# Scattering Amplitudes in QFT WS 2023/24

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**Sheet 09: BCFW Boundary terms & Dimensional Regularisation**

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## Exercise 1 - Boundary Terms for BCFW shifts of gluon amplitudes

Consider an $[i, j\rangle$-shift applied to an $n$-gluon amplitude $\mathcal{A}_n$. On the level of the external momenta, the shift acts as

$$\hat{p}_i^\mu = p_i^\mu + z\,\frac{1}{2}\,[j\,\gamma^\mu\,i\rangle\,, \qquad \hat{p}_j^\mu = p_j^\mu - z\,\frac{1}{2}\,[j\,\gamma^\mu\,i\rangle\,, \qquad \hat{p}_k^\mu = p_k^\mu\,, \quad k \neq i, j\,. \tag{1}$$

If the amplitude in terms of the shifted momenta goes to zero as $z \to \infty$, the shift is dubbed "good", as this implies that the amplitude can be fully constructed from lower-point amplitudes by mean of this shift and the BCFW approach. The goal of this exercise is to determine which shifts are good shifts for gluon amplitudes. To do so, we examine their large $z$-behaviour of gluonic scattering amplitudes through the background field method. The relevant part of the QCD Lagrangian for tree-level gluon amplitudes is the Yang-Mills Lagrangian,

$$\mathcal{L}_{YM} = -\frac{1}{4}\,F^a_{\mu\nu}\,F^{a,\mu\nu}\,, \tag{2}$$

where the field strength tensor reads $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu$. In the large $z$-limit, we can view the shifted gluon amplitude $\mathcal{A}_n(z)$ defined above, as an energetic gluon moving through a soft background field. This motivates to study the problem with the background field method, where we split the gluon field $A^a_\mu$ into a quantum field $Q^a_\mu$ carrying the large momentum modes $\propto z$ and a background field $B^a_\mu$ carrying the soft modes $\propto z^0$, $A^a_\mu = Q^a_\mu + B^a_\mu$.

1. Substitute this decomposition into the Lagrangian (2) and show that the terms quadratic in $Q^a_\mu$ take the form

$$\mathcal{L}_{Q^2} = -\frac{1}{2}\left\{(D_\mu Q_\nu)^a (D^\mu Q^\nu)^a - (D_\mu Q_\nu)^a (D^\nu Q^\mu)^a + g_s\, f^{abc} Q^b_\mu Q^c_\nu B^{a,\mu\nu}\right\}\,, \tag{3}$$

   where $(D_\mu Q_\nu)^a = (\delta^{ac}\partial_\mu + g_s f^{abc} B^b_\mu) Q^c_\nu$ and $B^a_{\mu\nu}$ is the field strength tensor of the background field $B^a_\mu$.

We are interested in Green's functions with exactly two external legs associated with the field $Q^a_\mu$ and momentum $\propto z$, which we denote by $\mathcal{M}_{2Q,(n-2)B}$[1]. At tree level, the only contributing terms in the Lagrangian are those quadratic in $Q^a_\mu$ [2]. All terms linear in $Q^a_\mu$ have to be neglected, which can be explained from two points of view. On the one hand, such vertices would make it possible for a particle associated with the quantum field $Q^a_\mu$ to propagate into particles associated with $B^a_\mu$, meaning that $B^a_\mu$ would not be just a background field. On the other hand, vertices involving just one particle associated with $Q^a_\mu$ cannot obey momentum conservation, as we take the background field to be soft and there is

[1] By construction, $\mathcal{M}_{2Q,(n-2)B} = \mathcal{A}_n(z)$ for large $z$.

[2] Also those independent of $Q^a_\mu$, but those don't add anything to the $z$-scaling behaviour of $\mathcal{M}_{2Q,(n-2)B}$, which is what we are ultimately interested in.

only one edge with momentum $\propto z$ entering these vertices[3]. Further, as we threw away the vertices linear in $Q^a_\mu$, all vertices cubic and quartic in $Q^a_\mu$ contribute to $\mathcal{M}_{2Q,(n-2)B}$ only at loop level.

