Advanced Methods for Collider Physics

WS 2024

Lorenzo Tancredi

Motivation: why a course on advanced methods for collider physics "today"?

Colliders are most naive & EFFECTIVE way to discover what our world is made of at the most fundamental level

$\Rightarrow$ historically extremely successful

EXAMPLES of most recent discoveries

1983 Z, W bosons @ $Sp\bar{p}S$, CERN (630 GeV)

1995 top quark @ Tevatron, FERMILAB (2 TeV)

2012 Higgs boson @ LHC, CERN (7–13.6 TeV)

$\downarrow$

?

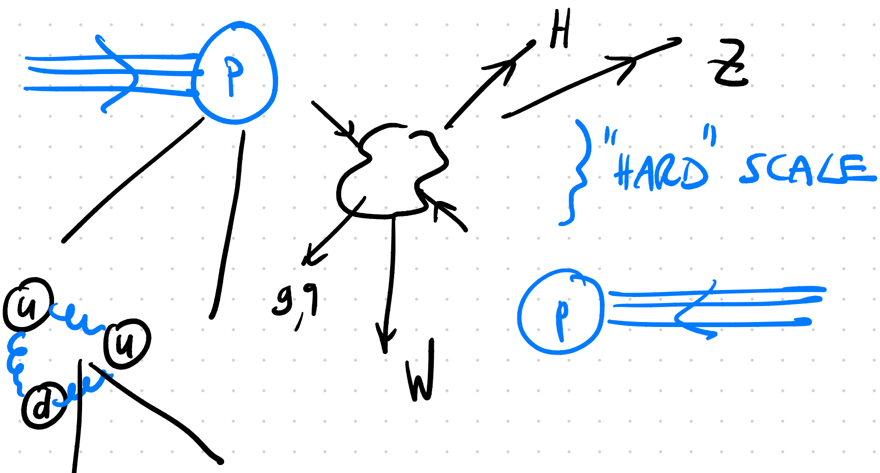
S.M. is "complete" as far as particle content

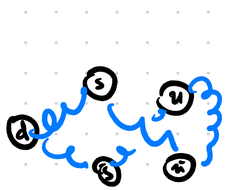
OPEN QUESTIONS $\left\{ \begin{array}{l} \text{Dark Matter, Dark Energy} \\ \text{Neutrino masses} \\ \text{Details of SSB, strong CP, ...} \end{array} \right.$

Discoveries often done at hadron colliders ⇒ they allow to reach next energy frontier

CONS Colliding "non-elementary" particles (QCD bound states) means much more "clutter" from "strong interactions". Collisions are less clear and more difficult to interpret

LHC collides protons @ $\lesssim$ 14 TeV





Scale-invariance breaking
the proton appears different
depending on energy at which is probed

⇒ to understand collider physics & model it with high precision we need to UNDERSTAND STRONG INTERACTIONS ACROSS DIFF. SCALES ⇒ different "physics", different methods developed to handle it.

In this course we will go deep into some of these methods - focus on

(*1) "Hard Scattering", High-energy collision that happens among elementary particles

⇒ QFT methods, Scattering Amplitudes, Feynman integrals ⇒ DIFF. EQS.

(*2) Combination of "Amplitudes" into IR-finite physically well-defined OBSERVABLES @ FIXED ORDER

⇒ IR divergences, subtraction & Factorization

(*3) Modelling realistic processes @ LHC

⇒ PDFs, MC integration, Jets & Hadrons, Parton showers, Fragmentation

A "diverse" set of lecturers with different expertise

Today we start looking at Building blocks of Fixed Order calculations ⇒ Scattering Amplitudes & Feynman Integrals

We work in QCD = Quantum Chromodynamics

YM theory based on SU(3) gauge group

$$\mathcal{L}_{QCD} = -\frac{1}{2} \mathrm{Tr}\left[F^{\mu\nu} F_{\mu\nu} \right] + \sum_{j=1}^{N_f} \bar{\psi}_{q_j} (i \not{D} - m_j) \psi_{q_j}$$

N_f quark flavours in SU(3) Fundamental representation

Typically $m_j = 0$

$$F^{\mu\nu} = F^{\mu\nu A} T^A$$

Fundam. generators

$$F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu - g_s f^{ABC} A^B_\mu A^C_\nu$$

$$D_\mu = \partial_\mu + i g_s A^A_\mu T^A$$

$$\mathrm{Tr}[T^A T^B] = T_F \delta^{AB} = \frac{1}{2} \delta^{AB}$$

$$\alpha_s = \frac{g_s^2}{4\pi}$$

4

QCD Feynman Rules $\quad \mathcal{L} = \mathcal{L}_{QCD} + \mathcal{L}_{GF}$

a] $\mathcal{L}_{GF} = -\frac{1}{2\lambda} \sum_A |\partial^\mu A^A_\mu|^2$ covariant gauges ($\lambda = \xi$)

We won't derive them here, but just list them

PROPAGATORS

Adjoint

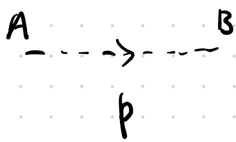
gluon = 

$$= \frac{i\delta^{AB}}{k^2+i\varepsilon}\left[-g^{\mu\nu} + (1-\lambda)\frac{k^\mu k^\nu}{k^2+i\varepsilon}\right]$$

Fundamental

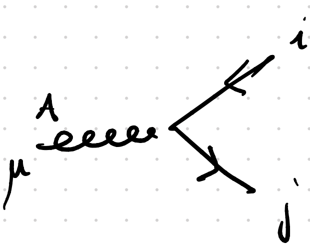
quark = 

$$= \frac{i\,\delta_{ij}}{\not{p} - m + i\varepsilon} \qquad \not{p} = p^\mu \gamma_\mu$$

ghost = 

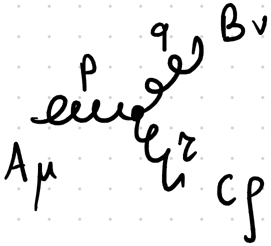
$$= \frac{i\,\delta_{AB}}{p^2+i\varepsilon}$$

VERTICES

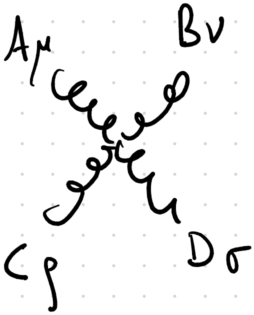


$$= -i g_s T^A_{ji}\,\gamma^\mu$$

"Like in QED"



$$= -g_s f^{ABC}\left[(p-q)^\rho g^{\mu\nu} + (q-r)^\mu g^{\nu\rho} + (r-p)^\nu g^{\rho\mu}\right]$$



$$= -i g_s^2 f^{XAC} f^{XBD}\left[g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}\right]$$

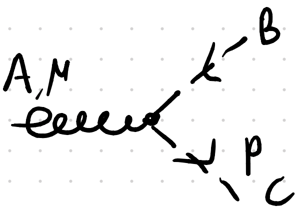
$$-i g_s^2 f^{XAD} f^{XBC}\left[g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}\right]$$

$$-i g_s^2 f^{XAB} f^{XCD}\left[g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right]$$

[new vertices due to

$$F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu - g_s f^{ABC} A^B_\mu A^C_\nu$$

]

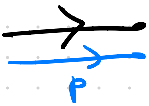


$$= g_s f^{ABC} p^\mu$$

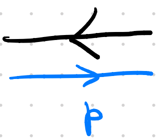
ghost vertex

WAVE FUNCTIONS

$\varepsilon^{\mu}_{A}(k)$ gluon polarisation



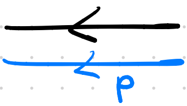
$u(p)$ Incoming quark



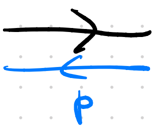
$\bar{v}(p)$ incoming anti quark

($= \bar{u}(p)$ if quark massless!)

