# Interaction picture

$\hat{H}(t) = \hat{H}_0 + \hat{V}(t)$ $\qquad \hat{O}$ operator

$$V_I = e^{iH_0 t/\hbar}\, V\, e^{-iH_0 t/\hbar} \qquad O_I = e^{iH_0 t/\hbar}\, O\, e^{-iH_0 t/\hbar}$$

$$\Rightarrow \frac{d}{dt} O_I(t) = \frac{iH_0}{\hbar} O_I - \frac{i}{\hbar} O_I H_0 = \frac{i}{\hbar}[H_0, O_I]$$

# 3 level system

Consider a system with energy levels $\hbar\omega_a, \hbar\omega_b, \hbar\omega_c$.

Free t-evolution

$$e^{it\frac{H_0}{\hbar}} = \begin{pmatrix} e^{it\omega_a} & 0 & 0 \\ 0 & e^{i\omega_b t} & 0 \\ 0 & 0 & e^{i\omega_c t} \end{pmatrix}$$

The perturbation is, in the $H_0$-eigenstates basis:

$$V(t) = \begin{pmatrix} 0 & \frac{1}{2}e^{-i\omega_1 t} W_{ab} & \frac{1}{2}e^{-i\omega_2 t} W_{ac} \\ \frac{1}{2}e^{i\omega_1 t} W_{ab} & 0 & 0 \\ \frac{1}{2}e^{i\omega_2 t} W_{ac} & 0 & 0 \end{pmatrix}$$

with $W_{ab}, W_{ac}$ real.

In the interaction picture

$$V_I(t) = U_0^\dagger(t) V(t) U_0(t) = \begin{pmatrix} 0 & x & y \\ x^* & 0 & 0 \\ y^* & 0 & 0 \end{pmatrix}$$

with

$$x = \frac{1}{2} W_{ab} e^{i\delta_1 t}, \quad y = \frac{1}{2} W_{ac} e^{i\delta_2 t} \qquad \begin{aligned} \delta_1 &= (\omega_a - \omega_b) - \omega_1 = \omega_{ab} - \omega_1 \\ \delta_2 &= (\omega_a - \omega_c) - \omega_2 = \omega_{ac} - \omega_2 \end{aligned}$$

The state in interaction picture at time $t$ is

$$|\psi(t)\rangle_I = U_0^\dagger(t)\,|\psi(t)\rangle_S$$

In general $|\psi(t)\rangle_I = \begin{pmatrix} a(t) \\ b(t) \\ c(t) \end{pmatrix}$ in the $H_0$-eigenstates basis.

and we have seen in class that

$$i\hbar \frac{\partial}{\partial t} c_n(t) = \sum_m V_{nm}(t)\, e^{i\omega_{nm} \cdot t}\, c_m(t)$$

where $\omega_{nm} = E_n - E_m$ (of $H_0$!)

and $V_{nm} = \langle n | V | m \rangle$

For our problem:

$$\begin{cases} i\hbar \dfrac{da(t)}{dt} = \dfrac{1}{2} W_{ab}\, e^{i\delta_1 t}\, b(t) + \dfrac{1}{2} W_{ac}\, e^{i\delta_2 t}\, c(t) \\ i\hbar \dfrac{db(t)}{dt} = \dfrac{1}{2} W_{ab}\, e^{-i\delta_1 t} \\ i\hbar \dfrac{dc(t)}{dt} = \dfrac{1}{2} W_{ac}\, e^{-i\delta_2 t}\, a(t) \end{cases}$$

assume:

$b(t) = b(0)\, e^{i(\Omega - \delta_1)t}$ $\qquad c(t) = c(0)\, e^{i(\Omega - \delta_2)t}$

$$a(t) = a(0)\, e^{i\Omega t}$$

$$\Downarrow$$

$$\begin{cases} -\hbar\Omega\, a(0) = \dfrac{1}{2} W_{ab}\, b(0) + \dfrac{1}{2} W_{ac}\, c(0) \\ -\hbar(\Omega - \delta_1)\, b(0) = \dfrac{1}{2} W_{ab}\, a(0) \\ -\hbar(\Omega - \delta_2)\, c(0) = \dfrac{1}{2} W_{ac}\, a(0) \end{cases}$$

This equation is difficult to solve.