2. Fix the gauge for $Q^a_\mu$ by adding a gauge-fixing term
$$\mathcal{L}_{gf} = -\frac{1}{2}(D_\mu Q^\mu)^a (D_\nu Q^\nu)^a \tag{4}$$
and show that the part quadratic in $Q^a_\mu$ of the gauge-fixed Lagrangian reads
$$\mathcal{L}^{gf}_{Q^2} = \mathcal{L}_{Q^2} + \mathcal{L}_{gf} = -\frac{1}{2}(D_\mu Q_\nu)^a (D^\mu Q^\nu)^a + g_s\, f^{abc} Q^b_\mu Q^c_\nu B^{a,\mu\nu}\,. \tag{5}$$

3. The first term has an extra "Lorentz"-like global internal symmetry, which the second term does not possess. Identify this symmetry and give the corresponding transformations of $Q^a_\mu$ and $B^a_\mu$.

4. Identify the vertices with the most divergent scaling behaviour in $z$ and deduce the class of Feynman diagrams giving the most divergent contribution to $\mathcal{M}_{2Q,(n-2)B}$ as $z \to \infty$.

5. We denote by $\mathcal{M}^{\mu\nu,ab}_{2Q,(n-2)B}$ the amplitude with the polarisation vectors of the two particles carrying the large momentum modes $\propto z$ stripped off and their colour indices as $a, b$. Combine your results from the previous two points to argue that its $z$-expansion takes the form
$$\mathcal{M}^{\mu\nu,ab}_{2Q,(n-2)B} = g^{\mu\nu}\,\delta^{ab}\, z \sum_{m=0}^{\infty} c_m\, z^{-m} + f^{abc} A^{c,\mu\nu} + \sum_{m=1}^{\infty} b^{ab,\mu\nu}_m\, z^{-m}\,, \tag{6}$$
where $A^{c,\mu\nu}$ is antisymmetric in $\mu \leftrightarrow \nu$, $A^{c,\mu\nu} = -A^{c,\nu\mu}$.

6. Finally, associate to the gluon with indices $(\mu, a)$ the momentum $\hat{p}_i$ and to the gluon with indices $(\nu, b)$ the momentum $\hat{p}_j$ and show by contracting eq. (6) with the external polarisation vectors that for $z \to \infty$ the helicity amplitudes scale as
$$\mathcal{M}^{++}_{2Q,(n-2)B} \sim \frac{1}{z}\,, \quad \mathcal{M}^{+-}_{2Q,(n-2)B} \sim \frac{1}{z}\,, \quad \mathcal{M}^{-+}_{2Q,(n-2)B} \sim z^3\,, \quad \mathcal{M}^{--}_{2Q,(n-2)B} \sim \frac{1}{z}\,, \tag{7}$$
where the first helicity belongs to the gluon with momentum $\hat{p}_i$ an the second one to the gluon with momentum $\hat{p}_j$.

   Hint: Use the Ward identities $\hat{p}_{i,\mu}\, \mathcal{M}^{\mu\nu,ab}_{2Q,(n-2)B}\, \varepsilon^{\pm}_\nu(\hat{p}_j, r_j) = 0 = \varepsilon^{\pm}_\mu(\hat{p}_i, r_i)\, \mathcal{M}^{\mu\nu,ab}_{2Q,(n-2)B}\, \hat{p}_{j,\nu}$.

## Exercise 2 - Dimensional Regularisation and partial fractioning

1. Compute the dimensionally regularised integrals
$$\mathcal{I}_1 = \int \frac{\mathrm{d}^D k}{\mathrm{i}\pi^{\frac{D}{2}}}\, \frac{1}{k^2 - m^2 + \mathrm{i}0^+}\,, \tag{8}$$
$$\mathcal{I}_2 = \int \frac{\mathrm{d}^D k}{\mathrm{i}\pi^{\frac{D}{2}}}\, \frac{1}{(k^2 + \mathrm{i}0^+)(k^2 - m^2 + \mathrm{i}0^+)}\,. \tag{9}$$

   Hint: Even $\mathcal{I}_2$ can be easily evaluated *without* introducing Feynman parameters.

[3] If you are familiar with SCET, you may notice the parallels in the construction.

2. Partial fraction the integrand of $\mathcal{I}_2$ with respect to $k^2$ and relate the result to $\mathcal{I}_1$. Check that this is consistent with the results from the previous points, given that scaleless integrals vanish in dimensional regularisation.