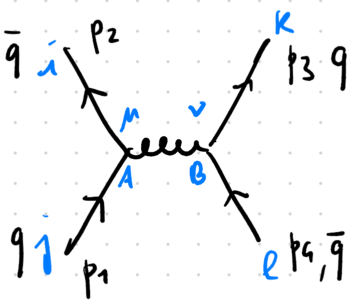
outgoing quark



$\bar{u}(p)$ outgoing quark



$v(p)$ outgoing anti quark



$$= \bar{v}(p_2)\left(-i g_s T^{A}_{ij}\gamma^{\mu}\right) u(p_1)\, D^{AB}_{\mu\nu}(q)$$

$$\bar{u}(p_3)\left(-i g_s T^{B}_{kl}\gamma^{\nu}\right) v(p_4)$$

ALTERNATIVE : quantize in "non covariant" gauge

b] $$\mathcal{L}_{GF} = -\frac{1}{2\lambda}\sum_A |n^\mu A^A_\mu|^2$$

is non-covariant (depends on direction n^μ)

if n^μ spacelike ($n^2<0$) = AXIAL GAUGE

if n^μ light like ($n^2=0$) = LIGHT-LIKE GAUGE

if n^μ time-like ($n^2>0$) = COULOMB GAUGE

In practice, people call all of them Axial Gauges (AG)

PROPAGATOR GLUON / PHOTON CHANGES

$$D^{AB}_{\mu\nu}(k) = \frac{i}{k^2+i\varepsilon}\left[-g_{\mu\nu}+\frac{n_\mu k_\nu + n_\nu k_\mu}{n\cdot k} - n^2\frac{k_\mu k_\nu}{n\cdot k}\right]\delta^{AB}$$

for arbitrary n^μ introduces an extra "non-physical" singularity!

most used light like $n^2=0$

But the pro is, ghosts decouple and can be ignored as in QED!

Depending on what we do, it might be more or less useful to use covariant or axial gauges

Gluon Propagator in axial (light-like) gauge

$$D^{AB}_{\mu\nu}(k) = \frac{i}{k^2+i\varepsilon}\left[-g_{\mu\nu} + \frac{n_\mu k_\nu + n_\nu k_\mu}{n\cdot k}\right]\delta^{AB}$$

We see that

$$k^\mu D^{AB}_{\mu\nu} = k^\nu D^{AB}_{\mu\nu} = 0$$

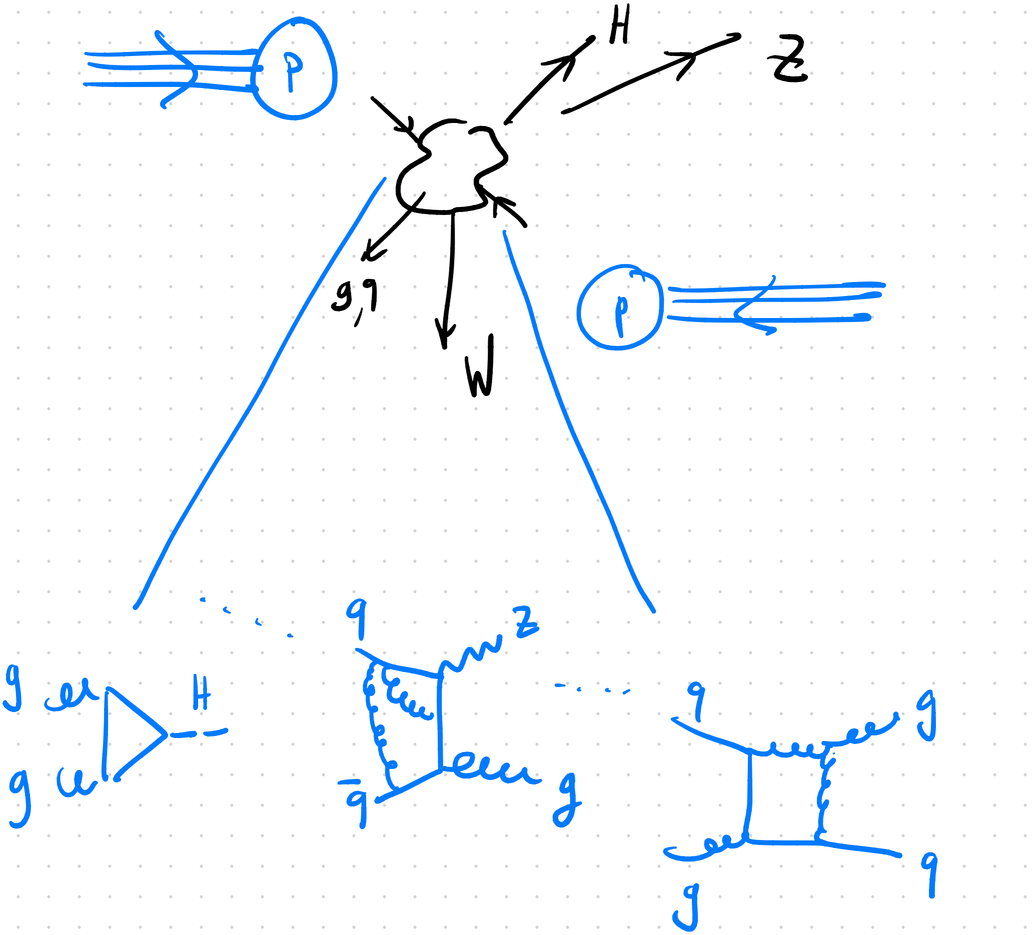
$$n^\mu D^{AB}_{\mu\nu} = n^\nu D^{AB}_{\mu\nu} = 0$$

$D_{\mu\nu}$ propagates only 2 physical polarization states $\Rightarrow$ orthogonal to k^μ & n^μ!

which is the reason why ghosts are not needed!

SCATTERING AMPLITUDES in QCD

With this let's go dive deep into hard scattering



$$\sigma_{partonic} = \int d\Phi \, |A|^2$$

scattering amplitude

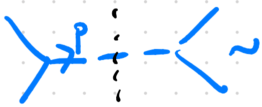
Some important facts

1] Amplitudes are complex functions, like in Quantum Mechanics

$\Rightarrow$ interference patterns crucial quantum phenom.

