Scattering Amplitudes in QFT WS 2023/24

Lecturer: Prof. Lorenzo Tancredi
Assistant: Fabian Wagner
https://www.ph.nat.tum.de/ttpmath/teaching/ws-2023-2024/

Sheet 04: Four-gluon amplitude squared and colour decomposition at higher points and loop level

---

Exercise 1 - Four gluon amplitude squared and summed over colours and polarisations

In the last exercise sheet, we derived the colour-decomposed tree-level four-gluon amplitude

$$\mathcal{M}_{4g} = 4 \sum_{\sigma \in S_3} \mathrm{tr}\left(T^{a_1} T^{a_{\sigma(2)}} T^{a_{\sigma(3)}} T^{a_{\sigma(4)}}\right) \mathcal{A}\left[1, \sigma(2), \sigma(3), \sigma(4)\right] , \tag{1}$$

where the MHV amplitudes read

$$\begin{aligned} \mathcal{A}\left[1^+, 2^+, 3^-, 4^-\right] &= g^2 \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} , \\ \mathcal{A}\left[1^+, 2^-, 3^+, 4^-\right] &= g^2 \frac{\langle 13 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} , \\ \mathcal{A}\left[1^+, 2^-, 3^-, 4^+\right] &= g^2 \frac{\langle 14 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} . \end{aligned} \tag{2}$$

In the lecture, it was discussed that we can extend $\mathrm{SU}(N) \to \mathrm{U}(N)$ for tree-level gluon amplitudes since they do not receive contributions from fermions. This allows us to use $T^a_{ij} T^a_{kl} = \delta_{il}\delta_{kj}/2$ when manipulating strings of colour generators.

1. First, show that the following identities hold:

$$\begin{aligned} \mathrm{tr}\left(T^a A\right) \left[\mathrm{tr}\left(T^a B\right)\right]^\dagger &= \frac{1}{2} \mathrm{tr}(A B^\dagger) , \\ \mathrm{tr}(T^a A\, T^a B) &= \frac{1}{2} \mathrm{tr}(A)\, \mathrm{tr}(B) . \end{aligned} \tag{3}$$

2. Using Eq.(3), show that squared traces in $\sum_{\mathrm{col}} |\mathcal{M}_{4g}|^2$ yield

$$\mathrm{tr}\left(T^{a_1} T^{a_{\sigma(2)}} T^{a_{\sigma(3)}} T^{a_{\sigma(4)}}\right) \left[\mathrm{tr}\left(T^{a_1} T^{a_{\sigma(2)}} T^{a_{\sigma(3)}} T^{a_{\sigma(4)}}\right)\right]^\dagger = \frac{N^4}{16} . \tag{4}$$

3. Using Eq.(3), show that interference terms of traces in $\sum_{\mathrm{col}} |\mathcal{M}_{4g}|^2$ yield

$$\mathrm{tr}\left(T^{a_1} T^{a_{\sigma(2)}} T^{a_{\sigma(3)}} T^{a_{\sigma(4)}}\right) \left[\mathrm{tr}\left(T^{a_1} T^{a_{\sigma'(2)}} T^{a_{\sigma'(3)}} T^{a_{\sigma'(4)}}\right)\right]^\dagger = \frac{N^2}{16} , \tag{5}$$

   for $\sigma \neq \sigma'$ different permutations.

4. Using the results for color traces in Eq.(4) and Eq.(5), we can write

$$\begin{aligned}\sum_{\text{col}} |\mathcal{M}_{4g}| =& N^4 \sum_{\sigma\in S_3} |\mathcal{A}\left[1,\sigma(2),\sigma(3),\sigma(4)\right]|^2 + \\ &+ N^2 \sum_{\substack{\sigma,\sigma'\in S_3\\ \sigma\neq\sigma'}} \mathcal{A}\left[1,\sigma(2),\sigma(3),\sigma(4)\right] \left(\mathcal{A}\left[1,\sigma'(2),\sigma'(3),\sigma'(4)\right]\right)^\dagger .\end{aligned} \tag{6}$$

Using the relation between the colour-ordered amplitudes,

$$\text{(a)} \quad \textit{Cyclicity}: \quad \mathcal{A}\left[a,b,c,d\right] = \mathcal{A}\left[d,a,b,c\right] = \mathcal{A}\left[c,d,a,b\right] = \mathcal{A}\left[b,c,d,a\right] , \tag{7}$$

$$\text{(b)} \quad \textit{Reflection symmetry}: \quad \mathcal{A}\left[a,b,c,d\right] = \mathcal{A}\left[d,c,b,a\right] , \tag{8}$$

$$\text{(c)} \quad \textit{Photon decoupling}: \quad \mathcal{A}\left[a,b,c,d\right] + \mathcal{A}\left[b,a,c,d\right] + \mathcal{A}\left[b,c,a,d\right] = 0 . \tag{9}$$

show that we can write

$$\sum_{\text{col}} |M_{4g}| = 2N^2(N^2-1)\left\{ |\mathcal{A}[1,2,3,4]|^2 + |\mathcal{A}[1,2,4,3]|^2 + |\mathcal{A}[1,3,2,4]|^2 \right\}. \tag{10}$$

5. Compute the sum over helicities for the first partial amplitude in Eq.(10) using the results in Eq.(2). You should find

$$\sum_{\text{pol}} |\mathcal{A}[1,2,3,4]|^2 = 2g^4 \left[\frac{s_{12}^2}{s_{23}^2} + \frac{s_{13}^4}{s_{12}^2 s_{23}^2} + \frac{s_{23}^2}{s_{12}^2} \right] . \tag{11}$$

6. Use the result in Eq.(11) to compute the final result

$$\sum_{\text{pol,col}} |M_{4g}(1,2,3,4)| = 2g^4 N^2(N^2-1) \times \frac{(s_{12}^2+s_{13}^2+s_{14}^2)^3}{s_{12}^2 s_{13}^2 s_{14}^2} . \tag{12}$$

Exercise 2 - Colour ordering in QCD at higher point and higher loops

1. Argue that tree-level n-gluon amplitudes admit for any n a colour decomposition of the form

$$\mathcal{M}_{ng}^{\text{tree}} = \sum_{\sigma\in S_{n-1}} \text{tr}\left(T^{a_1} T^{a_{\sigma(2)}} \ldots T^{a_{\sigma(n)}}\right) \mathcal{A}\left[1,\sigma(2),\ldots\sigma(n)\right] . \tag{13}$$

2. Argue that extending $SU(N) \to U(N)$ is no longer possible at loop level.

3. Argue that the colour basis must be enlarged at loop-level. Give an explicit example of a Feynman diagram leading to this enlargement at $n = 4$ and derive the type of new colour basis elements produced by it.