Assume perfect tuning, $\delta_1 = \delta_2 = 0$

$$\left\{\begin{array}{ll} -\hbar\Omega a(0) = \frac{1}{2} W_{ab} b(0) + \frac{1}{2} W_{ac} c(0) \\ -\hbar\Omega b(0) = \frac{1}{2} W_{ab} a(0) \\ -\hbar\Omega c(0) = \frac{1}{2} W_{ac} a(0) \end{array}\right] \mathrm{I}$$

multiply:

$$+\hbar^{\cancel{3}\,2}\Omega^3 \cancel{A(0)B(0)C(0)} = \left(\frac{1}{2}\right)^3 W_{ab} W_{ac} A^2(0) \left(W_{ab} B(0) + W_{ac} C(0)\right)$$

$$= \left(\frac{1}{2}\right)^2 \cancel{(-\hbar)}\,\Omega \left(W_{ab}^2 \cancel{A(0)B(0)C(0)} + W_{ac}^2 \cancel{A(0)B(0)C(0)}\right)$$

$$\left[\ \hbar^2 \Omega^3 = \left(\frac{1}{2}\right)^2 \left(W^2_{ab} + W^2_{ac}\right)\Omega\right.$$

Three solutions

$$\underbrace{\Omega = 0}, \quad \Omega = \pm r, \quad \text{where } r = \sqrt{\left(\frac{W_{ab}}{2\hbar}\right)^2 + \left(\frac{W_{ac}}{2\hbar}\right)^2}$$

↳ do not forget this

We can write a general solution as superposition

$$\left\{\begin{array}{l} a(t) = a_0 + a_+ e^{irt} + a_- e^{-irt} \\ b(t) = b_0 + b_+ e^{irt} + b_- e^{-irt} \\ c(t) = c_0 + c_+ e^{irt} + c_- e^{-irt} \end{array}\right. \quad \mathrm{II}$$

Eq $\mathcal{I}$ should actually be three eqs., that need to be solved independently (they must hold $\forall t$)

Let's fix $(a,b,c)_{0,+,-}$

$\boxed{\Omega=0}$ $\quad \mathcal{I} \Rightarrow \begin{cases} 0 = \frac{1}{2} W_{ab} b_0 + \frac{1}{2} W_{ac} c_0 \Rightarrow W_{ab} b_0 = -W_{ac} c_0 \\ 0 = a_0 \end{cases}$

For later convenience, we normalise the vectors separately

$$\begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix} = \begin{pmatrix} 0 \\ W_{ac}/2\hbar r \\ -W_{ab}/2\hbar r \end{pmatrix}$$

$\boxed{\Omega=r}$

$$\begin{cases} -\hbar r a_+ = \frac{1}{2} W_{ab} b_+ + \frac{1}{2} W_{ac} c_+ \\ -\hbar r b_+ = \frac{1}{2} W_{ab} a_+ \\ -\hbar r c_+ = \frac{1}{2} W_{ac} a_+ \end{cases}$$

$$\Downarrow$$

$$\begin{pmatrix} a_+ \\ b_+ \\ c_+ \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -\frac{W_{ab}}{2\hbar r} \\ -\frac{W_{ac}}{2\hbar r} \end{pmatrix}$$

$\boxed{\Omega=-r}$ same algebra

$$\begin{pmatrix} a_- \\ b_- \\ c_- \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \frac{W_{ab}}{2\hbar r} \\ \frac{W_{ac}}{2\hbar r} \end{pmatrix}$$

We can write the general solution as

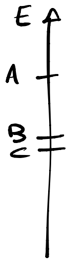
$$\begin{pmatrix} a(t) \\ b(t) \\ c(t) \end{pmatrix} = \alpha_0 \begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix} + \alpha_+ \begin{pmatrix} a_+ \\ b_+ \\ c_+ \end{pmatrix} e^{irt} + \alpha_- \begin{pmatrix} a_- \\ b_- \\ c_- \end{pmatrix} e^{-irt}$$

with the coeff. just determined.