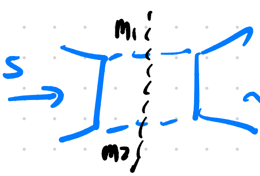
2] Amplitudes are on-shell, amputated correlators (LSZ formula)

$\Rightarrow$ they have poles & branch cuts in complex plane: poles $\Rightarrow$ single-particle intermediate states



$\sim \frac{1}{p^2}$ not necessarily elementary!

branch cuts $\Rightarrow$ multi-particle intermediate states



$\sim \ln(s-(m_1+m_2)^2)$

or $(s-(m_1+m_2)^2)^{1/2}$

3] Amplitudes are "ALMOST" Lorentz invariant

$\Rightarrow$ they transform according to Little group of external particles

Massless $\Rightarrow$ (I) $SO(2) \sim U(1) \sim e^{ih}$

$p^\mu = (E, 0, 0, E)$

h is helicity

$\pm \frac{1}{2}$ massless fermions

± 1 massless bosons

etc

Massive $\Rightarrow$ $SO(3) \sim SU(2)$ $\quad l = \frac{n}{2}$

$p^\mu = (m, \vec{0})$

$m = -\frac{n}{2}, -\frac{n}{2}+1, \dots$

$\dots + \frac{n}{2}$

$(2l+1)$ states

Crucial role played by polarization vectors / spinors $\Rightarrow$ they bi-transform

① Lorentz ② Little group

Example 1] $g(p_1) + g(p_2) \to H(q)$

$$A_{gg\to H} = \varepsilon_\mu(p_1)\, \varepsilon_\nu(p_2)\, A^{\mu\nu}$$

Lorentz indices contracted

"scalar under Lorentz"

but ε_μ still transforms under little group

Example 2] $q(p_1) + \bar{q}(p_2) \to g(p_3) + g(p_4)$

$$A_{q\bar{q}gg} = \underbrace{\bar{u}(p_2)\, \Gamma^{\mu\nu}\, u(p_1)}_{\text{scalar in spinor space}}\, \varepsilon_\mu(p_3)\, \varepsilon_\nu(p_4)$$

still transforms under little group due to $u, \bar{u}$!

⇒ Spinor helicity formalism makes this manifest, we won't use it here because it works well only for massless amplitudes and we want to be more general!

TENSOR DECOMPOSITION

We typically compute amplitudes in perturbation theory (tree-level, one-loop, ...)

There are FEW things, nevertheless, that we can say about amplitudes and that are true to all orders.

⇒ Parametrize their most general form allowed by all "symmetries"

- Lorentz + Little Group
- Covariance under SU(3) [Color]
- Gauge "covariance"
- Bose symmetry ...

Focus on Lorentz / Little Group

$\Rightarrow$ Tensor / Form Factor Decomposition

Start with our two examples 1. $gg \to H$

2. $q\bar{q} \to gg$

1. $A_{ggH} = \varepsilon_\mu(p_1)\, \varepsilon_\nu(p_2)\, A^{\mu\nu}$

what can we say about $A^{\mu\nu}$?

. rank-2 Lorentz tensor

. theory CP invariant $\Rightarrow$ $p_1^\mu,\ p_2^\mu,\ g^{\mu\nu}$

$$A^{\mu\nu} = F_{11}\, p_1^\mu p_1^\nu + F_{12}\, p_1^\mu p_2^\nu + F_{21}\, p_2^\mu p_1^\nu + F_{22}\, p_2^\mu p_2^\nu$$

$$+ F_{00}\, g^{\mu\nu}$$

5 objects $\{ p_i^{\mu} p_j^{\nu} \, ; \, g^{\mu\nu} \}$ are a "basis"

in a vector space, $A^{\mu\nu}$ lives in this space.

vectors & dual vectors $\{ p_i^{\mu} p_j^{\nu} , g^{\mu\nu} \}$ SAME!

METRIC in vector space $\boxed{g_{\mu\rho} \, g_{\nu\sigma}}$ scalar product!

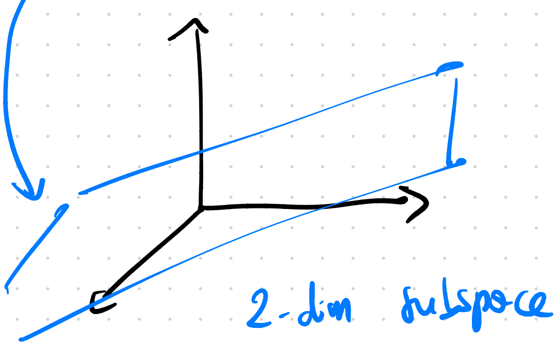
Do we know more? yes!

a) gluons are on-shell $\Rightarrow$ $\varepsilon_1 \cdot p_1 = 0$

$\varepsilon_2 \cdot p_2 = 0$

$$A^{\mu\nu} = \underbrace{\left(F_{00} \, g^{\mu\nu} + F_{21} \, p_2^{\mu} p_1^{\nu} \right)}_{\tilde{A}^{\mu\nu}} + \underbrace{A^{\mu\nu}_{n.p}}_{\text{rest}}$$

$$A^{\mu\nu}_{n.p.} \varepsilon_{1\mu} \varepsilon_{2\nu} = 0$$



I can constrain myself to work on this subspace by modifying METRIC

16

b) Further, I know that only physically relevant part of $\tilde{A}_{\mu\nu}$ is the one that fulfils WARD IDENTITIES

$$\tilde{A}^{\mu\nu} \cdot p_{1\mu} \varepsilon_{2\nu} = \tilde{A}^{\mu\nu} \tilde{p}_{2\nu} \varepsilon_{1\mu} = 0$$

$$\Rightarrow \left(F_{00} + F_{21}\, p_1 \cdot p_2 \right) \varepsilon_2 \cdot p_1 = 0$$

$$F_{00} = - p_1 \cdot p_2 \, F_{21}$$

$$A^{\mu\nu}_{phys} = F \left(- g^{\mu\nu} p_1 \cdot p_2 + p_1^{\mu} p_2^{\nu} \right)$$

$$A^{\mu\nu} = \underset{\uparrow}{A^{\mu\nu}_{phys}} + A^{\mu\nu}_{n.p}$$

"one-dimensional" in tensor space

from $5 \to 1$ tensors / form factor

$A^{\mu\nu}_{phys}$ so constructed is "gauge covariant"

$b - v_2]$ notice, equivalent result could have been achieved FIXING GAUGE of two gluons

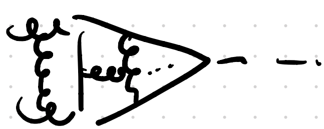
for example $\varepsilon_1 \cdot p_2 = \varepsilon_2 \cdot p_1 = 0$

this would have allowed me to write

$$A^{\mu\nu} = A^{\mu\nu}_{ALT} + A^{\mu\nu}_{\perp}$$

$$A^{\mu\nu}_{ALT} = F_{00}\, g^{\mu\nu} \quad ; \quad A^{\mu\nu}_{\perp} \cdot \varepsilon_{1\mu} \varepsilon_{2\nu} = 0$$

$\Rightarrow$ how do I get to this from the practice?

At l-loops  $= \int \prod_{j=1}^{L} \frac{d^D k_j}{(2\pi)^D} \frac{\mathcal{N}}{D_1 \dots D_N}$

$$\mathcal{N} = \varepsilon_1^{\mu} \varepsilon_2^{\nu} T_{\mu\nu} \qquad T_{\mu\nu} = \{ k_\mu, k_\nu, p_{1\mu}, p_{2\nu}]$$

arbitrary ways

Computing tensor integrals is complicated in general

$$\Rightarrow \quad \int \frac{d^D k}{(2\pi)^D} \frac{k^\mu k^\nu}{D_1 \cdots D_N} \quad \text{etc}$$

it is useful, once general form is known, to "project out" directly the scalar form factors that are physically relevant

PROJECTOR-FORM FACTOR METHOD (in 't HV)

Using "dual vectors", I can build a "projector operator", which is just a combination of dual vectors that singles out F_{00}, or F in previous representation

$$\underline{P \cdot A = F_{00}}$$

$\cdot$ = scalar product !

