Scattering Amplitudes in QFT WS 2023/24

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https://www.ph.nat.tum.de/ttpmath/teaching/ws-2023-2024/

Sheet 07: BCFW recursion relations

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1 Parke-Taylor formula for non-adjacent helicities

Use the BCFW recursion to prove the Parke-Taylor formula for the tree-level colour-ordered MHV n-gluon amplitudes in the case where the two equal helicity gluons are non-adjacent,

$$\mathcal{A}(1^-, ..i^+, ..., j^+, ...n^-) = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle...\langle n1\rangle}\,. \tag{1}$$

2 Split helicity NMHV six-gluon amplitude

In this exercise, we will use the BCFW recursion to compute the tree-level colour-ordered split-helicity NMHV 6-gluon amplitude $\mathcal{A}[1^-, 2^-, 3^-, 4^+, 5^+, 6^+]$. In what follows we shall denote as $[i, j\rangle$-shift the following:

$$|\hat{i}] = i] + z\, j]\,, \quad |\hat{j}] = j]\,, \quad |\hat{i}\rangle = i\rangle\,, \quad |\hat{j}\rangle = j\rangle - z\, i\rangle\,. \tag{2}$$

1. Choose the $[1, 2\rangle$-shift to derive for $\mathcal{A}[1^+, 2^+, 3^+, 4^-, 5^-, 6^-]$ the following representation:

$$A_6[1^+2^+3^+4^-5^-6^-] = \frac{\langle 3\,(\not{1}+\not{2})\,6]^3}{p_{126}^2\,[21]\,[16]\,\langle 34\rangle\,\langle 45\rangle\,\langle 5\,(\not{1}+\not{6})\,2]} + \frac{\langle 1\,(\not{5}+\not{6})\,4]^3}{p_{156}^2\,[23]\,[34]\,\langle 56\rangle\,\langle 61\rangle\,\langle 5\,(\not{1}+\not{6})\,2]}\,, \tag{3}$$

   where we used the abbreviation $p_{ijk} = p_i + p_j + p_k$.

2. Identify the non-physical pole in eq. (3) and prove that its residue equals zero.

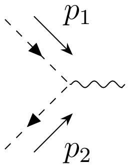
3. Focus on the physical poles and compare with the MHV n-gluon amplitudes. What type of new intermediate states can produce poles in NMHV amplitudes compared to MHV ones? Can you justify why these new poles can never appear in the MHV amplitudes?

3 Scalar QED

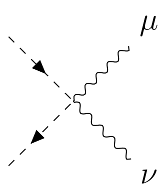
Consider the Lagrangian for massless scalar QED:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi\,(D^\mu\phi)^* - \frac{1}{4}\lambda(\phi\phi^*)^2 \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \partial_\mu\phi\,\partial^\mu\phi^* - \mathrm{i}\,eA^\mu\left[\phi\,\partial_\mu\phi^* - \phi^*\,\partial_\mu\phi\right] + e^2A^\mu A_\mu\phi^*\phi - \frac{1}{4}\lambda(\phi\phi^*)^2.\end{aligned} \tag{4}$$

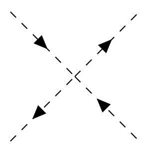
The Feynman rules for the interactions read



$= \mathrm{i}\,e\,(p_2 - p_1)^\mu\,,$



$= 2\,\mathrm{i}\,e\,g^{\mu\nu}\,,$


$= -\mathrm{i}\,\lambda\,.$ (5)

1. Compute the tree-level four-scalar amplitude $A_4(\phi\phi^*\phi\phi^*)$ the traditional way and show that it can be written as
$$A_4(\phi\phi^*\phi\phi^*) = -\lambda + 2\,e^2\left(1 + \frac{\langle 13\rangle^2\langle 24\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\right), \tag{6}$$
where p_1, p_3 are the momenta of the scalars and p_2, p_4 are the momenta of the anti-scalars (all momenta are assumed incoming).

2. Now compute $A_4(\phi\phi^*\phi\phi^*)$ from BCFW recursion, using a $[1,3\rangle$-shift. You should find
$$A_4(\phi\phi^*\phi\phi^*) = 2\,e^2\frac{\langle 13\rangle^2\langle 24\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} + B_4(\phi\phi^*\phi\phi^*)\,, \tag{7}$$
where $B_4(\phi\phi^*\phi\phi^*)$ denotes the boundary term.

3. If you were interested in computing higher-point tree-level amplitudes from BCFW recursion, which additional property would you like to impose on the theory defined by the Lagrangian in eq. (4)?