At $t=0$

$$\begin{cases} a(0) = \dfrac{1}{\sqrt{2}}(\alpha_+ + \alpha_-) \\[2ex] b(0) = \dfrac{W_{ac}}{2\hbar r}\alpha_0 - \dfrac{W_{ab}}{2\sqrt{2}\hbar r}\alpha_+ + \dfrac{W_{ab}}{2\sqrt{2}\hbar r}\alpha_- \\[2ex] c(0) = \dfrac{W_{ab}}{2\hbar r}\alpha_0 - \dfrac{W_{ac}}{2\sqrt{2}\hbar r}\alpha_+ + \dfrac{W_{ac}}{2\sqrt{2}\hbar r}\alpha_- \end{cases}$$

## DARK STATES




→ suppose $E_c \simeq E_b$ and $W_{ab} = W_{ac}$

We have:

- interaction connects $a \leftrightarrow b$, $a \leftrightarrow c$
- $E_b \simeq E_c$
- $W_{ab} = W_{ac}$ same intensity

Let's take a linear combination of $|b\rangle$ and $|c\rangle$ state in "equal" superposition

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|b\rangle - |c\rangle) \Rightarrow \begin{array}{l} a(0) = 0 \\ b(0) = -c(0) = \frac{1}{\sqrt{2}} \end{array}$$

Solving for $(\alpha_0, \alpha_+, \alpha_-)$

$$\alpha_0 = \frac{W_{ab} + W_{ac}}{2\sqrt{2}\hbar r} = \frac{W_{ab}}{\sqrt{2}\hbar r}$$

$$\alpha_+ = \alpha_- = 0 \Rightarrow \boxed{a(t) = 0 \quad \forall t}$$ $|a\rangle$ is not an accessible state.

Reason is that the probability of jumping from $b \to a$ or $c \to a$ interfere destructively

$|a\rangle$ is a <u>dark state</u>

## Electromagnetically induced transparency

Suppose now that $|a\rangle$ and $|b\rangle$ are strongly coupled, while $|a\rangle$ and $|c\rangle$ are weakly coupled. $|W_{ab}| \gg |W_{ac}|$

Take $a(0) = b(0) = 0$, $|c(0)| = 1$ $\Longrightarrow \begin{cases} \alpha_- = -\alpha_+ \\ \alpha_+ = \dfrac{W_{ac}}{\sqrt{2}W_{ab}}\alpha_0 \end{cases}$

$c(0) = \dfrac{2\hbar r}{W_{ab}}\alpha_0$ we take $\alpha_0 = \dfrac{W_{ab}}{2\hbar r}$. $\Longrightarrow$

$$\Rightarrow a(t) = \frac{\alpha_+}{\sqrt{2}}\left(e^{irt} - e^{-irt}\right) = i\frac{W_{ac}}{2\hbar r}\sin(rt)$$

$$P_a(t) = |a(t)|^2 = \frac{W_{ac}^2}{W_{ac}^2 + W_{ab}^2}\sin^2(rt) \ll 1$$

State $|a\rangle$ cannot be excited by photons of that frequency

# Second order coefficients

$$i\hbar \frac{\partial}{\partial t} U_I(t,t_0) = V_I(t,t_0)\, U_I(t,t_0)$$

Formal solution $U_I(t,t_0) = \mathbb{I} - \frac{i}{\hbar}\int_{t_0}^{t} dt'\, V_I(t')\,\mathcal{U}_I(t',t_0)$

$$\mathcal{U}_I(t,t_0) = \mathcal{U}_I^{(0)}(t,t_0) + \mathcal{U}_I^{(1)}(t,t_0) + \mathcal{U}_I^{(2)}(t,t_0) + \cdots$$