SUBTLETY: in general P built out of ALL 5
if I want to use STANDARD SCAL. PRODUCT

on full space (5 dimensional)

$$v = T^{\mu\nu} = \{ p_i^\mu p_j^\nu ,\ g^{\mu\nu} \} \ ; \quad v^* = T^{\mu\nu} = \{ \dots \}$$

$$\langle v^*, v \rangle = T^{\mu\nu} \left[g_{\mu\rho}\, g_{\nu\sigma} \right] T^{\rho\sigma}$$

on 1-dim subspace I need a new metric that lives on that space

$\Rightarrow$ remember this is what num of gluon prop in AXIAL gauge does!

$$g_{\mu\rho}\, g_{\nu\sigma} \longmapsto \left[-g_{\mu\rho} + \frac{p_{1\mu} n_{1\rho} + p_{1\rho} n_{1\mu}}{p_1 \cdot n_1} \right]$$

$$\times \left[-g_{\nu\sigma} + \frac{p_{2\nu} n_{2\sigma} + p_{2\sigma} n_{2\nu}}{p_2 \cdot n_2} \right]$$

$$\forall\ n_1^\mu, n_2^\nu \neq p_1^\mu, p_2^\nu \ \ \& \ \ n_1^2 = n_2^2 = 0$$

by fixing $n_1^\mu = p_2^\mu$ & $n_2^\mu = p_1^\mu$ I am selecting EXACTLY the 1-dim subspace $\supset A^{\mu\nu}_{ALT}$

$\Rightarrow$ in general, I would usually have to do this also for "gauge covariant" version, and in this case any n_1^μ, n_2^μ would work. Nevertheless in this case its not needed because $(-g^{\mu\nu} p_1 \cdot p_2 + p_1^\mu p_2^\nu)$ is by construction ORTHOGONAL to all other tensors!, adding $n_1^\mu p_1^\nu + n_1^\nu p_1^\mu$ etc has NO EFFECT !

In this case, convenient to proceed with

$$A^{\mu\nu} = F(-g^{\mu\nu} p_1 \cdot p_2 + p_1^\nu p_2^\mu)$$

$$\mathcal{P}^{\mu\nu} = c\left[-g^{\mu\nu}\, p_1\cdot p_2 + p_1^{\mu} p_2^{\mu}\right]$$

$$\mathcal{P}\cdot A = \mathcal{P}^{\mu\nu}\left[g_{\mu\rho}\, g_{\nu\sigma}\right] A^{\rho\sigma}$$

$$= c\cdot F\left[+D\,(p_1\cdot p_2)^2 - (p_1\cdot p_2)^2 - (p_1\cdot p_2)^2 + \cancel{p_1^2\, p_2^2}\right]$$

$$= c\cdot F\cdot (D-2)\,(p_1\cdot p_2)^2 \overset{!}{=} F$$

$$c = \frac{1}{(D-2)(p_1\cdot p_2)^2} = \frac{4}{(D-2)\left[q^2\right]^2}$$

$$p_1\cdot p_2 = \frac{1}{2}(p_1+p_2)^2 = \frac{1}{2}q^2$$

$$\mathcal{P} = \left[\frac{2}{D-2}\right]\frac{1}{q^2}\left(-g^{\mu\nu} + \frac{2\,p_1^{\mu} p_2^{\mu}}{q^2}\right)$$

$\Rightarrow$ D or 4 dimensions?

we do this algebra in D dimensions because loop integrals are typically divergent!

Still remember $\Rightarrow$ $A^{\mu\nu} \cdot \underbrace{\epsilon_\mu \epsilon_\nu}$

4-dim external states
ie 't Hooft - Veltman
regularization scheme!

Can we EXPLOIT THIS, similarly to the fact that $\epsilon_\mu \epsilon_\nu$ "cut out" some non-physical tensors due to ward identities etc?

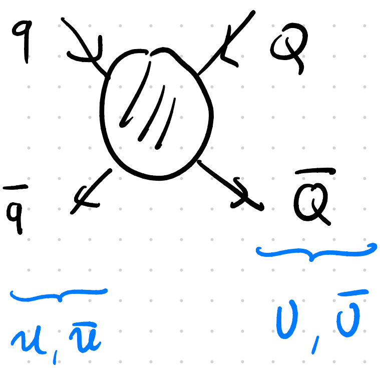
$\Rightarrow$ nothing to improve for $gg \to H$ (we are already down to 1 single Form Factor) but when dealing with $2 \to n$ scattering, it can reveal to be very useful [Peraro, Tancredi 19, 20]

Let's consider a case where the problem is especially "visible".

$q\bar{q} \to Q\bar{Q}$ scattering, two flavours of

MASSLESS QUARKS

$$q(p_1) + \bar{q}(p_2) \to Q(p_3) + \bar{Q}(p_4)$$



All incoming

$$A_{q\bar{q}Q\bar{Q}} \sim \bar{u}(p_2)\,\Gamma_1\,u(p_1)\,\bar{U}(p_4)\,\Gamma_2\,U(p_3)$$

For example:

$$\bar{u}(p_2)\,\gamma^\mu\,u(p_1)\,\bar{U}(p_4)\,\gamma_\mu\,U(p_3)$$

$$\bar{u}(p_2)\,\not{p}_3\,u(p_1)\,\bar{U}(p_4)\,\not{p}_1\,U(p_3)$$

$$\bar{u}(p_2)\, \not{p}_3 \gamma^\mu \gamma^\nu\, u(p_1)\;\; \bar{U}(p_4)\, \gamma_\mu \gamma_\nu\, U(p_3)$$

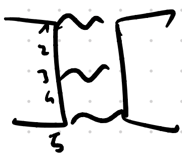
$$\bar{u}(p_2)\, \gamma^\mu \gamma^\nu \gamma^\rho\, u(p_1)\;\; \bar{U}(p_4)\, \gamma_\mu \gamma_\nu \gamma_\rho\, U(p_3)$$

etc

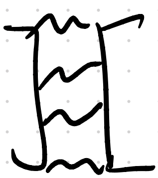
when do I stop ?

$\Rightarrow$ if q & Q both monless, I can never have $\Gamma_j \sim$ even number of γ^μ $\Rightarrow$ helicity conservation !

$\Rightarrow$ Apart from this to compute this amplitude to l-loops in CDR eventually all these structures can contribute !



$\Rightarrow$ 2 loops $\Rightarrow$ 5 γ^μ per line etc



$\Rightarrow$ 3 loops $\Rightarrow$ 7 γ^μ per line etc these are all independent if μ is a D-dim index

$\Rightarrow$ γ-algebra is not closed in D dimensions!

BUT : we can work in 't Hooft - Veltman !

