### Advanced Quantum Mechanics SS 2024

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Sheet 06: Partial Wave Expansion and Sommerfeld Enhancement (to be handed in by 17.07.2024)

#### **1** Partial Wave Expansion

The goal of this exercise is to prove the partial wave expansion

$$e^{i\vec{k}\cdot\vec{r}} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i^{l} j_{l}(kr) Y_{lm}^{*}(\theta_{k},\phi_{k}) Y_{lm}(\theta_{r},\phi_{r}), \qquad (1)$$

where in spherical coordinates

$$\vec{k} = k \begin{pmatrix} \cos \phi_k \sin \theta_k \\ \sin \phi_k \sin \theta_k \\ \cos \theta_k \end{pmatrix}, \qquad \vec{r} = r \begin{pmatrix} \cos \phi_r \sin \theta_r \\ \sin \phi_r \sin \theta_r \\ \cos \theta_r \end{pmatrix}.$$
(2)

*Hint*: In the appendix, you will find formulas useful throughout the problem.

1. First, show that the partial wave expansion is equivalent to the statement

$$e^{i p \cos \theta} = \sum_{l=0}^{\infty} (2l+1) i^l j_l(p) P_l(\cos \theta), \qquad (3)$$

where p = kr and  $\theta$  the angle between the two vectors,  $\theta = \langle (\vec{k}, \vec{r}) \rangle$ . The functions  $P_l$  are the Legendre polynomials, while  $j_l$  denotes the spherical Bessel functions given by *Rayleigh's formula*,

$$j_l(p) = (-p)^l \left(\frac{1}{p} \frac{\mathrm{d}}{\mathrm{d}p}\right)^l \frac{\sin p}{p} \,. \tag{4}$$

2. Next, show that eq. (3) is, in turn, equivalent to the spherical Bessel functions admitting the representation

$$j_l(p) = \frac{(-i)^l}{2} \int_{-1}^{1} \mathrm{d}s \, P_l(s) \, e^{i \, p \, s} \,. \tag{5}$$

*Hint*: Multiply both sides of eq. (3) by  $P_{l'}(\cos \theta)$ ,  $l' \in \mathbb{N}_0$ , and integrate over  $\cos \theta$  from -1 to 1.

3. To show the equivalence of eq. (5) to the definition in eq. (4), prove the identity

$$\frac{\sin p}{p} = \frac{1}{2} \int_{-1}^{1} \mathrm{d}s \, e^{i \, p \, s} \tag{6}$$



and plug it into eq. (4) to show that

$$j_l(p) = \frac{(-p)^l}{2^{l+1} l!} \int_{-1}^1 \mathrm{d}s \, (s^2 - 1)^l \, e^{i \, p \, s} \,. \tag{7}$$

*Hint*: Notice that  $\frac{d}{ds}(s^2 - 1) = 2s$  and use integration-by-parts.

4. Finally, use integration-by-parts once again to show that the integral in eq. (7) can be brought to the form of eq. (5).

## 2 Phase Shifts for the potential well

Consider an incoming particle with mass m and momentum  $\vec{k} = (0, 0, k)^T$  scattering off a target sitting at the origin. Suppose our particle is subject to the spherical potential well,

$$V(\vec{r}) = V_0 \,\theta(R - r). \tag{8}$$

1. Explain that the wave function of the particle must take the form

$$\psi(r,\theta,\phi) = \begin{cases} \sum_{l=0}^{\infty} c_l \, j_l(k'r) \, Y_{l0}(\theta,\phi) & \text{for } r < R \,, \\ \sum_{l=0}^{\infty} \left[ a_l \, j_l(kr) + b_l \, y_l(kr) \right] \, Y_{l0}(\theta,\phi) & \text{for } r > R \,, \end{cases}$$
(9)

where  $k' = \sqrt{k^2 - 2mV_0/\hbar^2}$  and  $j_l$  and  $y_l$  denote the spherical Bessel functions.

2. Impose that the wave function should be continuous and differentiable at r = R to show that

$$\alpha_l \equiv \frac{b_l}{a_l} = \frac{k \, j_l'(kR) \, j_l(k'R) - k' \, j_l'(k'R) j_l(kR)}{k' \, j_l'(k'R) \, y_l(kR) - k \, y_l'(kR) j_l(k'R)} \,. \tag{10}$$

3. Examine the asymptotic behaviour of eq.(9) as  $r \to \infty$  to show that the phase shift  $\delta_l$  is given by

$$e^{2i\delta_l} = \frac{1 - i\alpha_l}{1 + i\alpha_l}.$$
(11)

*Hint*: The spherical Bessel functions exhibit the asymptotic behaviour

$$j_l(x) \sim \frac{1}{x} \cos\left(x - \frac{(l+1)\pi}{2}\right), \qquad y_l(x) \sim \frac{1}{x} \sin\left(x - \frac{(l+1)\pi}{2}\right),$$
 (12)

as  $x \to \infty$ .

