Scattering Amplitudes in Quantum Field Theory SS 2022

Prof. Lorenzo Tancredi
https://www.ph.nat.tum.de/ttpmath/teaching/ss-2022/

Sheet 03: Differential equations for Feynman integrals
Discussion: 24/06/2022

Introduction

In the lectures you have seen that one-loop amplitudes can be written as a linear combination of specific Feynman integrals, usually called *master integrals*. These are the box, the triangle, the bubble and the tadpole. In this exercise we will apply the method of differential equations, which relies on Integration-By-Part (IBP) identities, to solve scalar one-loop master integrals in $D = 4 - 2\epsilon$ spacetime dimensions.

Master integrals for $gg \to H$ via a top quark loop

As a working example, we consider the integrals appearing in the amplitude for the production of a Higgs boson in gluon fusion, $gg \to H$. They can be incorporated into a single *integral family*

$$I(a_1, a_2, a_3) = \int \frac{\mathrm{d}^D k}{i\pi^{D/2}} \frac{\mathrm{e}^{\epsilon\gamma_E}}{[k^2 - m^2]^{a_1}[(k-p_1)^2 - m^2]^{a_2}[(k-p_1-p_2)^2 - m^2]^{a_3}}. \tag{1}$$

All external particles are considered incoming and the kinematics is specified as follows:

$$p_1^2 = 0,\ p_2^2 = 0,\ p_1 \cdot p_2 = s_{12}/2\,.$$

1. Starting from (1) and the general IBP formula

$$\int \frac{\mathrm{d}^D k}{i\pi^{D/2}} \left[\frac{\partial}{\partial k^\mu} \frac{u^\mu}{[k^2 - m^2]^{a_1}[(k-p_1)^2 - m^2]^{a_2}[(k-p_1-p_2)^2 - m^2]^{a_3}} \right] = 0 \tag{2}$$

derive the relevant IBP identities for $u^\mu = k^\mu, p_1^\mu, p_2^\mu$. You should find

$$\begin{aligned}
u^\mu = k^\mu: &\quad (\mathrm{D} - 2a_1 - a_2 - a_3)\,\mathbb{1} - a_2 1^- 2^+ - a_3 1^- 3^+ - 2a_1 1^+ m^2 \\
&\quad - 2a_2 2^+ m^2 - 2a_3 3^+ m^2 + a_3 s_{12} 3^+ \qquad (3)\\
u^\mu = p_1^\mu: &\quad (-a_1 + a_2)\mathbb{1} + a_1 2^- 1^+ - a_2 1^- 2^+ - a_3 1^- 3^+ + a_3 2^- 3^+ + a_3 s_{12} 3^+ \qquad (4)\\
u^\mu = p_2^\mu: &\quad (-a_2 + a_3)\mathbb{1} + a_2 3^- 2^+ - a_1 2^- 1^+ + a_1 3^- 1^+ - a_3 2^- 3^+ - a_1 s_{12} 1^+ \qquad (5)
\end{aligned}$$

where the three equations should be interpreted as operator equations acting on a generic integral o f the family under consideration. In particular, $\mathbb{1}$ is the identity operator, while $j^\pm$ increases or reduces the power of the corresponding propagator

$$\begin{aligned}
\mathbb{1} I(a_1, a_2, a_3) &= I(a_1, a_2, a_3) \\
j^\pm I(a_1, ..., a_j, ..., a_3) &= I(a_1, ..., a_j \pm 1, ..., a_3) \quad \text{with} \quad j = 1, 2, 3\,.
\end{aligned} \tag{6}$$

2. Using the ibps derived above, prove that the integral family defined in (1) has three master integrals, which can be chosen to be

$$\mathbf{I} = \{I(0,0,1), I(1,0,1), I(1,1,1)\}. \tag{7}$$

3. In general, since the master integrals are scalar, they can only depend on all possible scalar products $p_i \cdot p_j$ between the external momenta. Show that

$$\frac{\partial}{\partial s_{12}} = \frac{1}{2s_{12}} \left(p_1^\mu \frac{\partial}{\partial p_1^\mu} + p_2^\mu \frac{\partial}{\partial p_2^\mu} \right). \tag{8}$$