3. Use Feynman parameterisation to show that

$$\mathcal{I}_3(p^2) = \int \frac{\mathrm{d}^D k}{\mathrm{i}\pi^{\frac{D}{2}}} \frac{1}{(k^2 - m^2 + \mathrm{i}0^+)((k-p)^2 - m^2 + \mathrm{i}0^+)} = \tag{10}$$

$$= \Gamma\left(2 - \frac{D}{2}\right) \int_0^1 \mathrm{d}x \left[m^2 - x(1-x)p^2 - \mathrm{i}0^+\right]^{\frac{D}{2}-2} . \tag{11}$$

The result of the previous point implies that

$$\mathcal{I}_3(p^2 = 0) = \left(m^2\right)^{\frac{D}{2}-2} \Gamma\left(2 - \frac{D}{2}\right) . \tag{12}$$

Like $\mathcal{I}_2$, also $\mathcal{I}_3$ can be partial-fractioned with respect to $k^2$,

$$\mathcal{I}_3(p^2) = \int \frac{\mathrm{d}^D k}{\mathrm{i}\pi^{\frac{D}{2}}} \frac{1}{2k\cdot p - p^2} \left(\frac{1}{(k-p)^2 - m^2 + \mathrm{i}0^+} - \frac{1}{k^2 - m^2 + \mathrm{i}0^+}\right) . \tag{13}$$

For $p^2 = 0$, shifting the loop momentum in the first term in the brackets according to $k \to k + p$ suggests that $\mathcal{I}_3(p^2 = 0) = 0$ in contradiction to the result in eq. (12).

4. Show that the two individual integrals appearing in eq. (13),

$$\mathcal{I}^a(m^2) = \int \frac{\mathrm{d}^D k}{\mathrm{i}\pi^{\frac{D}{2}}} \frac{1}{(2k\cdot p)[(k - a\,p)^2 - m^2 + \mathrm{i}0^+]} , \qquad a \in \{0;1\} , \, p^2 = 0 . \tag{14}$$

   are divergent for any (complex) value of $D$.

That dimensional regularisation fails to regularise a Feynman integral is an issue that can arise when *eikonal* propagators[4] appear in the integrand and an external leg goes on-shell[5]. As a consequence, performing the shift $k \to k + p$ only in one of the two integrals in eq. (13) is not a well-defined operation. One way to resolve the whole issue is by treating the Feynman prescription with more care and write

$$\mathcal{I}_3(p^2 = 0) = \int \frac{\mathrm{d}^D k}{\mathrm{i}\pi^{\frac{D}{2}}} \frac{1}{(k^2 - m^2 + \mathrm{i}\epsilon_1)((k-p)^2 - m^2 + \mathrm{i}\epsilon_2)} = \tag{15}$$

$$= \int \frac{\mathrm{d}^D k}{\mathrm{i}\pi^{\frac{D}{2}}} \frac{1}{2k\cdot p - \mathrm{i}(\epsilon_2 - \epsilon_1)} \left(\frac{1}{(k-p)^2 - m^2 + \mathrm{i}\epsilon_2} - \frac{1}{k^2 - m^2 + \mathrm{i}\epsilon_1}\right) , \tag{16}$$

[4] Propagators that are *linear* in the loop momentum instead of quadratic.

[5] Common lingo refers to an external leg with momentum $p$ as *on-shell* if $p^2 = 0$ and *off-shell* if $p^2 \neq 0$.

where we now keep track of the limits $\epsilon_1 \to 0^+$ and $\epsilon_2 \to 0^+$ independently. For $\epsilon_1 \neq \epsilon_2$ and at least one of the two $\epsilon_1, \epsilon_2 > 0$, the two individual integrals

$$\mathcal{J}_1 = \int \frac{\mathrm{d}^D k}{\mathrm{i}\pi^{\frac{D}{2}}} \frac{1}{[2k \cdot p - \mathrm{i}(\epsilon_2 - \epsilon_1)](k^2 - m^2 + \mathrm{i}\epsilon_1)}, \tag{17}$$

$$\mathcal{J}_2 = \int \frac{\mathrm{d}^D k}{\mathrm{i}\pi^{\frac{D}{2}}} \frac{1}{[2k \cdot p - \mathrm{i}(\epsilon_2 - \epsilon_1)][(k-p)^2 - m^2 + \mathrm{i}\epsilon_2]}, \tag{18}$$

with $p^2 = 0$ are now well-defined.

5. Use Feynman parameterisation to calculate the integrals $\mathcal{J}_{1,2}$ and show that

$$\mathcal{I}_3(p^2 = 0) = \lim_{\epsilon_{1,2} \to 0^+} (\mathcal{J}_2 - \mathcal{J}_1) = \left(m^2\right)^{\frac{D}{2}-2} \Gamma\left(2 - \frac{D}{2}\right) \tag{19}$$

   in agreement with the result from eq. (12).

   Hint: One of the two limits can be taken before evaluating the Feynman parameter integral(s).