$u, \bar{u}, U, \bar{U}$ live in 4-dim space

only 4-dim component of γ^μ matters

for external states

then STARTING from $2 \to 2$ I can build

a BASIS for the 4-dim space as follows

$$\underbrace{p_1^\mu, p_2^\mu, p_3^\mu + w^\mu}_{} = \varepsilon^{p_1 p_2 p_3 \mu}$$

AXIAL VECTOR ORTHOGONAL TO p_1 p_2 p_3

$$\gamma^\mu = \sum_{i=1}^{3} \alpha_i \, p_i^\mu + \beta \, w^\mu$$

$\alpha_i, \beta \sim \{ \not{p}_i, \not{w} \}$

immaterial explicit expression!

this is an easy way to see that any

$$\bar{u}(p_2)\gamma^\mu\gamma^\nu\cdots\gamma^\rho u(p_1) \to \bar{u}(p_2)\not{q}_i\not{q}_j\cdots\not{q}_k u(p_1)$$

where $q_i^\mu = \{p_1^\mu, p_2^\mu, p_3^\mu, W^\mu\}$

then using $\bar{u}(p_2)\not{p}_2 = \not{p}_1 u(p_1) = 0$

& $p_4 = -(p_1+p_2+p_3)$ we see that only 2

possibilities remain

$\bar{u}(p_2)\not{p}_3 u(p_1)$ $\Rightarrow$ $\not{p}_3\not{W}$ forbidden

$\bar{u}(p_2)\not{W} u(p_1)$ $\quad 3\ \not{q}_i$ impossible since only 2 momenta can be used! $\underline{p_3, W}$!

$\Rightarrow$ similarly for other fermion line

$\bar{U}(p_4)\not{p}_1 U(p_3)$

$\bar{U}(p_4)\not{W} U(p_3)$ $\to$ $\not{p}_2$ not indep by momentum conservation!

$$
\begin{aligned}
A^{tHV}_{q\bar{q}Q\bar{Q}} = {} & F_1\, \bar{u}(p_2)\, \not{p}_3\, u(p_1)\, \bar{U}(p_4)\, \not{p}_1\, U(p_3) \\
& + F_2\, \bar{u}(p_2)\, \gamma^\mu\, u(p_1)\, \bar{U}(p_4)\, \gamma_\mu\, U(p_3)
\end{aligned}
\Bigg]\ \text{CP even}
$$

$$
\begin{aligned}
& + F_3\, \bar{u}(p_2)\, \not{p}_3\, u(p_1)\, \bar{U}(p_4)\, \gamma^\mu\, U(p_3) \\
& + F_4\, \bar{u}(p_2)\, \gamma^\mu\, u(p_1)\, \bar{U}(p_4)\, \not{p}_1\, V(p_3)
\end{aligned}
\Bigg]\ \text{CP odd}
$$

4 independent ones $\Rightarrow$ as many as the possible helicity (or spin!) configurations LL, LR, RL, RR

in $D=4$ external dimensions ('tHV)

$\Rightarrow$ now assume we see in QCD, theory is CP even (as far as we know)

$LL \leftrightarrow (RR)^*$ $\quad (LR)^* \leftrightarrow RL$ $\quad$ CP transf

so only 2 structures should matter $\Rightarrow$ in fact

$F_3\ \&\ F_4 \equiv 0$ in QCD !

note also that working

$$A_{q\bar{q}Q\bar{Q}} = \sum_{i=1}^{4} F_i \underbrace{T_i}_{\text{Tensor structures}}$$

then in this case the "scalar product" & the "dual vectors" require defining

T_i^{+} = dual vectors

$$\langle T_i^{+}, T_i \rangle = \sum_{pol} \bar{u}(p_1)\Gamma_i \underbrace{u(p_2)\bar{u}(p_2)}_{\not{p}_2}\Gamma_j u(p_1)$$

$$\bar{U}(p_3)\Gamma_k \underbrace{U(p_4)\bar{U}(p_4)}_{\not{p}_4}\Gamma_e U(p_3)$$

$$= \mathrm{Tr}\left[\not{p}_1 \Gamma_i \not{p}_2 \Gamma_j \not{p}_3 \Gamma_k \not{p}_4 \Gamma_e\right]$$

$\Rightarrow$ this implies that $T_1, T_2 \perp T_3, T_4$

once $\underline{\underline{w \cdot p_i = 0}}$!

this means that in CP even theory I can completely throw away T_3, T_4

Also if I don't like $\omega^\mu = \varepsilon^{p_1 p_2 p_3 \mu}$ I can write

$$\bar{u}(p_2)\gamma^\mu u(p_1)\,\bar{U}(p_4)\gamma^\nu U(p_3)\underbrace{\omega_\mu \omega_\nu}_{\sim \underline{\underline{g_{\mu\nu}}} + \sum_{ij} c_{ij} \underbrace{p_i^\mu p_j^\nu}}$$

directly then ignore!

$$\underline{\sim}\ \underbrace{\bar{u}(p_2)\gamma^\mu u(p_1)\,\bar{U}(p_4)\gamma_\mu U(p_3)}$$

Equivalent choice for T_2 ! (not same but spans same space!)

$q\bar{q} \to Q\bar{Q}$ massless quarks has 2 tensor structures in $D=4$ EXTERNAL dimensions (tHV-scheme)

IMPORTANT: imagine I insist to work in CDR

$$A = \sum_{i=1}^{2} F_i T_i + \underbrace{\sum_{i=3}^{N} F_i T_i}_{\text{independent only in } D \neq 4}$$

then I can do a Gram-Schmidt orthogonalization

$$A = \sum_{i=1}^{2} F_i T_i + \sum_{i=3}^{N} \tilde{F}_i \tilde{T}_i$$

$$\tilde{T}_i = T_i - \underbrace{\sum_{j=1}^{2} [P_j T_i] T_j}_{\text{substract their projection along the first 2}}$$

now compute F_i, $\tilde{F}_i$ (ALL!), then substract UV & IR poles (HARD FUNCTION), finally send

$$\underline{\varepsilon \to 0 \Rightarrow \tilde{T}_i \to 0 \quad \tilde{F}_i \to \text{FINITE!}}$$

$\Rightarrow$ Finite remainder in tHV & CDR is the same !!

General Formalism :

$$A = \sum_i F_i \, T_i \leftarrow \text{tensors}$$

$\uparrow$ form factors

T_i should be thought of as elements of a vector space

we can define "dual vectors" $\Rightarrow$ $T_i^\dagger$ (for ex build out of ε_j^*)

a "scalar product" in this vector space

using $\sum_{pol} \varepsilon_j^\mu \varepsilon_j^{*\nu} = -g^{\mu\nu} \pm \dots$

Depending on conditions we used to define T_i (restrict vector space!)

$T_i^\dagger \cdot T_j$ implies $\sum_{pol}$ $\Leftarrow$

then

$$P_j = \sum_k c_k^{(j)} T_k^\dagger \qquad \text{where} \qquad c_k^{(j)} = [M]^{-1}_{jk}$$

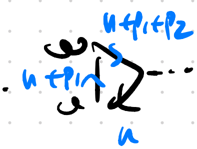
check! Linear Algebra $\rightarrow$ $M_{ij} = [T_i^\dagger \cdot T_j]$

SCALAR INTEGRALS

by applying projectors on Feynman Diagrams representation of an amplitude, at arbitrary # of loops, we SATURATE all LORENTZ & LITTLE GROUP covariance indices $\Rightarrow$ result must be combination of REALLY SCALAR OBJECTS

For $gg \to H$

$$P = \left[\frac{2}{D-2}\right]\frac{1}{q^2}\left(-g^{\mu\nu} + \frac{2\,p_1^{\nu} p_2^{\mu}}{q^2}\right)$$

$P \cdot$ 

$$= P \cdot \int \frac{d^D k}{(2\pi)^D} \frac{\mathrm{Tr}\left[\gamma^{\mu}(\not{k}+m)(\not{k}+\not{p}_{12}+m)\gamma^{\nu}(\not{k}+\not{p}_1+m)\right]}{\underbrace{(k^2-m^2)}_{D_1}\underbrace{((k+p_1)^2-m^2)}_{D_2}\underbrace{((k+p_{12})^2-m^2)}_{D_3}}$$

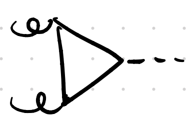
$$\sim \int \frac{d^D k}{(2\pi)^D} \frac{N\{k\cdot k,\, k\cdot p_1,\, k\cdot p_2\}}{D_1 D_2 D_3}$$

$\uparrow$ scalar integrals!