$\mathcal{U}_I^{(0)}(t,t_0) = \mathbb{I}$ (Drop $V_I(t)$)

$$\mathcal{U}_I^{(0)}(t,t_0) + \mathcal{U}^{(1)}(t,t_0) + \mathcal{U}^{(2)}(t,t_0) + \cdots =$$

$$= \mathbb{I} - \frac{i}{\hbar}\int_{t_0}^{t} dt'\, V_I(t')\left(\mathcal{U}_I^{(0)}(t',t_0) + \mathcal{U}_I^{(1)}(t',t_0) + \mathcal{U}_I^{(2)}(t',t_0) + \cdots\right)$$

$$= \mathbb{I} - \frac{i}{\hbar}\int_{t_0}^{t} dt'\, V_I(t') - \frac{i}{\hbar}\int_{t_0}^{t} dt'\, V_I(t')\,\mathcal{U}_I^{(1)}(t',t_0) + O(V_I^3)$$

comparing $\Rightarrow \mathcal{U}_I^{(1)}(t,t_0) = -\frac{i}{\hbar}\int_{t_0}^{t} dt'\, V_I(t')$

and inserting back

$$\mathcal{U}_I^{2}(t,t_0) = \left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, V_I(t')\,V_I(t'')$$

$$c_f(t) = c_f^{(0)} + c_f^{(1)} + c_f^{(2)} + \cdots$$

$\hookrightarrow \langle f|\mathcal{U}_I(t,t_0)|i\rangle =$

$$= \langle f|\mathbb{I}|i\rangle + \quad = \delta_{fi} = c_f^{(0)}$$

$$+ -\frac{i}{\hbar}\int_{t_0}^{t} dt'\, \langle f|V_I(t')|i\rangle = -\frac{i}{\hbar}\int_{t_0}^{t} dt'\, e^{\frac{i}{\hbar}(E_f - E_i)t'} \langle f|V(t')|i\rangle = c_f^{1}(t)$$

$$= \left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, \langle f|V_I(t')\,V_I(t'')|i\rangle \;\; \star =$$

$$* = \left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \sum_n \langle f | V_I(t') | n \rangle \langle n | V_I(t'') | i \rangle =$$

$$= \left(-\frac{i}{\hbar}\right)^2 \sum_n \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \; e^{i\omega_{fn} t'} \; e^{-i\omega_{ni} t''} \langle f | V(t') | n \rangle \langle n | V(t'') | i \rangle$$

# Two level system in perturbation theory

$$H_0 = E_1 |1\rangle\langle 1| + E_2 |2\rangle\langle 2|$$

$$V(t) = \gamma e^{i\omega t} |1\rangle\langle 2| + \gamma e^{-i\omega t} |2\rangle\langle 1|$$

$$c_n^{(1)}(t) = -\frac{i}{\hbar}\int_{t_0}^{t} dt' \langle n| V_I(t')|i\rangle = -\frac{i}{\hbar}\int_{t_0}^{t} dt' \, e^{i\omega_{ni} t'} V_{ni}(t)$$

$$c_n^{(2)}(t) = \left(-\frac{i}{\hbar}\right)^2 \sum_m \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \, e^{i\omega_{nm} t'} e^{i\omega_{mi} t''} V_{nm}(t') V_{mi}(t'')$$

$$E_1 = \hbar\omega_1 \qquad E_2 = \hbar\omega_2$$

$$V_{11} = V_{22} = 0 \qquad V_{12} = \gamma e^{i\omega t} \qquad V_{21} = \gamma e^{-i\omega t}$$

At $t=0$, state $|1\rangle$

$\sim$

$$c_1^{(1)}(t) = 0 \qquad \text{I ORDER}$$

$$c_1^{(2)}(t) = \left(-\frac{i}{\hbar}\right)^2 \int_0^t dt' \int_0^{t'} dt'' \, e^{i\overset{-\omega_{21}}{\overset{\parallel}{\omega_{12}}} t'} e^{i\omega_{21} t''} \gamma^2 e^{i\omega t'} e^{-i\omega t''} =$$

$$= \left(-\frac{i}{\hbar}\right)^2 \gamma^2 \int_0^t dt' \int_0^t dt'' \, e^{-i\omega_{21}(t'-t'')} e^{i\omega(t'-t'')} =$$

$$= \left(-\frac{i}{\hbar}\right)^2 \gamma^2 \int_0^t dt' \int_0^{t'} dt'' \, e^{i(\omega_{12}+\omega)(t'-t'')} =$$

$$= \left(-\frac{i}{\hbar}\right)^2 \gamma^2 \left( \frac{1 - e^{-it(\omega-\omega_{21})} - it(\omega-\omega_{21})}{(\omega-\omega_{21})^2} \right) \qquad \text{II order}$$

notice that only <u>even</u> coefficient of the perturbative series are non zero for $c_1^{(n)}(t)$.

