\bar{R})\, e^{-i\bar{k}\cdot\bar{R}}\, \varphi_0(\bar{R}) \right|^2$$

$$\varphi_0(\vec{R}) = \left(\frac{M_N\omega_0}{\pi\hbar}\right)^{3/4} e^{-m\omega_0\vec{R}^2/2\hbar}$$

$$\Rightarrow \left| \left(\frac{M_N W_0}{\pi\hbar}\right)^{3/2} \int d^3\vec{R}\, e^{-M_N W_0 \vec{R}^2/\hbar}\, e^{-i\vec{k}\cdot\vec{R}} \right|^2 =$$

$$= e^{-\hbar^2 k^2/(2M_N\hbar\omega_0)} = e^{-\text{recoil energy}/\text{level spacing}}$$

$\Rightarrow$ for stiff spring (ω_0 large) $\rightarrow$ recoilless emission are more probable

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It can be shown that in some realistic model,

$\omega_0\,(\omega_D) \sim \dfrac{L}{a\tau}$ $\quad L \sim v_s \cdot \tau$ $\Rightarrow \sim \dfrac{v_s}{a}$ $\leadsto$ speed of sound.

Spontaneous emission: $2p \rightarrow 1s$ Hydrogen transition

$$H = H_0 + H_{int} \qquad H_{int} = -\frac{e}{mc}\vec{A}(\vec{r},t)\cdot\hat{\vec{p}}$$

H_0 ↳ hydrogen

$$\hat{\vec{A}}_{quant} = \sum_{\underline{k},\alpha} c\sqrt{\frac{\hbar}{2\omega_{\underline{k}} V}}\left[\hat{a}_{\underline{k},\alpha}\vec{\varepsilon}_\alpha(\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega_{\underline{k}}t)} + \hat{a}^{+}_{\underline{k},\alpha}\vec{\varepsilon}^{\,*}_\alpha(\vec{k})e^{-i(\vec{k}\cdot\vec{x}-\omega_{\underline{k}}t)}\right]$$

where
$$\left.\begin{aligned}&[\hat{a}_{\underline{k},\alpha},\hat{a}^{+}_{\underline{k}',\beta}] = \delta_{\alpha\beta}\delta_{\underline{k},\underline{k}'}\\ &[\hat{a},\hat{a}] = [\hat{a}^{+},\hat{a}^{+}] = 0\end{aligned}\right\}\text{ counting algebra}$$

$$|\{n_{\underline{k},\alpha}\}\rangle = \prod_{\underline{k},\alpha}\frac{(a^{+}_{\underline{k},\alpha})^{n_{\underline{k},\alpha}}}{\sqrt{n_{\underline{k},\alpha}}}|0\rangle$$

$n_{\underline{k},\alpha}$ photons with momentum $\underline{k}$ and pol α.

Fermi Golden Rule:

$$w_{i\to f} = \frac{2\pi}{\hbar}|M_{fi}|^2\,\delta(E_f + \hbar\omega_{\underline{k}} - E_i)$$

where $|i\rangle = |E_i, n_{\underline{k},\alpha}\rangle$

E_i ↳ $2p$ state; $n_{\underline{k},\alpha}$ ↳ photons going around

$$|f\rangle = |E_f, n_{\underline{k},\alpha}+1\rangle$$

$+1$ ↳ important! ↳ $+1$ photon

$$a^{+}_{\underline{k},\alpha}|n_{\underline{k},\alpha}\rangle = \sqrt{n_{\underline{k},\alpha}+1}\,|n_{\underline{k},\alpha}+1\rangle$$

$$M_{fi} = \langle E_f, n_{\underline{k},\alpha}+1|\, -\frac{e}{mc}c\sqrt{\frac{\hbar}{2\omega V}}\,a^{+}_{\underline{k},\alpha}e^{i\vec{k}\cdot\vec{r}}\,\varepsilon^{(\alpha)}\cdot\hat{p}\,|E_i; n_{\underline{k},\alpha}\rangle$$

$$= -\frac{e}{m}\sqrt{\frac{(n_{\underline{k},\alpha}+1)\hbar}{2\omega V}}\,\langle E_f|e^{i\vec{k}\cdot\vec{r}}\varepsilon^{(\alpha)}\cdot\hat{\vec{p}}\,|E_i\rangle$$

we computed the density of states for massless particles in ex. sheet 3:

$$\rho(\omega)d(\hbar\omega)d\Omega = \frac{V\omega^3}{(2\pi)^3\hbar c^3}\,d(\hbar\omega)\,d\Omega$$

$$W\,d\Omega = w_{i\to f}\,\rho(\omega)\,d(\hbar\omega)\,d\Omega = \frac{e^2}{4\pi\hbar c}\,\frac{\omega(n_{\underline{k},\alpha}+1)}{2\pi m^2c^2}\,|\langle E_f|e^{i\vec{k}\cdot\vec{r}}\varepsilon^{\alpha}\cdot\hat{\vec{p}}\,|E_i\rangle|^2 \cdot d\Omega$$

In dipole approx:

$$M_{fi} = \langle E_f | \vec{\varepsilon}^{(\alpha)} \cdot \hat{p} | E_i \rangle = -\frac{i m (\overset{\omega}{\overset{\shortparallel}{E_f - E_i}})}{\hbar} \langle E_f | \varepsilon^{(\alpha)} \cdot \hat{r} | E_i \rangle =$$

$$\frac{d\hat{r}}{dt} \sim [\hat{H}, \hat{r}]$$

$$= i m \omega \vec{\varepsilon}^{\alpha} \cdot \underbrace{\langle E_f | \hat{r} | E_i \rangle}_{\vec{r}_{fi}}$$

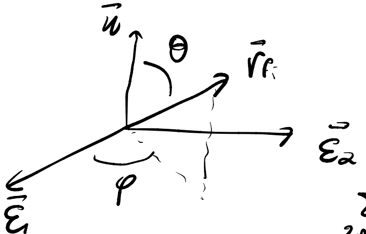
define $\cos\theta^{(\alpha)} = \vec{\varepsilon}^{(\alpha)} \cdot \vec{r}_{fi} / |\vec{r}_{fi}|$

$$\Rightarrow \vec{\varepsilon}^{\alpha} \cdot \vec{r}_{fi} = \cos\theta^{(\alpha)} |\vec{r}_{fi}|$$

spontaneous emission: $n_{\underline{k},\alpha} = 0$.

$$W_{d\Omega} = \frac{e^2}{4\pi\hbar c} \frac{\omega^3}{2\pi c^2} \cos^2(\theta^{(\alpha)}) |\vec{r}_{fi}|^2 d\Omega$$

Integrate over emission angles and sum over two polarisations



$$\cos\theta^{(1)} = \sin\theta\cos\varphi$$

$$\cos\theta^{(2)} = \sin\theta\sin\varphi$$

$$\Downarrow$$

$$\sum_{2\,pol} (\cos^2\theta^{(1)} + \cos^2\theta^{(2)}) = \sin^2\theta$$

$$\Rightarrow \int d\Omega \sin^2\theta = 2\pi \int_{-1}^{+1} d\cos\theta\, (1 - \cos^2\theta) = 2\pi\left(2 - \frac{2}{3}\right) = \frac{8}{3}\pi$$

$$W_{d\Omega} = \frac{e^2}{4\pi\hbar c^2} \frac{4\omega^3}{3} |\vec{r}_{fi}|^2$$

$$|\vec{r}_{fi}|^2 = \Big| \underbrace{\int_0^{+\infty} dr\, R_{10}(r) R_{21}(r) r^3}_{\frac{2^{15}}{3^9} a_0^2} \Big|^2 \left(\int d\Omega\, Y_0^{0*}(\theta,\phi)\, \hat{r}\, Y_1^{m}(\theta,\phi) \right)^2$$

$$\hat{r} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

$$\Rightarrow W = \frac{e^2}{4\pi\hbar c} \frac{4\omega^3}{3c^2} \cdot \frac{2^{15}}{3^9} a_0^2 \cdot \frac{1}{3} = \frac{e^2}{4\pi\hbar c} \frac{4}{9} \omega^3 \frac{2^{15}}{3^9} \frac{a_0^2}{c^2}$$

radial angular

$$\omega = E_2 - E_1 = \frac{3}{8} \frac{mc^2\alpha^2}{\hbar} \qquad \alpha^2 = \frac{e^2}{\hbar c}$$

$$W = \frac{\Gamma}{\hbar} = \frac{1}{\tau} \Rightarrow \tau \sim 1{,}59 \cdot 10^{-9}\, s$$