IRREDUCIBLE SCALAR PRODUCTS :

@ 1 loop, all scalar products can always be rewritten in terms of the propagators of the problem

EXAMPLE ABOVE :

- 

$$\begin{cases} D_1 = k^2 - m^2 \\ D_2 = (k+p_1)^2 - m^2 \\ D_3 = (k+p_1+p_2)^2 - m^2 \end{cases} \qquad \begin{cases} k\cdot k \\ k\cdot p_1 \\ k\cdot p_2 \end{cases}$$

$$\Rightarrow \quad k\cdot k = D_1 + m^2$$

$$k\cdot p_1 = \frac{1}{2}\left[D_2 - D_1 - p_1^2\right]$$

$$k\cdot p_2 = \frac{1}{2}\left[D_3 - D_2 - p_2^2 - 2p_1\cdot p_2\right]$$

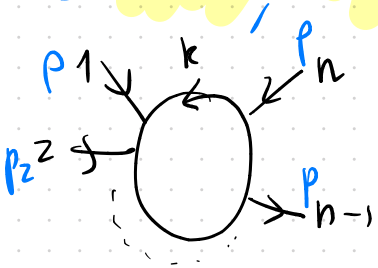
so substituting these, all scalar ints become of the type

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{D_1^{a_1} D_2^{a_2} D_3^{a_3}} = I(a_1, a_2, a_3)$$

$$a_i \in \mathbb{Z}$$

1-LOOP CASE IS SPECIAL :

@ 1 loop n points → n "propagators" $D_1 \ldots D_n$



$$\Rightarrow \begin{cases} D_1 = (k+p_1)^2 - m_1^2 \\ \vdots \\ D_n = k^2 - m_n^2 \end{cases}$$

n - scalar products $\begin{Bmatrix} k \cdot k \\ k \cdot p_i \end{Bmatrix}$

scalar integrals will always be of the type

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{D_1^{a_1} \cdots D_n^{a_n}} = I(a_1 \ldots a_n)$$

$$a_i \in \mathbb{Z}$$

SCALAR FEYNMAN INTEGRALS @ L-loops

@ L loops, $L \geq 2$, not all scalar products can be expressed in terms of D_i, we need to generalise the notation:

COMBINATORIC EXERCISE SHOWS THAT L loops, N points

SCAL PRODS ρ is:

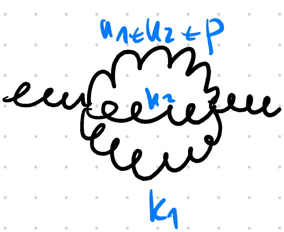
$$\rho = L\left(N + \frac{L}{2} - \frac{1}{2}\right) \implies L=1 \quad \boxed{\rho = N}$$

\# of legs, not indep momenta

@ 1 Loop $\rho = N$

(can you prove it?)

EXAMPLE: Two loop gluon propagator



$l = 2$

$N = 2$

$$\rho = 2\left(2 + 1 - \frac{1}{2}\right) = 5$$

3 propagators, 5 scalar products

$$S_i = \{ k_1^2, k_2^2, k_1 \cdot k_2, k_1 \cdot p_1, k_2 \cdot p \}$$

given propagators $\left\{\begin{array}{c} k_1^2 \\ k_2^2 \\ (k_1+k_2+p)^2 \end{array}\right\}$ a possible choice for ISPs $k_1\cdot p$, $k_2\cdot p$

$$\begin{cases} k_1\cdot k_1 = D_1 = k_1^2 \\ k_2\cdot k_2 = D_2 = k_2^2 \\ k_1\cdot k_2 = \frac{1}{2}\Big[D_3 - 2\underbrace{k_1\cdot p}_{S_1} - 2\underbrace{k_2\cdot p}_{S_2} - D_1 - D_2 - p^2\Big] \end{cases}$$

these are my two ISPs

FAMILY OF INTEGRALS : $\Rightarrow$ our object of study!

$$\int \frac{d^D k_1}{(2\pi)^D}\frac{d^D k_2}{(2\pi)^D}\,\frac{S_1^{b_1}\,S_2^{b_2}}{D_1^{a_1}\,D_2^{a_2}\,D_3^{a_3}} =$$

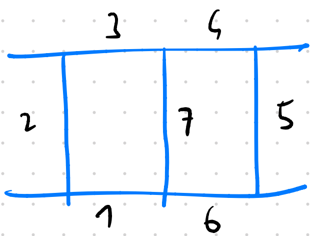
$$= I(a_1, a_2, a_3; \underbrace{-b_1, -b_2}_{\text{numerators}})$$

General Nomenclature:

$$I(a_1,\dots,a_t;\underbrace{-b_1,\dots,-b_\sigma}_{\text{negative indices mean numerators!}}) = \int \prod_{l=1}^{L} \frac{d^D k_l}{(2\pi)^D} \, \frac{S_1^{b_1}\cdots S_\sigma^{b_\sigma}}{D_1^{a_1}\cdots D_t^{a_t}}$$

$I(1,\dots,1;0,\dots,0)$ = defines the TOP SECTOR or TOP-TOPOLOGY, i.e the graph we are considering

EXAMPLE: a 2-loop double box for $gg \to gg$ in QCD

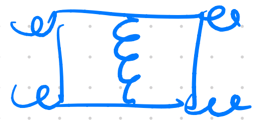


$$= \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{1}{D_1 \dots D_7}$$

$\Uparrow$

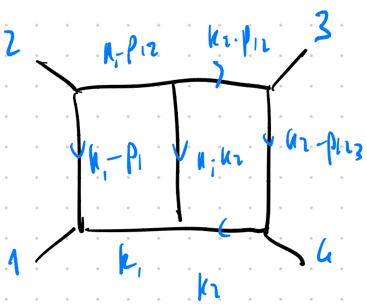
We draw graph associated to scalar integral, we mean

— no Feynman Rules —

contributes to these dig $\longrightarrow$  etc

SKIP

counting shows 9 scalar products, 7 propagators $\Big\}$ <u>2 ISPs</u>



$p_{ijk} = p_i + p_j + p_k$ etc

FAMILY

$$= \left\{ \begin{array}{l} k_1^2 \\ k_2^2 \\ (k_1-k_2)^2 \\ (k_1-p_1)^2 \\ (k_1-p_{12})^2 \\ (k_2-p_{12})^2 \\ (k_2-p_{123})^2 \\ \text{ADD extra props} \\ (k_2-p_1)^2 \\ (k_1-p_{123})^2 \end{array} \right.$$

So Integrals we are interested in will be

$I(n_1, \dots, n_7 ; n_8, n_9)$ $\left\{ \begin{array}{l} n_1, \dots, n_7 \geq 0 \\ n_8, n_9 \leq 0 \end{array} \right\}$