4. Calculate the phase shift for the l = 0 partial wave and expand the result in the limit of small potential,  $V_0 \ll (\hbar^2 k^2)/(2m)$ . *Hint*:  $j_0(x) = \sin(x)/x$ ,  $y_0(x) = -\cos(x)/x$ . 5. Compute the contribution of the l = 0 partial wave to the scattering amplitude  $f_k(\theta, \phi)$ . You should find

$$f_k(\theta,\phi) = -\frac{mV_0}{\hbar^2} \frac{kR - \cos\left(kR\right)\sin\left(kR\right)}{k^3} + \mathcal{O}\left(V_0^2\right) + \{\text{partial waves } l \ge 1\} .$$
(13)

On the last sheet, you calculated the full scattering amplitude in Born approximation (up to  $\mathcal{O}(V_0^2)$ ). The result you obtained was

$$f_k(\theta,\phi) = -\frac{2mV_0}{\hbar^2} \frac{|\Delta \vec{k}|R - \cos\left(|\Delta \vec{k}|R\right)\sin\left(|\Delta \vec{k}|R\right)}{|\Delta \vec{k}|^3},\tag{14}$$

where  $|\Delta \vec{k}| = k\sqrt{2(1 - \cos\theta)}$ . Integrating over the angles, the full cross section, expanded in the low energy limit  $kR \ll 1$ , reads

$$\sigma_F = \int d\Omega \ |f_k(\theta, \phi)|^2 = \frac{16\pi \, m^2 \, V_0^2}{9\hbar^4} R^6 + \mathcal{O}\left(kR\right) \,. \tag{15}$$

6. Compute the contribution of the l = 0 partial wave to the cross section and expand it in the low energy limit,  $kR \ll 1$ . Compare the result to the full cross section (15).

#### **3** Sommerfeld Enhancement

Consider an incoming, non-relativistic free particle with mass m, momentum  $\vec{k} = (0, 0, k)^T$  and wave function  $\psi_k^{(0)}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}$  and a target sitting at the origin, which can interact with the particle. Assuming the interaction to be point-like (for example an annihilation process), the probability for the interaction to take place is proportional to the probability to find the particle at the origin,  $|\psi_k^{(0)}(0)|^2$ .

Suppose now, that we turn on some central potential V(r) and the wave function of the particle scattering off to the potential is  $\psi_k(\vec{r})$ . The interaction probability is now proportional to  $|\psi_k(0)|^2$ . The Sommerfeld enhancement factor  $S_k$  is defined as the ratio of the interaction probability with the potential being turned on compared to the potential being turned off,

$$S_k = \frac{|\psi_k(0)|^2}{|\psi_k^{(0)}(0)|^2} = |\psi_k(0)|^2.$$
(16)

With the boundary condition, that scattering off the potential can only produce outgoing spherical waves as  $r \to \infty$ , you derived in the lecture that the wave function  $\psi_k(\vec{r})$  can be expanded in terms of spherical harmonics as

$$\psi_k(\vec{r}) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) \, i^l \, e^{i\delta_l} \, P_l(\cos\theta) \, R_{k,l}(r) \,, \tag{17}$$

where  $\delta_l$  is the phase shift and  $R_{k,l}(r)$  are the solutions to the radial part of the Schrödinger equation,

$$\frac{\hbar^2}{2m} \frac{\mathrm{d}^2(rR_{k,l}(r))}{\mathrm{d}r^2} = \left[ V(r) - \frac{k^2}{2m} + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} \right] rR_{k,l}(r) \,. \tag{18}$$

1. Determine the behaviour of  $R_{k,l}(r)$  as  $r \to 0$ , assuming that the potential does not blow up faster than 1/r as  $r \to 0$ .

*Hint*:  $R_{k,l}(r)$  should be regular at r = 0.

2. Show that the Sommerfeld enhancement factor can be calculated as

$$S_k = \left|\frac{R_{k,0}(0)}{k}\right|^2.$$
(19)

3. In the case of the Coulomb potential

$$V_c(r) = \frac{\alpha}{r} \,, \tag{20}$$

one finds

$$|R_{k,0}(0)|^2 = k^2 \left| \frac{\eta}{e^{\eta} - 1} \right|, \quad \text{where} \quad \eta = \frac{2\pi m\alpha}{k}.$$
 (21)

Discuss the qualitative behaviour of the Sommerfeld enhancement factor for the two cases of a repulsive potential  $\alpha > 0$  and an attractive potential,  $\alpha < 0$  as a function of the particle's speed v = k/m.

# Appendix

Spherical harmonics,  $Y_{l,m}(\theta, \phi)$ , satisfy

Orthogonality: 
$$\int_{0}^{2\pi} \mathrm{d}\phi \int_{-1}^{1} \mathrm{d}\cos(\theta) Y_{lm}^{*}(\theta,\phi) Y_{kn}(\theta,\phi) = \delta_{lk}\delta_{mn}, \qquad (22)$$

Addition: 
$$\frac{4\pi}{2l+1} \sum_{m=-l}^{m} Y_{lm}(\theta,\phi) Y_{lm}(\theta',\phi') = P_l(\hat{\vec{r}} \cdot \hat{\vec{r}}'), \qquad (23)$$

where in spherical coordinates

$$\hat{\vec{r}} = \begin{pmatrix} \cos\phi \sin\theta\\ \sin\phi \sin\theta\\ \cos\theta \end{pmatrix}, \qquad \hat{\vec{r}}' = \begin{pmatrix} \cos\phi' \sin\theta'\\ \sin\phi' \sin\theta'\\ \cos\theta' \end{pmatrix}.$$
(24)

and  $P_l(x)$  denotes the Legendre polynomials, given by *Rodrigues' formula*,

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} \left(x^2 - 1\right)^l \,. \tag{25}$$

. They satisfy the orthogonality relation

$$\int_{-1}^{1} \mathrm{d}x \, P_l(x) \, P_{l'}(x) = \frac{2 \, \delta_{ll'}}{2l+1} \,. \tag{26}$$