4. Apply the differential operator from the previous step to the master integrals (7) and compute the derivative with respect to s_{12}. Compute also the derivative of the master integrals with respect to m^2. Show that the result of the differentiation can be written in terms of integrals that belong to family (1),

$$\partial_{s_{12}} I(0,0,1) = \frac{\mathrm{I}(0,-1,2)}{s_{12}} - \frac{\mathrm{I}(0,0,1)}{s_{12}} \tag{9}$$

$$\partial_{s_{12}} I(1,0,1) = \frac{\mathrm{I}(1,-1,2)}{s_{12}} - \frac{\mathrm{I}(1,0,1)}{s_{12}} \tag{10}$$

$$\partial_{s_{12}} I(1,1,1) = \frac{\mathrm{I}(1,0,2)}{s_{12}} - \frac{\mathrm{I}(1,1,1)}{s_{12}} \tag{11}$$

$$\partial_{m^2} I(0,0,1) = \mathrm{I}(0,0,2) \tag{12}$$

$$\partial_{m^2} I(1,0,1) = \mathrm{I}(1,0,2) + \mathrm{I}(2,0,1) \tag{13}$$

$$\partial_{m^2} I(1,1,1) = \mathrm{I}(1,1,2) + \mathrm{I}(1,2,1) + \mathrm{I}(2,1,1). \tag{14}$$

5. Use IBP identities (3), (4), (5) to recast the result of the differentiation in terms of the master integrals (7). The goal is to obtain an expression of the form

$$\partial_{s_{12}} \mathbf{I} = M_{s_{12}} \mathbf{I}, \quad \partial_{m^2} \mathbf{I} = M_{m^2} \mathbf{I}. \tag{15}$$

6. Compute $(s_{12}\partial_{s_{12}} + m^2 \partial_{m^2})\mathbf{I}$. What does the result tell you?

7. To determine the unique physical solution to these differential equations we need to specify suitable boundary terms. Focus on the differential equation of s_{12} and identify its poles. Using the fact that the chosen master integrals (7) are finite in the limit $s_{12} \to 0$, argue that at that limit the relevant boundary terms for $\{I(1,0,1), I(1,1,1)\}$ are

$$I(1,0,1)|_{s_{12}\to 0} = -\frac{(\epsilon - 1)\mathrm{I}(0,0,1)}{m^2} \tag{16}$$

$$I(1,1,1)|_{s_{12}\to 0} = \frac{(\epsilon - 1)\epsilon \mathrm{I}(0,0,1)}{2\left(m^2\right)^2} \tag{17}$$

   where

$$I(0,0,1) = -e^{\epsilon\gamma}\Gamma(\epsilon - 1)\left(m^2\right)^{1-\epsilon}. \tag{18}$$

8. Instead of working with the integrals basis (7), switch to the following basis of master integrals,

$$
\begin{aligned}
f_1 =& \epsilon (m^2)^\epsilon I(2,0,0) \\
f_2 =& \epsilon s_{12} (m^2)^\epsilon \sqrt{1 - \frac{4m^2}{s_{12}}} I(1,0,2) \\
f_3 =& \epsilon s_{12} (m^2)^\epsilon I(1,1,1).
\end{aligned}
\tag{19}
$$

Use IBP identities to express $I(2,0,0), I(1,0,2)$ in terms of $I(0,0,1), I(1,0,1)$, so that $\mathbf{f} = \mathbf{T}\mathbf{I}$, where $\mathbf{f} = \{f_1, f_2, f_3\}$ and $\mathbf{T}$ is the transformation matrix that connects the two bases.

9. The square root which appears in the differential equation of the previous step can be rationalised by the change of variables

$$
\frac{m^2}{-s_{12}} = \frac{x}{(1-x)^2}. \tag{20}
$$

Perform this change of variables assuming for $0 < x < 1$: what does this correspond in terms of the original variable s_{12}, which analytical region does it represent? Verify that in this new variable the differential equation takes the form

$$
\partial_x \mathbf{f} = \epsilon \left(\frac{\mathbf{a}}{x} + \frac{\mathbf{b}}{x+1} \right) \mathbf{f} \tag{21}
$$

with $\mathbf{a}, \mathbf{b}$ being numerical matrices.

10. Assuming a power series solution for (21), i.e. $f_i = \sum_{n \geq 0} \epsilon^n f_i^{[n]}$, solve (21) using the boundary conditions that you obtained in previous steps. You should find

$$
f_1 = 1 + \frac{\pi^2 \epsilon^2}{12} \tag{22}
$$

$$
f_2 = \epsilon \log(x) + \epsilon^2 \left(-2\text{Li}_2(-x) + \frac{\log^2(x)}{2} - 2\log(x+1)\log(x) \right) \tag{23}
$$

$$
f_3 = -\frac{1}{2}\epsilon^2 \log^2(x) \tag{24}
$$

where the dilogarithm $\text{Li}_2(-x)$ is defined as

$$
\text{Li}_2(-x) = -\int_0^x \frac{\log(1+t)}{t}. \tag{25}
$$