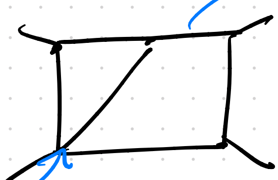
- $I(1,1,1,1,1,1,1,0,0)$ is <u>TOP SECTOR</u>

  <u>TOP TOPOLOGY</u>

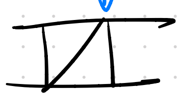
- we call all ints obtained removing one or more propagators in all possible ways SUBSECTORS or SUB TOPOLOGIES

$\Rightarrow$ they give subtopology <u>tree</u>

For example $I(0,1,1,1,1,1,1,0,0)$

$=$  it's a sub sector

pinch pinch

$I(0,1,1,1,1,0,1,0,0) =$  subsector of

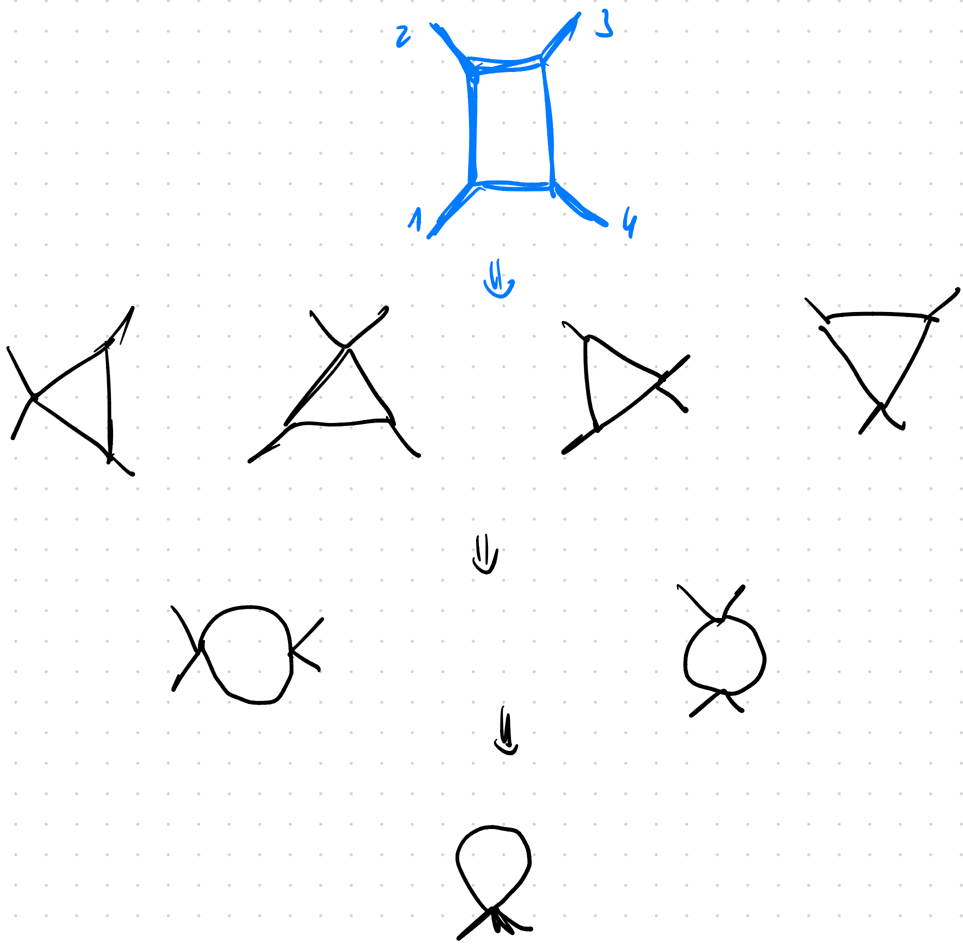
<u>etc</u>

40

→ HERE

In these cases we often say that subsectors are obtained by PINCHING propagators of top sector

INTEGRAL FAMILY & SUBTOPOLOGY TREE



Now, as you can imagine, if we start writing down **all Feynman diagrams** for a given problem, perform projection to compute scalar form factors, and finally **collect all scalar integrals**, we will in general find **huge number** of apparently different

integrals ⇒

$gg \to gg$ (in QCD)

$\left\{ \begin{array}{l} @ \text{ 1 loop } \; \mathcal{O}(100) \text{ ints} \\ @ \text{ 2 loops } \; \mathcal{O}(10000) \text{ ints} \\ @ \text{ 3 loops } \; \mathcal{O}(10^7) \text{ ints} ! \end{array} \right.$

simple combinatorics — clearly hopeless to compute all of them one by one —

Luckily, not all these integrals are independent!

INTEGRATION BY PARTS & MASTER INTEGRALS

We work in Dim Regularization to regularize UV & IR singularities.

The axiom of dim reg imply that we can perform a generic transformation on loop momenta

INFINITESIMALLY $\quad k_i^\mu \to k_i^\mu + \alpha v_j^\mu \qquad v_j^\mu = \{k_j^\mu, p_j^\mu\}$

complete set of momenta

$$\Rightarrow f(\vec{k},\vec{q}) \to f(\vec{k},\vec{q}) + \alpha v_j^\mu \frac{\partial}{\partial k_i^\mu} f(\vec{k},\vec{q})$$

plus:

$$d^D k_i \to (1+\alpha D)\, d^D k_i \qquad \text{if } v_j^\mu = k_i^\mu$$

invariance of integral implies then

$$\int \prod_{\ell=1}^{L} \frac{d^D k_\ell}{(2\pi)^D}\, O_{ij}\, f(\vec{k},\vec{q}) = 0$$

where $O_{ij} = \partial_i \cdot v_j \qquad 1 \le L, \quad j \ge i$

used $\partial_i k_i = D + k_i \partial_i$ to reabsorb jacobian!

Our generate a Lie Algebra —

Let's write these identities in a human friendly form — given a FAMILY OF INTEGRALS:

$$\int \prod_{l=1}^{L} \frac{d^D k_l}{(2\pi)^D} \left[\frac{\partial}{\partial k_i^\mu} v_j^\mu \frac{S_1^{b_1} \cdots S_\sigma^{b_\sigma}}{D_1^{a_1} \cdots D_\tau^{b_\tau}} \right] = 0$$

it's nothing but generalization of 1-dimensional

$$\int_{-\infty}^{+\infty} dx \frac{\partial}{\partial x} f(x) = 0 \quad \text{if} \quad \int_{-\infty}^{+\infty} f(x)\, dx < \infty$$

$\Rightarrow$ usually referred to as INTEGRATION BY PARTS identities (IBPs)

[Chetyrkin, Tkachov '81]

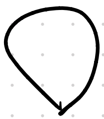
- by inspection, it is clear that by differentiating we generate integrals in the same FAMILY so we expect that IBPs relate apparently different integrals in same family

- Using Lie Group property, one can prove formally that all integrals can be expressed in terms of a FINITE NUMBER OF MASTER INTS.
  $\Rightarrow$ they are a BASIS of all integrals

  [A. V. Smirnov, A. V. Petukhov 2010]

Proof above is NOT CONSTRUCTIVE —
Let's see how this works in practice:

TADPOLE

 $= \displaystyle\int \frac{d^Dk}{(2\pi)^D} \frac{1}{k^2 - m^2 + i\varepsilon}$

$\xrightarrow[\text{Rotation}]{\text{Wick}} \propto \displaystyle\int \frac{d^Dk}{(2\pi)^D} \frac{1}{k^2 + m^2}$

In lectures I will work in Euclidean km. for convenience!

