Scattering Amplitudes in QFT WS 2023/24

Lecturer: Prof. Lorenzo Tancredi
Assistant: Fabian Wagner
https://www.ph.nat.tum.de/ttpmath/teaching/ws-2023-2024/

Sheet 08: Gravity Amplitudes

In this exercise sheet, we are going to investigate tree-level three- and four-point graviton amplitudes with on-shell methods *without* resorting to an explicit Lagrangian. It suffices to know that massless spin-two particles have two helicity states (as all massless non-scalar particles) and their polarisations are rank-two *tensors* that can be constructed as

$$\varepsilon^{\mu\nu}_{\pm}(p,r) = \varepsilon^{\mu}_{\pm}(p,r)\varepsilon^{\nu}_{\pm}(p,r)\,, \tag{1}$$

with $\varepsilon^{\pm}(p,r)$ the polarisation vectors of massless spin-one particles. Further, the three-graviton vertex is proportional to the coupling constant of gravity $\kappa \propto \sqrt{G_N}$, which has mass dimension -1 in natural units. In the following, we will denote an n-graviton amplitude with incoming momenta $\{p_1, \dots, p_n\}$ and helicities $\{\lambda_1, \dots, \lambda_n\}$ as $\mathcal{M}(1^{\lambda_1}, \dots, n^{\lambda_n})$.

Exercise 1 - Three-graviton helicity amplitudes

As for gluons, the tree-level three-graviton amplitude can be determined from Little group scaling arguments alone (up to normalisation).

1. In the lecture, you found the Little group scaling relation for any amplitude $\mathcal{A}$ to read[1]

$$\mathcal{A}(p_1, ..., W_i^{\mu} p_{i\mu}, ..., p_N) = z^{+2h_i}\mathcal{A}(p_1, ..., p_i, ..., p_N)\,, \tag{2}$$

   where h_i is the helicity of the particle with momentum p_i and W_i is a Lorentz transformation which leaves p_i invariant (a Little group transformation). Check that eq. (2) remains valid for massless particles with spin-two.

2. Now use eq. (2), together with dimensional analysis and three-particle kinematics, to compute all three-graviton helicity amplitudes up to overall numerical factors. Concretely, show that

$$\mathcal{M}(1^{\pm}, 2^{\pm}, 3^{\pm}) = 0\,, \quad \mathcal{M}(1^{+}, 2^{+}, 3^{-}) = c_+\,\kappa\,\frac{\langle 12\rangle^6}{\langle 23\rangle^2\langle 31\rangle^2}\,, \quad \mathcal{M}(1^{-}, 2^{-}, 3^{+}) = c_-\,\kappa\,\frac{[12]^6}{[23]^2\,[31]^2}\,, \tag{3}$$

   where $c_{\pm}$ are numbers. Parity invariance implies $c_+ = c_- \equiv c$ and we may eliminate the constants by redefining $\kappa \to \kappa/c$.

   Hint: An n-point amplitude has mass dimension $4 - n$ in natural units.

Exercise 2 - Four-graviton helicity amplitudes

The four-graviton helicity amplitudes can be determined from the three-graviton ones by means of the BCFW approach.

1. Show that for gravitons the all-plus and single-minus amplitudes vanish, just as for gluons.

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[1] We associate massless left-handed fermions with helicity $-1/2$ and massless right-handed fermions with helicity $+1/2$.

2. Using the BCFW recursion with a valid shift, show that the MHV graviton amplitude is given by

$$\mathcal{M}(1^+, 2^+, 3^-, 4^-) = \kappa^2 \frac{\langle 12\rangle^7 [12]}{\langle 13\rangle \langle 14\rangle \langle 23\rangle \langle 24\rangle \langle 34\rangle^2} \,. \tag{4}$$

As shift, you can use $[1, 2\rangle$ and verify a posteriori that this is indeed a valid shift.

