Scattering Amplitudes in QFT WS 2023/24

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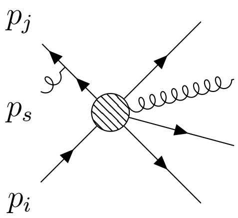
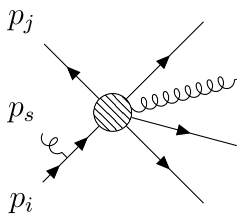
https://www.ph.nat.tum.de/ttpmath/teaching/ws-2023-2024/

Sheet 06: Gluon emission in soft and collinear limits

Exercise 1 - Universality of the eikonal factor in QCD

We are interested in the behaviour of arbitrary tree-level n-point (all incoming) QCD amplitudes as an external gluon becomes soft. After performing a colour decomposition, we may restrict our analysis to colour-ordered amplitudes, which we denote by $\mathcal{A}[\dots]$. In the lecture, you argued that if the soft gluon, which we label with s, is adjacent to massless quark lines, the leading behaviour of the colour-ordered amplitude comes from the diagrams





Further, you showed that the amplitude behaves in this limit as

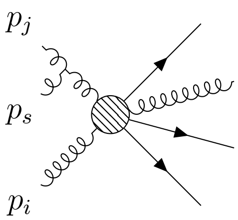
$$\lim_{p_s \to 0} \mathcal{A}[1, \dots, i, s, j, \dots, n] \propto g_s \left\{ \frac{\varepsilon_s \cdot p_i}{p_s \cdot p_i} - \frac{\varepsilon_s \cdot p_j}{p_s \cdot p_j} \right\} \mathcal{A}[1, ..., i, j, ..., n] \,, \tag{1}$$

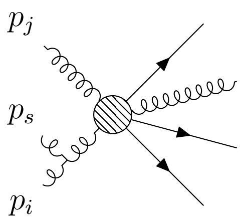
where $\mathcal{A}[1, ..., i, j, ..., n]$ is the colour ordered amplitude without the soft gluon.

1. Show that if the quark lines have a mass m, we nevertheless find the same behaviour.

2. Repeat the exercise for the case, where the adjacent legs are (physical) gluons instead to convince yourself that the eikonal factor is universal.

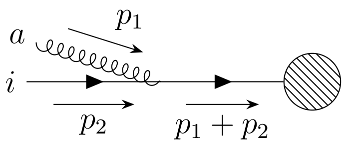
   Hint: The leading contributions come from the diagrams





Exercise 2 - Collinear limits in QCD

We consider again generic tree-level QCD n-point amplitudes, which we denote by $\mathcal{M}$. In the lecture, you studied the limit of an external gluon attached to a massless quark line becoming very collinear to that quark, $gq \to q$.



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1. Convince yourself that there is no singular collinear limit if the quark is massive.

In the rest of this exercise, we will consider the two remaining scenarios admitting collinear limits, $q\bar{q} \to g$ and $gg \to g$. We parameterise the momenta p_1 and p_2 in Sudakov decomposition,

$$p_1 = x_1\, p + y_1\, \bar{p} + p_\perp\,, \tag{3}$$

$$p_2 = x_2\, p + y_2\, \bar{p} - p_\perp\,, \tag{4}$$

where p and $\bar{p}$ denote light-like momenta to which $p_\perp$ is orthogonal, $p \cdot p_\perp = \bar{p} \cdot p_\perp = 0$. The collinear limit corresponds to $\vec{p}_{1,2} \parallel \vec{p}$, and the decomposition is realised such that $x_1 + x_2 \to 1$ in the collinear limit.

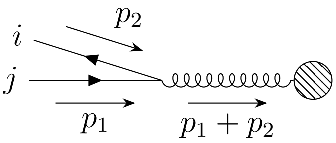
2. Check that the collinear limit can be studied by taking $p_\perp \to 0$, by showing that

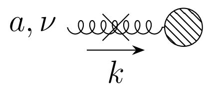
$$y_{1,2} = -\frac{p_\perp^2}{2x_{1,2}(p\cdot\bar{p})}\,, \qquad (p_1+p_2)^2 = -\frac{(x_1+x_2)^2}{x_1 x_2} p_\perp^2\,. \tag{5}$$

Consider first the case $q\bar{q} \to g$ with a massless quark-antiquark pair. We write an associated amplitude as

$$\mathrm{i}\,\mathcal{M}(q(p_1), \bar{q}(p_2), \dots) = g_s T^a_{ij} \bar{V}(p_2)\gamma_\mu U(p_1) \frac{-g^{\mu\nu}}{(p_1+p_2)^2} \widetilde{\mathcal{M}}^a_\nu(g(p_1+p_2), \dots)\,, \tag{6}$$

where $\widetilde{\mathcal{M}}_\nu$ is the amplitude stripped of the quark-antiquark pair,

$\mathrm{i}\,\mathcal{M} =$ 

, $\qquad \widetilde{\mathcal{M}}^a_\nu(g(k), \dots) =$ 

. (7)

The cross means to omit the polarisation vector associated with the external line.