Family $\displaystyle\int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2+m^2)^n} = I(n)$

1 IBP $\displaystyle\int \frac{d^Dk}{(2\pi)^D} \frac{\partial}{\partial k^\mu}\left[k^\mu \frac{1}{(k^2+m^2)^n}\right] = 0$

$$\frac{\partial}{\partial k^\mu} \frac{k^\mu}{(k^2+m^2)^n} = \frac{D}{(k^2+m^2)^n} - \frac{n\, k^\mu}{(k^2+m^2)^{n+1}} 2k_\mu$$

$$= \frac{D}{(k^2+m^2)^n} - 2n \frac{k^2}{(k^2+m^2)^{n+1}} = \frac{D-2n}{(k^2+m^2)^n} + \frac{2n\, m^2}{(k^2+m^2)^{n+1}}$$

which implies

$$(D-2n)\, I(n) + 2n\, m^2\, I(n+1) = 0$$

$$I(n+1) = -\frac{(D-2n)}{2n m^2}\, I(n)$$

$$I(2) = -\frac{D-2}{2m^2}\, I(1)$$

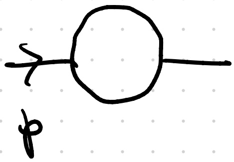
$$I(3) = -\frac{D-4}{4m^2}\, I(2) = \frac{(D-4)(D-2)}{8m^4}\, I(1)$$

we say that tadpole family has ONE master integral, can be chosen as $I(1)$

In this case, easy to solve IBP for generic "n"

in general this will not be possible → we can instead "generate" and "solve" IBPs for specific choices of indices $a_1, \dots, a_n$

. ONE LOOP BUBBLE (Euclidean signature)



$$= \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2+m^2)^a \left((k+p)^2+m^2\right)^b}$$

$$= I(a, b) \quad \text{family}$$

I can derive 2 IBPs now :

$$\text{①} \quad \int \frac{d^D k}{(2\pi)^D} \frac{\partial}{\partial k^\mu} \left[k^\mu \frac{1}{D_1^a D_2^b} \right] = 0$$

$$\text{②} \quad \int \frac{d^D k}{(2\pi)^D} \frac{\partial}{\partial k^\mu} \left[p^\mu \frac{1}{D_1^a D_2^b} \right] = 0$$

Derive them for specific values of $(a, b) = \{(1,1); \cdots\}$

Prove that

$$I(1,2) = I(2,1) = -\frac{(D-2)}{2m^2(p^2+4m^2)} I(1,0) - \frac{D-3}{p^2+4m^2} I(1,1)$$

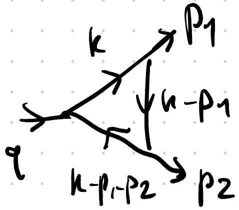
$\Rightarrow$ this problem has 2 master integrals

$$\left[\begin{array}{lll} I(1,0) & = & \text{the Tadpole} \\ I(1,1) & = & \text{the one loop bubble} \end{array}\right]$$

REDUCIBLE INTEGRALS $\Rightarrow$ QUOTE RESULT

Look paper + LAPORTA

Consider the massless triangle



$$= \int \frac{d^Dk}{(2\pi)^D} \frac{1}{\underbrace{(k^2)^a}_{D_1^a} \underbrace{((k-p_1)^2)^b}_{D_2^b} \underbrace{((k-p_1-p_2)^2)^c}_{D_3^c}}$$

with $p_1^2 = p_2^2 = 0$; $q^2 = (p_1+p_2)^2 = s$

three IBPs

$$\int \frac{d^Dk}{(2\pi)^D} \frac{\partial}{\partial k^\mu} \left\{ \begin{array}{c} p_1^\mu \\ p_2^\mu \\ k^\mu \end{array} \right\} \frac{1}{D_1^a D_2^b D_3^c} = 0$$

EXERCISE Prove

$$\begin{cases} s\,I(1,1,2) + (D-4)\,I(1,1,1) = 0 \\ s\,I(1,1,2) + I(1,0,2) + I(2,0,1) = 0 \\ s\,I(2,1,1) + I(1,0,2) + I(2,0,1) = 0 \end{cases}$$

NOTICE

- not all IBPs are independent! $\underline{2 \equiv 3}$

- solving 1)

$$I(1,1,2) = -\frac{D-4}{s}\, I(1,1,1)$$

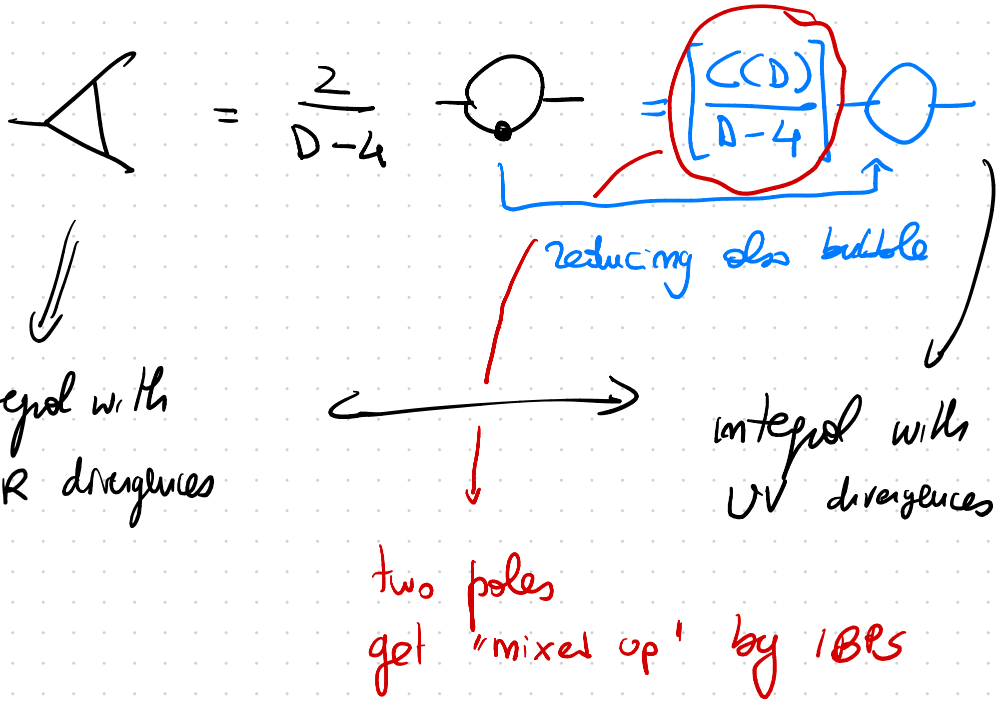
putting it into 2)

$$(D-4)\,I(1,1,1) = I(1,0,2) + I(2,0,1)$$

triangle gets "reduced" to bubbles!

noting $I(1,0,2) = I(2,0,1)$ we find

$$I(1,1,1) = \frac{2}{D-4}\, I(2,0,1)$$



In general, we solve IBPs in this way:

generate all of them starting from "seed" integrals

take big linear system, solve it

$\Rightarrow$ [LAPORTA ALGORITHM '00]