Exercise 3 - Colour-Kinematics Duality and Double Copy

Examining eq. (3), you may notice that the three-graviton amplitudes are just squares of (colour-ordered) three-gluon amplitudes with the couplings stripped off,

$$\widetilde{\mathcal{M}}(1^+, 2^+, 3^-) = \widetilde{\mathcal{A}}[1^+, 2^+, 3^-]^2 \,, \qquad \widetilde{\mathcal{M}}(1^-, 2^-, 3^+) = \widetilde{\mathcal{A}}[1^-, 2^-, 3^+]^2 \,, \tag{5}$$

where the $\tilde{\cdot}$ denotes that the coupling constants are stripped off. This observation can be seen as the starting point of a procedure referred to as *double copy*, which allows to construct graviton amplitudes from "squares of gluon amplitudes". At tree-level, this goes as follows: given a general n-gluon amplitude $\mathcal{A}(1, \ldots, n)$, start by writing it in the form

$$\mathcal{A}(1, \ldots, n) = g_s^{n-2} \sum_i \frac{c_i\, n_i}{D_i} \,, \tag{6}$$

where the c_i are colour factors, the D_i are products of inverse propagators and the n_i denote the remaining numerator factors as coming from Feynman rules. The amplitude is said to obey colour-kinematics duality if the numerator factors n_i obey the same linear relations as the colour factors c_i. In that case, it can be shown that the n-graviton amplitude $\mathcal{M}(1, \ldots, n)$ is given by

$$\mathcal{M}(1, \ldots, n) = \mathcal{N}\, \kappa^{n-2} \sum_i \frac{n_i^2}{D_i} \,, \tag{7}$$

where $\kappa = \sqrt{8\pi^2 G_N}$ and $\mathcal{N}$ is a constant depending on normalisation conventions between the colour factors c_i and numerator factors n_i. Let's make this concrete for the case $n = 4$. As done in an earlier exercise, the four-gluon amplitude can be written as

$$\mathcal{M}(1, 2, 3, 4) = g_s^2 \left(\frac{n_s\, c_s}{s} + \frac{n_t\, c_t}{t} + \frac{n_u\, c_u}{u} \right) \,, \tag{8}$$

where $s = (p_1 + p_2)^2,\ t = (p_1 + p_3)^2,\ u = (p_1 + p_4)^2$ are the usual Mandelstam variables and the colour factors are given by

$$c_s = f^{a_1 a_2 b} f^{b a_3 a_4}, \quad c_t = f^{a_1 a_3 b} f^{b a_4 a_2}, \quad c_u = f^{a_1 a_4 b} f^{b a_2 a_3} \,. \tag{9}$$

The amplitude obeys colour-kinematics duality[2] in form of the Jacobi identity $c_s + c_t + c_u = 0$ and for numerator factors $n_s + n_t + n_u = 0$. The four-graviton amplitude is then given by

$$\mathcal{M}(1, 2, 3, 4) = -\kappa^2 \left(\frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u} \right) \,. \tag{10}$$

[2] In fact, all gluon amplitudes do after performing appropriate generalised gauge transformations $n_i \to n_i + \Delta_i$ with gauge functions Δ_i subject to the constraint $\sum_i \frac{c_i\, \Delta_i}{D_i} = 0$. It is easy to check that this transformations leaves the amplitude invariant.

1. Show that eq.(10) can be rewritten in the form

$$\mathcal{M}(1,2,3,4) = \kappa^2 \frac{s\,u}{t}\,\mathcal{A}[1,2,3,4]^2\,, \tag{11}$$

   where $\mathcal{A}[1,2,3,4]$ denotes a colour-ordered gluon amplitude.

   Hint: Recall the relation between the numerator factors and colour-ordered amplitudes we found in a previous exercise[3]:

$$\mathcal{A}[1,2,3,4] = \frac{n_u}{u} - \frac{n_s}{s}\,. \tag{12}$$

2. Verify that eq.(11) reproduces eq. (4) for the MHV helicity configuration.

[3]The factor-of-2 difference comes from a slightly different normalisation convention for the colour factors, such that $\mathcal{A}[1^+,2^+,3^+,4^+]$ is given by the Parke-Taylor formula.