3. Argue that one can make the replacement

$$g^{\mu\nu} \to -\sum_\lambda \varepsilon^\mu_\lambda(p,\bar{p})\,[\varepsilon^\nu_\lambda(p,\bar{p})]^* + \mathcal{O}\big(p_\perp^2\big) \tag{8}$$

in Eq.(6), where p and $\bar{p}$ are the two light-like momenta used for the Sudakov decomposition. Hence, the scattering amplitude can be written as

$$\mathrm{i}\,\mathcal{M}(q(p_1), \bar{q}(p_2), \dots) = \frac{g_s}{(p_1+p_2)^2}\, T^a_{ij} \sum_\lambda \bar{V}(p_2)\, \not{\varepsilon}_\lambda(p,\bar{p})\, U(p_1) \widetilde{\mathcal{M}}(g^{-\lambda}(p), \dots) + \mathcal{O}\big(p_\perp^2\big)\,. \tag{9}$$

4. Take the collinear limit in Eq.(9). You should find[1]

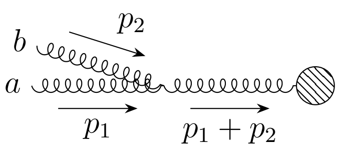
$$\mathrm{i}\,\mathcal{M}(q_L(p_1), \bar{q}_R(p_2), \dots) \to -\sqrt{2} g_s\, T^a_{ij} \left\{ \frac{z}{[12]} \widetilde{\mathcal{M}}(g^-(p), \dots) + \frac{1-z}{\langle 12 \rangle} \widetilde{\mathcal{M}}(g^+(p), \dots) \right\}, \tag{10}$$

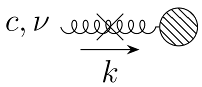
$$\mathrm{i}\,\mathcal{M}(q_R(p_1), \bar{q}_L(p_2), \dots) \to \sqrt{2} g_s\, T^a_{ij} \left\{ \frac{1-z}{[12]} \widetilde{\mathcal{M}}(g^-(p), \dots) + \frac{z}{\langle 12 \rangle} \widetilde{\mathcal{M}}(g^+(p), \dots) \right\}, \tag{11}$$

where parameterised $x_1 = z$ and $x_2 = 1 - z$ in the collinear limit.

Consider now the last case, $gg \to g$. An associated amplitude reads

$$\begin{aligned} \mathrm{i}\,\mathcal{M}(g(p_1), g(p_2), \dots) =& \mathrm{i}\, f^{abc}\, g_s\, \varepsilon^{\lambda_1}_{\mu_1} \varepsilon^{\lambda_2}_{\mu_2} \Big[g^{\mu_1\mu_2} (p_2 - p_1)^{\mu_3} + g^{\mu_2\mu_3} (-p_1 - 2p_2)^{\mu_1} + g^{\mu_3\mu_1} (2p_1 + p_2)^{\mu_2} \Big] \\ & \times \frac{-g_{\mu_3\nu}}{(p_1+p_2)^2} \widetilde{\mathcal{M}}^{c,\nu}(g(p_1+p_2), \dots)\,, \end{aligned} \tag{12}$$

$\mathrm{i}\,\mathcal{M} =$ 

, $\widetilde{\mathcal{M}}^c_\nu(g(k), \dots) =$ 

. (13)

5. Argue that we can again make the replacement in eq. (8), such that the amplitude in eq. (12) can be written as

$$\mathrm{i}\,\mathcal{M}(g(p_1), g(p_2), \dots) = \mathrm{i}\, f^{abc}\, g_s \sum_\lambda \mathrm{Split}_\lambda(g^{\lambda_1}(p_1), g^{\lambda_2}(p_2))\, \widetilde{\mathcal{M}}(g^{-\lambda}(p_1+p_2), \dots) + \mathcal{O}\big(p_\perp^2\big) \tag{14}$$

where we introduced the so-called *splitting function*

$$\begin{aligned} \mathrm{Split}_\lambda(g^{\lambda_1}(p_1), g^{\lambda_2}(p_2)) =& \frac{\varepsilon^{\lambda_1}_{\mu_1}(p_1, r_1)\, \varepsilon^{\lambda_2}_{\mu_2}(p_2, r_2)\, \varepsilon^{\lambda}_{\mu_3}(p, \bar{p})}{(p_1+p_2)^2} \times \\ & \times \Big[g^{\mu_1\mu_2} (p_2 - p_1)^{\mu_3} - 2 g^{\mu_2\mu_3} p_2^{\mu_1} + 2 g^{\mu_3\mu_1} p_1^{\mu_2} \Big]. \end{aligned} \tag{15}$$

6. Discuss, why in the collinear limit only $r_1 = r_2 = \bar{p}$ is a sensible choice of reference vectors.

7. Compute the collinear limit of $\mathrm{Split}_\lambda(g^{\lambda_1}(p_1), g^{\lambda_2}(p_2))$ in eq. (15) for all independent helicity configurations. You should find

$$\mathrm{Split}_+(g^+(p_1), g^+(p_2)) = 0\,, \quad \mathrm{Split}_-(g^+(p_1), g^+(p_2)) = -\frac{\sqrt{2}}{[12]} \frac{1}{\sqrt{z}\sqrt{1-z}}\,, \tag{16}$$

$$\mathrm{Split}_+(g^+(p_1), g^-(p_2)) = -\frac{\sqrt{2}}{[12]} \frac{(1-z)^2}{\sqrt{z}\sqrt{1-z}}\,, \quad \mathrm{Split}_-(g^+(p_1), g^-(p_2)) = \frac{\sqrt{2}}{\langle 12 \rangle} \frac{z^2}{\sqrt{z}\sqrt{1-z}}\,. \tag{17}$$

[1] We use the convention that a left-handed anti-quark spinor corresponds to a right-handed quark spinor and vice versa.