Scattering Amplitudes in QFT WS 2023/24

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Sheet 05: Relations among colour-ordered amplitudes

Exercise 1 - Kleiss-Kuijf relations and photon decoupling identity

In the previous exercises we have seen that the tree-level four-gluon amplitude in QCD admits a colour decomposition of the form

$$\mathcal{M}_{4g} = 4 \sum_{\sigma \in S_3} \mathrm{tr}\left(T^{a_1} T^{a_{\sigma(2)}} T^{a_{\sigma(3)}} T^{a_{\sigma(4)}}\right) \mathcal{A}\left[1, \sigma(2), \sigma(3), \sigma(4)\right] . \tag{1}$$

The partial amplitudes $\mathcal{A}$ satisfy several linear relations, for example the *photon decoupling identity*,

$$\mathcal{A}\left[1, \sigma(2), \sigma(3), \sigma(4)\right] + \mathcal{A}\left[1, \sigma(3), \sigma(4), \sigma(2)\right] + \mathcal{A}\left[1, \sigma(4), \sigma(2), \sigma(3)\right] = 0\,, \tag{2}$$

where $\sigma \in S_3$.

1. We argued that we can extend $SU(N) \to U(N)$ without changing $\mathcal{M}_{4g}$ as the extra generator is proportional to the identity, which implies that the extra gauge boson (the "photon") doesn't couple directly to the gluons. Use this to prove eq. (2).

The photon decoupling identity might also be seen as a consequence of the fact that the colour basis in eq. (1) is overcomplete. Instead of the trace basis, the amplitude may be decomposed in the so-called *multiperipheral* colour basis given by products of structure constants that are independent under the Jacobi identity. For a tree-level n-point gluon amplitude, this basis can be written as

$$\left\{ f^{a_1 a_{\sigma(2)} b_1} f^{b_1 a_{\sigma(3)} b_2} \cdots f^{b_{n-3} a_{\sigma(n-1)} a_n} \,|\, \sigma \in S_{n-2} \right\} . \tag{3}$$

Relating the two colour bases leads to the so-called Kleiss-Kuijf relations among the colour-ordered amplitudes $\mathcal{A}$. One way to write these relations is

$$\mathcal{A}[1, \{\alpha\}, n, \{\beta\}] = (-1)^{|\beta|} \sum_{\sigma \in \{\alpha\} \sqcup\!\sqcup \{\beta^T\}} \mathcal{A}[1, \sigma, n]\,, \tag{4}$$

where $\{\beta^T\}$ denotes the reverse ordering of the labels $\{\beta\}$ and $|\beta|$ is the number of labels in $\{\beta\}$. The sum runs over all "shuffles" of the labels $\{\alpha\}$ and $\{\beta^T\}$. That is over all permutations of labels in the joined set $\{\alpha\} \cup \{\beta^T\}$ which respect the relative ordering within $\{\alpha\}$ and $\{\beta^T\}$ (i.e. all ways to slide two decks of cards into each other).

2. Write down all the Kleiss-Kuijf relations for $n = 4$ and show that they reproduce the reversal and photon decoupling identities.

3. To familiarise yourself with the shuffle product, deduce the Kleiss-Kuijf relation for $n = 6$ with $\alpha = \{2, 3\}$ and $\beta = \{4, 5\}$.

There is another way to formulate the Kleiss-Kuijf relations, which has a natural generalisation to colour decomposition beyond tree level. Collecting the colour trace basis into a $(n-1)!$-dimensional vector $\vec{t}$ and the multiperipheral basis into a $(n-2)!$-dimensional vector $\vec{c}$, we may write the n-gluon amplitude in either colour decomposition as

$$\mathcal{M}_{ng} = \sum_{i=1}^{(n-1)!} t_i\, \mathcal{A}_i = \sum_{i=1}^{(n-2)!} c_i\, \tilde{\mathcal{A}}_i\,. \tag{5}$$

We may relate the two by expressing $\vec{c}$ in terms of the traces $\vec{t}$,

$$\vec{c} = M \cdot \vec{t} \qquad \Rightarrow \qquad \mathcal{A}_i = \sum_{i=1}^{(n-2)!} \tilde{\mathcal{A}}_j M_{ji}\,, \tag{6}$$

where M is a $((n-2)! \times (n-1)!)$-dimensional matrix. The matrix M has $((n-1)! - (n-2)!)$ right null eigenvectors $\vec{r}^{\,\lambda}$, which we can use to derive relations among the colour-ordered amplitudes $\mathcal{A}_i$,

$$M \cdot \vec{r}^{\,\lambda} = 0 \qquad \Rightarrow \qquad \sum_{i=1}^{(n-1)!} \mathcal{A}_i\, r_i^{\lambda} = 0\,. \tag{7}$$

4. Check for $n = 4$ that this construction reproduces the Kleiss-Kuijf relations you already found above.

Exercise 2 - BCJ relations

We have seen that the Kleiss-Kuijf relations reduce the number of independent n-gluon tree-level partial amplitudes to $(n-2)!$. However, there exist even more relations, which reduce this number further to $(n-3)!$. We will study these so-called *BCJ*[1] relations for the concrete case $n = 4$ in this exercise. We start from the expression for the amplitude as given by Feynman rules

$$\mathcal{M}_{4g} = \frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\,. \tag{8}$$

Here, $s = (p_1 + p_2)^2,\ t = (p_1 + p_3)^2,\ u = (p_1 + p_4)^2$ are the usual Mandelstam variables, the c_i are the colour factors

$$c_s \equiv f^{a_1 a_2 b} f^{b a_3 a_4}, \quad c_t \equiv f^{a_1 a_3 b} f^{b a_4 a_2}, \quad c_u \equiv f^{a_1 a_4 b} f^{b a_2 a_3}\,, \tag{9}$$

and the n_i collect all the other factors, which depend on the kinematics and the polarisations.

1. Relate the three colour factors by the Jacobi identity. Use it to show that the amplitude $\mathcal{M}_{4g}$ is invariant under

$$n_s \to n_s + s\,f\,, \quad n_t \to n_t + t\,f\,, \quad n_u \to n_u + u\,f\,, \tag{10}$$

   where f is an arbitrary function. This symmetry transformation is often referred to as *generalised gauge transformation.*

[1]After Z. Bern, J. J. M. Carasso and H. Johannson (2008).

2. Use the Jacobi identity once more to obtain the multiperipheral colour-decomposition of the amplitude.

3. Use your results from point four of the previous exercise and eq. (6) to express the partial amplitudes $\mathcal{A}$ from eq. (1) in terms of the Mandelstam variables and the factors n_i. Verify the Kleiss-Kuijf relations explicitly.

4. Colour-Kinematics Duality states that the numerator factors satisfy a "Jacobi identity" of their own, analogous to the colour factors,

$$n_s + n_t + n_u = 0\,. \tag{11}$$

   Verify this result from explicit expressions in terms of Feynman rules.

5. Use eq. (11) to verify the BCJ relation

$$u\,\mathcal{A}[1,2,3,4] = t\,\mathcal{A}[1,2,4,3]\,. \tag{12}$$