$$c_2^{(1)}(t) = -\frac{i}{\hbar}\int_0^t dt'\, e^{i\omega_{21}t'}\,\gamma\, e^{-i\omega t'} =$$

$$= -\frac{\gamma}{\hbar}\,\frac{\left(1-e^{-it(\omega-\omega_{21})}\right)}{\omega-\omega_{21}}$$ I ORDER

$$c_2^{(2)}(t) = 0$$ II ORDER

only odd coefficient are non zero for $c_2^{(n)}(t)$.

$$P_2(t) = \left|c_2^{(0)} + c_2^{(1)}\right|^2 = \frac{\gamma^2}{\hbar^2}\,\frac{2\left(1-\cos((\omega-\omega_{21})t)\right)}{(\omega-\omega_{21})^2} = \frac{4\gamma^2/\hbar^2}{(\omega-\omega_{21})^2}\sin^2\left(\left(\frac{\omega-\omega_{21}}{2}\right)t\right)$$

This is the result $O(\gamma^2)$ $\rightarrow$ $2\mathrm{Re}(c_2^{(2)*}\cdot c_2^{(0)})$ is $O(\gamma^2)$
but $c_2^{(2)} = 0$.

$$P_1(t) = 1 - P_2(t)$$

Now, if we compute $P_1(t)$ from $c_1^{(n)}$ and we use the same order we would get:

$$P_1(t) = |\underset{1}{c_1^{(0)}} + \underset{0}{c_1^{(1)}}|^2 = 1$$ but this looks inconsistent.

The problem is realising that this is only $\gamma^0$ accurate.

$$P_1(t) = |\underset{\sim\gamma^0}{c_1^{(0)}} + \underset{\gamma^1}{c_1^{(1)}} + \underset{\gamma^2}{c_1^{(2)}}|^2 = \underset{1}{|c_1^{(0)}|^2} + 2\mathrm{Re}(c_1^{(2)*}c_1^{(0)}) + O(\gamma^4)$$

0

$$= 1 - 2\mathrm{Re}\left(\frac{\gamma^2}{\hbar^2}\left(\frac{1-e^{-it(\omega-\omega_{21})} - it(\omega-\omega_{21})}{(\omega-\omega_{21})^2}\right)\right) =$$

$$= 1 - 2\,\frac{\gamma^2}{\hbar^2}\,\frac{1}{(\omega-\omega_{21})^2}\left(1-\cos(t(\omega-\omega_{21}))\right) = 1 - \frac{4\gamma^2/\hbar^2}{(\omega-\omega_{21})^2}\sin^2\left(\left(\frac{\omega-\omega_{21}}{2}\right)t\right)$$

consistent with $P_2(t)$ ✓

Let's focus on $P_2(t)$

$$P_2(t) = \frac{4\gamma^2/\hbar^2}{(\omega_{21}-\omega)^2} \sin^2\left(\frac{\omega-\omega_{21}}{2} t\right)$$

For $\omega$ very different from $\omega_{21}$ $P_2(t)$ oscillate very rapidly in time and it is also suppressed by $\frac{1}{(\omega_{21}-\omega)^2}$.

For $\omega \sim \omega_{12}$

$P_2(t) \sim \frac{\gamma^2 t^2}{\hbar^2}$ and after some time $P_2(t) > 1$.

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In class we computed $P_2(t)$ exactly (Rabi Formula)

$$P_2(t) = \frac{\gamma^2/\hbar^2}{\gamma^2/\hbar^2 + \left(\frac{\omega-\omega_{21}}{2}\right)^2} \sin^2\left(\sqrt{\frac{\gamma^2}{\hbar^2} + \left(\frac{\omega-\omega_{21}}{2}\right)^2}\, t\right)$$

which, if expanded to second order in $\gamma$, gives $P_2(t)$ as computed in perturbation theory.

In the resonant case: $P_2(t) \simeq \sin^2\left(\frac{\gamma}{\hbar} t\right)$ which does not grow as $t$.

but shows you that the perturbative calculation cannot be trusted for big $t$ in this case