Scattering Amplitudes in Quantum Field Theory SS 2022

Prof. Lorenzo Tancredi
https://www.ph.nat.tum.de/ttpmath/teaching/ss-2022/

Sheet 06: Multiple polylogarithms
Discussion: 29/07/2022

Introduction

A very important class of special functions that appear in many calculations of multiloop Feynman integrals are the so-called multiple polylogarithms (MPLs), defined as

$$G(a_1, a_2, \ldots, a_n; x) = \int_0^x \frac{\mathrm{d}t}{t - a_1} G(a_2, \ldots, a_n; t). \tag{1}$$

The vector $\vec{a} = (a_1, \ldots, a_n)$ constitutes the *indices* of the MPLs and is also called the *vector of singularities*. There are specific cases where MPLs can be written as more basic functions, such as logarithms, classical polylogarithms $\mathrm{Li}_n(x)$ and Nielsen polylogarithms $\mathrm{S}_{n,p}(x)$

$$G(\vec{0}_n; x) = \frac{1}{n!} \log^{(n)}(x) \tag{2}$$

$$G(\vec{a}_n; x) = \frac{1}{n!} \log^{(n)}\left(1 - \frac{x}{a}\right) \tag{3}$$

$$G(\vec{0}_{n-1}, a; x) = -\mathrm{Li}_n\left(\frac{x}{a}\right) \tag{4}$$

$$G(\vec{0}_n, \vec{a}_p; z) = (-1)^p \mathrm{S}_{n,p}\left(\frac{x}{a}\right). \tag{5}$$

In the following we will denote with $f(a; x) = \frac{1}{x-a}$ the integration kernel. We will also denote as $G(\vec{a}_w; x)$ a weight-w MPL, i.e. a MPL whose vector of singularities has w entries. Unless stated otherwise we will consider $0 < x < 1$.

Argument transformations, derivatives and analytic continuation

1. Show that
$$G(0, \vec{a}_{w-1}; 1 - x) = G(0, \vec{a}_{w-1}; 1) + \int_0^x \frac{\mathrm{d}y}{y - 1} G(\vec{a}_{w-1}; 1 - y) \tag{6}$$
and then use the transformation to express $G(0, 0, 1, 1; 1 - x)$ in terms of MPLs with argument x.

2. Show that
$$G\left(\vec{a}; \frac{1}{x}\right) = G(a_1, \vec{a}_{w-1}; 1) - \int_1^x \frac{\mathrm{d}}{t^2} f\left(a_1; \frac{1}{t}\right) G\left(\vec{a}_{w-1}; \frac{1}{t}\right). \tag{7}$$
Argue that we can in general assume that $a_1 \neq 1$. In this case $G(a_1, \vec{a}_{w-1}; 1)$ is a finite constant[1]. For this particular transformation, one must also pay attention to the branch points of the MPLs. Using $\log(-x) = \log(x) - i\pi$ for the analytic continuation of the logarithm, apply the above transformation to $G\left(0, -1, 1; \frac{1}{x}\right)$.

[1]When we say constant, we mean in the variable x. the vector $\vec{a}$ could in general depend on other variables.

3. Show that

$$G\Big(0,\vec{a}_{w-1};\frac{1-x}{1+x}\Big) = G(0,\vec{a}_{w-1};1) - \int_0^x \mathrm{d}t\Big(\frac{1}{t-1}+\frac{1}{t+1}\Big)G\Big(\vec{a}_{w-1};\frac{1-t}{1+t}\Big) \tag{8}$$

$$G\Big(-1,\vec{a}_{w-1};\frac{1-x}{1+x}\Big) = G(-1,\vec{a}_{w-1};1) - \int_0^x \frac{\mathrm{d}t}{t+1}G\Big(\vec{a}_{w-1};\frac{1-t}{1+t}\Big) \tag{9}$$

and use these transformations to express $G\big(-1,-1,1;\frac{1-x}{1+x}\big)$ in terms of MPLs with argument x.

4. Assuming that all arguments of the MPLs are generic, prove that

$$\mathrm{d}G(a_1;x) = \mathrm{dlog}\Big(\frac{x-a_1}{-a_1}\Big) \tag{10}$$

$$\mathrm{d}G(a_1,a_2;x) = G(a_2;x)\mathrm{dlog}\Big(\frac{x-a_1}{a_2-a_1}\Big) + G(a_1;x)\mathrm{dlog}\Big(\frac{a_1-a_2}{-a_2}\Big). \tag{11}$$

What is the total differential of a weight three MPL, i.e. $\mathrm{d}G(a_1,a_2,a_3;x)$? Starting from the explicit cases above, can you write down a general formula for the total differential of a MPL of arbitrary weight?

5. Consider the solution of the canonical massive one loop bubble in $D=4-2\epsilon$ dimensions up to weight two,

$$I(x,\epsilon) = \epsilon\, G(0;x) + \epsilon^2\big[G(0,0;x) - 2G(-1,0;x)\big]. \tag{12}$$

To write the analytic result, we used the Landau variable defined in the euclidean region

$$p^2 = m^2\frac{(1-x)^2}{x}, \quad \text{where} \quad p^2 \geq 0 \;\;\rightarrow\;\; 0 \leq x \leq 1\,.$$

We would like now to continue the result *above threshold*, i.e. for $s=-p^2 \leq 4m^2$. To do that, we continue $x \rightarrow -y \pm i\delta$, where δ is a small, positive imaginary part, such that

$$s = m^2\frac{(1+y)^2}{y}, \quad \text{where} \;\; s \geq 4m^2 \;\;\rightarrow\;\; 0 \leq y \leq 1\,.$$

(a) Determine the sign of the imaginary part associated to y, by using the fact that s must take a positive imaginary part when above threshold $(s+i\delta)$.

(b) With the proper sign for the imaginary part, perform the analytic continuation for the one-loop bubble and prove that

$$I(y,\epsilon) = \epsilon\big[G(0;y) - i\pi\big] + \epsilon^2\Big[-i\pi G(0;y) + 2i\pi G(1;y) + G(0,0;y) - 2G(1,0;y) - \frac{\pi^2}{2}\Big]. \tag{13}$$